Appendix to Chapter 1

We sketch the proofs of the fundamental existence and uniqueness theorems for systems of complex first order differential equations. For more details and alternate proofs see [Ince, 1] or [Cohn, 1].

Theorem A: Let $\frac{d\underline{x}}{dt} = \underline{f}(t,\underline{x})$ or

$$\frac{dx_j}{dt} = f_j(t, x_1, \cdots, x_m), \quad j = 1, \cdots, m \tag{A.1}$$

be a system of n first order equations and let $\underline{x}^0 = (x_1^0, \cdots, x_m^0)$ be complex constants. Suppose the f_j are analytic in the domain $|t| < a$, $|x_k - x_k^0| < b$, $1 \le k \le n$, where a and b are positive constants. Then there exists a unique solution $\underline{x}(t) = (x_1(t), \cdots, x_n(t))$ of (A.1) such that 1) $\underline{x}(t)$ is analytic in a neighborhood of $t = 0$ and 2) $\underline{x}(0) = \underline{x}^0$.

Proof: Uniqueness. Suppose $\underline{x}(t)$ is an analytic solution of (A.1) with $\underline{x}(0) = \underline{x}^0$. Expanding $\underline{x}(t)$ in a Taylor series about $t = 0$ we have

$$\begin{aligned} \underline{x}(t) &= \underline{x}(0) + \underline{x}'(0)t + \underline{x}''(0)\frac{t^2}{2} + \cdots \\ &= \sum_{j=0}^{\infty} \frac{d^j \underline{x}(s)}{ds^j}\bigg|_{s=0} \frac{t^j}{j!}. \end{aligned} \tag{A.2}$$

However, from (A.1) and the initial conditions it follows that $\underline{x}(0) = \underline{x}^0$, $\frac{d\underline{x}(0)}{dt} = \underline{f}(0,\underline{x}^0)$, $\frac{d^2}{dt^2}\underline{x}(0) = \frac{\partial f}{\partial t}(0,\underline{x}^0) + \sum_{l=1}^{n} \frac{\partial f}{\partial x_l}(0,\underline{x}^0) f_l(0,\underline{x}^0)$ and in general that each of the coefficients $\frac{d^m}{dt^m}\underline{x}(0)$ can be evaluated directly in terms of $\underline{x}^0$ and the functions $\underline{f}(t,\underline{x})$ by means of the chain rule. At each step of the process we determine the coefficients by addition and multiplication together of various derivatives of $\underline{f}(t,\underline{x})$, but never by subtraction. Since the coefficients are uniquely determined by (A.1) and the initial conditions it follows that the solution (A.2) is unique. Furthermore, (A.2) formally satisfies (A.1).

Existence: If a solution $\underline{x}(t)$ exists it must be given by the Taylor series expansion (A.2) with uniquely determined coefficients. We will show that this formal expansion always converges to an analytic function $\underline{x}(t)$.

Set $\underline{y} = \underline{x} - \underline{x}^0$ and $\underline{g}(t,\underline{y}) = \underline{f}(t,\underline{x})$. Then the system (A.1) becomes $\frac{d\underline{y}}{dt} = \underline{g}(t,\underline{y})$, $\underline{y}(0) = \underline{\theta}$. Thus, without loss of generality we can assume that the initial conditions 2) are $\underline{x}(0) = \underline{\theta}$.

Since $\underline{f}(t,\underline{x})$ is analytic we have

(A.3) $$F_j(t,\underline{x}) = \sum_{l,k_1,\cdots,k_m=0}^{\infty} C^{(j)}_{lk_1\cdots k_m} t^l x_1^{k_1}\cdots x_m^{k_m}, \quad j=1,\cdots,m$$
$$|t|<a,\ |x_k|<b$$

where $l!\,k_1!\cdots k_m!\,C^{(j)}_{lk_1\cdots k_m} = \dfrac{\partial^{l+k_1+\cdots+k_m}}{\partial t^l \partial x_1^{k_1}\cdots\partial x_m^{k_m}} F_j(t,\underline{x})\Big|_{t=0,\ \underline{x}=\underline{\theta}}$.

Choose a constant c such that $0<c<a$, $0<c<b$. Now the power series (A.3) is absolutely convergent for $t = x_1 = \cdots = x_m = c$ so it follows that there is a fixed constant $M>0$ such that

$$\sum_{lk_1\cdots k_m} |C^{(j)}_{lk_1\cdots k_m}|\, c^{l+k_1+\cdots+k_m} < M, \quad j=1,\cdots,m.$$

In particular,

(A.4) $$|C^{(j)}_{lk_1\cdots k_m}| < \frac{M}{c^{l+k_1+\cdots+k_m}}.$$

Consider the system

(A.5) $$\frac{d\underline{X}}{dt} = \underline{F}(t,\underline{X}),\quad \underline{X}(0) = \underline{\theta}$$

where

$$\underline{F}(t,\underline{X}) = (F(t,\underline{X}),\cdots,F(t,\underline{X}))$$

and

(A.6) $$F(t,\underline{X}) = \frac{M}{(1-\frac{t}{c})(1-\frac{X_1}{c})\cdots(1-\frac{X_m}{c})}.$$

The coefficient of $t^{l}X_1^{l_1}\cdots X_m^{l_m}$ in the Taylor series expansion of $F(t,\underline{X})$ about $(0,\underline{0})$ is just $M/c^{l+l_1+\cdots+l_m}$. It follows from the uniqueness theorem and (A.4) that if $\underline{X}(t)$ is a solution of (A.5) then

(A.7) $$\underline{X}(t) = \sum_{j=0}^{\infty} \frac{d^j}{dt^j}\underline{X}(t)\Big|_{t=0}\frac{t^j}{j!}$$

where $$\frac{d^j}{dt^j}X_l(t)\Big]_{t=0} \geq \left|\frac{d^j}{dt^j}x_l(t)\Big]_{t=0}\right|, \quad j=0,1,\cdots.$$

We say that the power series (A.7) <u>dominates</u> (A.2). Clearly, if (A.7) converges for some $t=t_0>0$ then (A.2) converges absolutely for all $|t|<t_0$. We shall show that (A.7) converges by explicitly solving (A.5).

Since (A.5) is symmetric in $X_1,\cdots,X_m$ we must have $X_1(t)=\cdots=X_m(t) = X(t)$. Therefore, this system is equivalent to the single equation

(A.8) $$\frac{d X(t)}{dt} = \frac{M}{(1-\frac{t}{c})(1-\frac{X}{c})^m}, \quad X(0)=0,$$

which has the explicit solution

$$X(t) = c - c\left[1+(m+1)M\ln\left(1-\frac{t}{c}\right)\right]^{\frac{1}{m+1}},$$

analytic for $|t|<c(1-e^{-M^{-1}(1+m)^{-1}})$. Q.E.D.

<u>Theorem B:</u> Let

(A.9) $$\frac{\partial x_j}{\partial t_l} = f_{jl}(t_1,\cdots,t_n,x_1,\cdots,x_m), \quad \begin{matrix} j=1,\cdots,m \\ l=1,\cdots,n \end{matrix},$$

be a system of first order partial differential equations and let $\underline{X}^0 = (x_1^0,\cdots,x_m^0)$ be complex constants. Suppose the f_{jl} are analytic in the domain $|t_l|<a$, $|x_j-x_j^0|<b$, $1\leq l\leq n$, $1\leq j\leq m$, where a and b are positive constants. Then there exists a unique solution $\underline{x}(\underline{t}) = (x_1(\underline{t}),\cdots,x_m(\underline{t}))$ of (A.9) with the properties

1) $\underline{X}(\underline{t})$ is analytic in a neighborhood of $\underline{t}=\underline{0}$.

2) $\underline{X}(0)=\underline{X}^0$

if the $f_{j\ell}$ satisfy the integrability conditions

(A.10) $$\sum_{s=1}^{m}\left(\frac{\partial f_{j\ell}}{\partial x_s} f_{sk} - \frac{\partial f_{jk}}{\partial x_s} f_{s\ell}\right) = \frac{\partial f_{jk}}{\partial t_\ell} - \frac{\partial f_{j\ell}}{\partial t_k}, \quad j=1,\cdots,m \quad k,\ell=1,\cdots,n.$$

Conversely, if there is a solution $\underline{X}(\underline{t})$ of (A.9) satisfying properties 1) and 2) then the f_{jk} satisfy (A.10)

Proof: Suppose $\underline{X}(\underline{t})$ is a solution of (A.9) satisfying properties 1) and 2). As the reader can easily verify, the equality of the second partials $\frac{\partial^2 x_i}{\partial t_j \partial t_k} = \frac{\partial^2 x_i}{\partial t_k \partial t_j}$ is equivalent to (A.10), i.e., the integrability conditions are $\longrightarrow \frac{\partial}{\partial t_k}(f_{j\ell}) = \frac{\partial}{\partial t_\ell}(f_{jk})$

Conversely, suppose conditions (A.10) are satisfied. We will try to determine a solution $\underline{X}(\underline{t})$ of (A.9) by finding the coefficients in the Taylor series expansion

(A.11) $$\underline{X}(\underline{t}) = \sum_{k=0}^{\infty} \sum_{j_1+\cdots+j_n=k} \frac{\partial^k \underline{X}(\underline{0})}{\partial t_1^{j_1}\cdots \partial t_n^{j_n}} \, \frac{t_1^{j_1}\cdots t_n^{j_n}}{j_1!\cdots j_n!} .$$

Now $\underline{X}(0)=\underline{X}^0$ and $\frac{\partial x_j}{\partial t_\ell}(\underline{0}) = f_{j\ell}(\underline{0},\underline{X}^0)$. The remaining coefficients can be determined recursively from (A.9) by differentiation with respect to the t_ℓ. The integrability conditions (A.10) guarantee that the coefficients $\frac{\partial^2 \underline{X}(0)}{\partial t_\ell \partial t_k}$ are uniquely determined, i.e., that we would not obtain different values for the coefficients if we evaluated them in a different order. Similarly, by differentiating (A.10) recursively with respect to the t_ℓ one can check that the higher order constants are also uniquely determined. We conclude that if (A.9) has a solution satisfying 1) and 2), then the solution is unique. ~~Furthermore, it is easy to show that our formal solution is unique.~~ Furthermore, it is easy to show that our formal solution formally satisfies the system of differential equations. It only remains to verify that the series (A.11) actually converges. This can be done by the method of majorants. The details of the proof are almost identical to the analogous proof of Theorem A so we omit them. Q.E.D.

Although the above proofs have been sketched for complex-valued functions, they also apply to real-valued analytic functions. Furthermore, the solutions $\underline{x}(t, \underline{x}^0)$ of (A.1) and (A.9) are analytic functions of the initial parameters $\underline{x}^0 = (x_1^0, \cdots, x_m^0)$. To see this, one can expand the functions $\underline{x}(\underline{t}, \underline{x}^0)$ as Taylor series in t_j and x_k^0, and show that the coefficients are uniquely determined by (A.1) or (A.9). Then the majorant method can be applied to show that the formal Taylor series actually converges to an analytic solution, [Cohn, 1].