CHAPTER 7

The Factorization Method

As presented in this book the procedure for associating special functions with a Lie algebra $\mathcal{G}$ divides naturally into three parts: (1) Determine an abstract irreducible representation ρ of $\mathcal{G}$. (2) Find a realization of ρ acting on a space $\mathcal{O}\!\ell$ of analytic functions. (3) Compute the multiplier representation of the local Lie group G induced by ρ on $\mathcal{O}\!\ell$. The practical importance of this procedure is demonstrated by numerous examples in Chapters 3–6.

In the next two chapters, however, we shall discontinue the multiplication of examples and concentrate instead on an analysis of the procedure itself.

The functions of hypergeometric type (hypergeometric, confluent hypergeometric, and Bessel functions) are solutions of linear second order ordinary differential equations and satisfy differential recurrence relations. In fact, corresponding to each of the operator *types* A, B, C', C'', D', E, F we have been able to derive a set of recurrence relations for these functions. Conversely, in this chapter, it will be shown that under rather general conditions the only "reasonable" families of functions satisfying second order linear differential equations and differential recurrence relations are the functions of hypergeometric type listed above. Furthermore, all of the recurrence relations obeyed by such functions can be obtained as combinations of recurrence relations derived earlier in this book. (There are families of functions other than functions of hypergeometric type, which satisfy second order equations and differential recursion relations, but they are too complicated to be of much practical importance.)

In other words special functions associated with the Lie algebras $\mathcal{G}(a, b)$ and $\mathcal{T}_6$ completely exhaust the supply of functions with the simple properties listed above. The material of Chapters 3–6 describes a one-to-one relationship between the representation theory of these Lie algebras and recurrence formulas of functions of hypergeometric type. In this sense Chapters 3–6 form a complete unit.

The technique for proving the above results is the factorization method, originated by Schroedinger and due in its definitive form to Infeld and Hull [1]. This technique was developed to solve eigenvalue problems appearing in quantum theory but is also a very powerful tool for studying recurrence formulas obeyed by special functions. The version presented here is a slight modification of Infeld and Hull [1]. For more details, especially concerning applications to physics, the reader should consult this reference.

7-1 Recurrence Relations

Let $X_m = -d^2/dx^2 + D_m(x)\, d/dx + E_m(x)$ be a sequence of second order ordinary differential operators defined for $m \in S = \{m_o, m_o \pm 1, m_o \pm 2, \ldots\}$, where m_o is an arbitrary complex number; and analytic for x in a neighborhood N of $x^o \in \mathcal{C}$. We shall be interested in solving the eigenvalue problems

$$X_m Y_\lambda(m, x) = \lambda Y_\lambda(m, x) \tag{7.1}$$

for all $m \in S$ and $x \in N$. (In the following the dependence of the function $Y_\lambda(m)$ on x will be supressed.)

If there exist linear differential operators

$$L_m^+ = A_m^+(x) \frac{d}{dx} + B_m^+(x), \qquad L_m^- = A_m^-(x) \frac{d}{dx} + B_m^-(x)$$

defined for all $m \in S$, $x \in N$, such that

$$L_{m+1}^- L_{m+1}^+ + a_{m+1} \equiv X_m, \qquad L_m^+ L_m^- + a_m \equiv X_m \tag{7.2}$$

where the a_m are constants, we say the operators X_m admit a **factorization**. In this case the eigenvalue equation (7.1) is equivalent to the two equations

$$L_{m+1}^- L_{m+1}^+ Y_\lambda(m) = (\lambda - a_{m+1})\, Y_\lambda(m), \qquad L_m^+ L_m^- Y_\lambda(m) = (\lambda - a_m)\, Y_\lambda(m) \tag{7.3}$$

for all $m \in S$. The significance of the existence of a factorization for the solution of the eigenvalue problem is given by:

Lemma 7.1 Let $Y_\lambda(l)$ be a solution of (7.1) for $m = l$. Then $L_{l+1}^+ Y_\lambda(l)$ is a solution of (7.1) for $m = l + 1$. Similarly $L_l^- Y_\lambda(l)$ is a solution for $m = l - 1$.

PROOF

$$\begin{aligned} X_{l+1}(L_{l+1}^+ Y_\lambda(l)) &= [L_{l+1}^+ L_{l+1}^- + a_{l+1}]\, L_{l+1}^+ Y_\lambda(l) \\ &= L_{l+1}^+[L_{l+1}^- L_{l+1}^+ Y_\lambda(l)] + a_{l+1} L_{l+1}^+ Y_\lambda(l) \\ &= \lambda L_{l+1}^+ Y_\lambda(l). \end{aligned}$$

The second assertion is proved in exactly the same way. Q.E.D.

Thus the operators $L^\pm$ satisfy the properties: (1) the "raising operators" L_{l+1}^+ map a solution of (7.1) for $m = l$ into a solution for $m = l + 1$; (2) the "lowering operators" L_l^- map a solution for $m = l$ into a solution for $m = l - 1$; and (3) if we first raise and then lower (or vice versa) we obtain our original function multiplied by a fixed constant. (The possibility that $L_{l+1}^+ Y_\lambda(l) \equiv 0$ or $L_l^- Y_\lambda(l) \equiv 0$ is not excluded.) Conversely, it is not difficult to show that, by suitable renormalization, any set of linear differential operators $L_m^\pm$, $m \in S$, satisfying properties (1), (2), and (3) can be assumed to satisfy (7.2) for some choice of constants a_m.

If the operators X_m admit a factorization, the raising and lowering operators can be used to derive recurrence relations for the functions $Y_\lambda(m)$. For example, if $Y_\lambda(l)$ is a solution of (7.1) for $m = l$, a ladder of solutions $Y_\lambda(l + n)$ can be defined recursively by

$$Y_\lambda(l + n + 1) = (\lambda - a_{l+n+1})^{-1/2} L_{l+n+1}^+ Y_\lambda(l + n), \qquad n = 0, 1, 2, \ldots .$$

Then Eqs. (7.2) imply

$$Y_\lambda(l + n) = (\lambda - a_{l+n+1})^{-1/2} L_{l+n+1}^- Y_\lambda(l + n + 1), \qquad n = 0, 1, 2, \ldots .$$

These expressions are differential recurrence relations for the functions $Y_\lambda(l + n)$, $n \geqslant 0$. Similarly we can derive recurrence relations for functions $Y_\lambda(l - n)$ with $n \geqslant 0$:

$$\begin{aligned} Y_\lambda(l - n - 1) &= (\lambda - a_{l-n})^{-1/2} L_{l-n}^- Y_\lambda(l - n), \\ Y_\lambda(l - n) &= (\lambda - a_{l-n})^{-1/2} L_{l-n}^+ Y_\lambda(l - n - 1), \qquad n = 0, 1, 2, \ldots . \end{aligned}$$

From this point of view the existence of differential recurrence relations satisfying properties (1), (2), and (3) listed above is equivalent to the existence of a factorization. Moreover, all of the differential recurrence relations derived for special functions in Chapters 3–6 can be obtained as factorizations of operators X_m by raising and lowering operators $L_m^{\pm}$. In fact, all of the known recurrence relations for functions of hypergeometric type listed in the Bateman Project (Erdélyi *et al.* [1]) can be derived either as factorizations or as combinations of recurrence relations obtained from factorizations. It is thus a problem of considerable importance in special function theory to classify all possible factorizations.

At first glance this classification problem appears very difficult—we must find all second order operators X_m, first order operators L_m^+, L_m^-, and constants a_m for all $m \in S$ such that Eqs. (7.2) hold identically. However, the problem can be drastically simplified by transforming the operators X_m into the standard form

$$X'_m = -\frac{d^2}{dx^2} + V_m(x), \qquad m \in S. \tag{7.4}$$

To obtain these operators choose a family of analytic functions $\{\varphi_m(x)\}$ such that $\varphi_m(x) \neq 0$ for all $m \in S$, $x \in N$, and define the mappings

$$\boldsymbol{\varphi}_m^{-1}\colon \quad Y_\lambda(m, x) \to Y'_\lambda(m, x) = (\varphi_m(x))^{-1} Y_\lambda(m, x), \qquad m \in S.$$

The domain of $\boldsymbol{\varphi}_m^{-1}$ is the space $\mathscr{M}(\lambda, m)$ of all solutions of the equation

$$X_m Y_\lambda(m, x) = \lambda Y_\lambda(m, x).$$

Denote the range of $\boldsymbol{\varphi}_m^{-1}$ by $\mathscr{M}'(\lambda, m)$. Then the elements of $\mathscr{M}'(\lambda, m)$ are eigenfunctions of the second order differential operator

$$X'_m = \varphi_m^{-1}(x)\, X_m \varphi_m(x), \qquad [\varphi_m^{-1}(x) = (\varphi_m(x))^{-1}].$$

In fact,

$$\begin{aligned} X'_m Y'_\lambda(m, x) &= \varphi_m^{-1}(x)\, X_m \varphi_m(x)\, \varphi_m^{-1}(x)\, Y_\lambda(m, x) = \varphi_m^{-1}(x)\, X_m Y_\lambda(m, x) \\ &= \lambda \varphi_m^{-1}(x)\, Y_\lambda(m, x) = \lambda Y'_\lambda(m, x). \end{aligned}$$

Similarly the first order operators

$$L_{m+1}^+\colon\ \mathscr{M}(\lambda, m) \to \mathscr{M}(\lambda, m+1), \qquad L_m^-\colon\ \mathscr{M}(\lambda, m) \to \mathscr{M}(\lambda, m-1)$$

are transformed into the operators

$$L_{m+1}^{+\prime}\colon\ \mathscr{M}'(\lambda, m) \to \mathscr{M}'(\lambda, m+1), \qquad L_m^{-\prime}\colon\ \mathscr{M}'(\lambda, m) \to \mathscr{M}'(\lambda, m-1)$$

where

$$L_{m+1}^{+\prime} = \varphi_{m+1}^{-1}(x)\, L_{m+1}^+ \varphi_m(x), \qquad L_m^{-\prime} = \varphi_{m-1}^{-1}(x)\, L_m^- \varphi_m(x).$$

As a consequence of these definitions the transformed operators $L_m^{\pm\prime}$ form a factorization of the operators X_m':

$$L_{m+1}^{-\prime}L_{m+1}^{+\prime} + a_{m+1} \equiv L_m^{+\prime}L_m^{-\prime} + a_m \equiv X_m', \qquad m \in S.$$

If we choose $\varphi_m(x) = \exp \frac{1}{2}\int_{x^o}^{x} D_m(t)\,dt$ for all $m \in S$, the transformed operators X_m' take the form (7.4) in some neighborhood of $x^o \in \mathcal{C}$. In fact, $X_m' = -d^2/dx^2 + E_m(x) + \frac{1}{4}D_m^2(x) - \frac{1}{2}\,dD_m(x)/dx$. [This is a standard computation (see Petrovsky [1], Chapter 2).] Thus, without loss of generality it can be assumed that the differential operators are

$$X_m = -\frac{d^2}{dx^2} + V_m(x), \qquad L_m^{\pm} = A_m^{\pm}(x)\frac{d}{dx} + B_m^{\pm}(x), \tag{7.5}$$

where

$$L_{m+1}^{-}L_{m+1}^{+} + a_{m+1} \equiv L_m^{+}L_m^{-} + a_m \equiv X_m \tag{7.6}$$

for all $m \in S$. Necessary and sufficient conditions for the validity of (7.6) are

$$\begin{aligned}
&(1)\quad A_m^{+}A_m^{-} = -1,\\
&(2)\quad A_m^{-}A_m^{+\prime} + A_m^{-}B_m^{+} + B_m^{-}A_m^{+} = 0,\\
&(3)\quad A_m^{+}A_m^{-\prime} + A_m^{-}B_m^{+} + B_m^{-}A_m^{+} = 0,\\
&(4)\quad B_{m+1}^{-}B_{m+1}^{+} + A_{m+1}^{-}B_{m+1}^{+\prime} + a_{m+1} = B_m^{+}B_m^{-} + A_m^{+}B_m^{-\prime} + a_m = V_m(x)
\end{aligned} \tag{7.7}$$

for all $m \in S$, where now the prime denotes differentiation with respect to x. Differentiation of (1) yields $A_m^{+\prime}A_m^{-} = -A_m^{+}A_m^{-\prime}$. Comparing this expression with conditions (2) and (3) we obtain the requirements

$$A_m^{+\prime} = A_m^{-\prime} = 0, \qquad A_m^{-}B_m^{+} + B_m^{-}A_m^{+} = 0. \tag{7.8}$$

Thus, A_m^{+} and A_m^{-} are constants such that $A_m^{+}A_m^{-} = -1$. Since (7.6) imposes restrictions only on products of the operators L_m^{+}, L_m^{-}, there is no loss of generality involved if the operator L_m^{-} is assumed to be normalized so that $A_m^{-} = +1$ for all $m \in S$. From this assumption there follow the conditions

$$A_m^{-} = -A_m^{+} = 1, \qquad B_m^{+}(x) = B_m^{-}(x) = B_m(x)$$

and

$$B_{m+1}^2(x) + B_{m+1}'(x) + a_{m+1} = V_m(x), \qquad B_m^2(x) - B_m'(x) + a_m = V_m(x). \tag{7.9}$$

The last two equations imply the equality

$$B_{m+1}^2 - B_m^2 + \frac{dB_{m+1}}{dx} + \frac{dB_m}{dx} = a_m - a_{m+1} \tag{7.10}$$

where the constants a_m are independent of x. Formula (7.10) is obviously a necessary condition for the existence of a factorization. However, it is also sufficient since functions $B_m(x)$ and constants a_m satisfying (7.10) can be used to unambiguously define the functions $V_m(x)$ by (7.9), hence, to define operators X_m which can be factored. Thus to find all possible factorization types it is sufficient to find all possible solutions of (7.10).

7-2 The Factorization Types

As a first trial solution of (7.10) choose $B_m(x) = f(m)$ where f is an arbitrary function of m. Then (7.10) is satisfied for $a_m = -f(m)^2$ and (7.1) becomes the equation of simple harmonic motion

$$-\frac{d^2}{dx^2} Y_\lambda(m) = \lambda Y_\lambda(m). \tag{7.11}$$

This factorization is of such a trivial nature that we will not examine it in any detail.

The trial solution $B_m(x) = j(x) + mk(x)$ yields more useful results. Substituting this solution into (7.10) and equating powers of m on both sides of the resulting expression we obtain the conditions

$$k^2 + k' = -a^2, \qquad j' + kj = -b\mu/2, \qquad a_m = a^2m^2 + b\mu m, \tag{7.12}$$

where a, b, μ are constants. (Since (7.10) depends only on the difference $a_m - a_{m+1}$, if the constants a_m satisfy (7.10), so do the constants $a_m + c$. However, it is easy to show that without loss of generality we can set $c = 0$. Indeed, c can be absorbed in the constant λ, Eq. (7.1).)

For $a \neq 0$ the possible solutions of (7.12) are:

Type (A) $\quad k(x) = a \cot a(x + p), \qquad j(x) = \dfrac{b\mu}{2a} \cot a(x + p) + \dfrac{q}{\sin a(x + p)};$

Type (B) $\quad k(x) = ia, \qquad j(x) = \dfrac{ib\mu}{2a} + qe^{-ia(x+p)},$

where p and q are constants. If $a = 0$, $b \neq 0$, the possible solutions are:

Type (C') $\quad k(x) = \dfrac{1}{x + p}, \qquad j(x) = -\dfrac{b\mu(x + p)}{4} + \dfrac{q}{x + p};$

Type (D') $\quad k(x) = 0, \qquad j(x) = -\dfrac{\mu b x}{2} + q.$

Finally, if $a = b = 0$ the solutions are:

Type (C'') $\quad k(x) = \dfrac{1}{x+p}, \qquad j(x) = \dfrac{q}{x+p};$

Type (D'') $\quad k(x) = 0, \qquad j(x) = q.$

Clearly, these factorizations correspond exactly to the *type A–D''* operators classified in Chapter 2. Indeed Eqs. (7.12) and (2.30) are identical. The type A and B factorizations can be normalized so that $a = 1$, $b = 0$ by replacing $a(x+p)$ with $x+p$ and m with $m' = m - b\mu/2a^2$. In this case the expressions for $k(x)$, $j(x)$ corresponding to type A and B factorizations coincide with those for the *type A* and *B* operators. Similarly the expressions for $k(x)$, $j(x)$ corresponding to type C' and D' factorizations coincide with those for *type C'* and *D'* operators when $a = 0$, $b = 1$. If $a = b = 0$ the expressions for $k(x)$, $j(x)$ corresponding to type C'' and D'' factorizations and *type C''* and *D''* operators coincide.

Each of the factorization types listed above determines a family of second order ordinary differential equations

$$\left[-\frac{d^2}{dx^2} + V_m(x)\right] Y_\lambda(m, x) = \lambda Y_\lambda(m, x), \qquad m \in S, \tag{7.13}$$

$$\begin{aligned} V_m(x) &= (j + (m+1)\,k)^2 + j' + (m+1)\,k' + a^2(m+1)^2 + b\mu(m+1) \\ &= (j + mk)^2 - j' - mk' + a^2m^2 + b\mu m, \end{aligned}$$

which admit a factorization by the linear differential operators

$$L_m^+ = -\frac{d}{dx} + j + mk, \qquad L_m^- = \frac{d}{dx} + j + mk. \tag{7.14}$$

The differential equations and recurrence relations for special functions given by these factorizations are almost exactly the same as those obtained from realizations of irreducible representations of the Lie algebras $\mathscr{G}(a, b)$ by the corresponding operator types. The difference is merely one of normalization.

For example if $a = 1$, $b = 0$, the type A factorizations are associated with the differential equations

$$\left[-\frac{d^2}{dx^2} + \frac{1}{\sin^2(x+p)}\,(2q(m+\tfrac{1}{2})\cos(x+p) + m^2 + q^2 + m)\right] \cdot Y_\lambda(m, x) = \lambda Y_\lambda(m, x). \tag{7.15}$$

Set $m' = m + \frac{1}{2}$, $\lambda' = \lambda - \frac{1}{4}$, $q' = -q$, and map the space $\mathscr{M}(\lambda, m)$ into the new space of solutions $Y'_{\lambda'}(m, x) = \varphi_m^{-1}(x)\, Y_\lambda(m, x)$ where $\varphi_m^{-1}(x) = (\sin(x + p))^{1/2}$ (see Section 7-1). We find that the new solutions satisfy the transformed equations

$$-\frac{1}{\sin(x+p)}\frac{d}{dx}\left[\sin(x+p)\frac{d}{dx}\, Y'_{\lambda'}(m, x)\right]$$
$$+\left[\frac{m'^2 + q'^2 - 2q'm'\cos(x+p)}{\sin^2(x+p)}\right] Y'_{\lambda'}(m, x) = \lambda' Y'_{\lambda'}(m, x)$$

which are identical with the equations for eigenfunctions of *type A* operators derived in Section 2-7. Also, the recursion relations obtained from factorization by the operators L_m^+, L_m^- transform into recursion relations for *type A* eigenfunctions obtained from the operators J^+, J^-. Similar remarks hold for the other factorization types.

For more details concerning the derivation of identities for special functions from the factorization types A–D'' the reader should consult Infeld and Hull [1]. Clearly, the theory of these factorization types is completely equivalent to the representation theory of the Lie algebras $\mathscr{G}(a, b)$. Hence, the principal results of Infeld and Hull related to such factorizations are already contained in Chapters 3–5. (Infeld and Hull combine the type C' and type C'' factorizations to form type C factorizations. Similarly, they combine types D' and D'' to form type D factorizations. However, from the Lie algebraic viewpoint adopted in this book, each of the type C and type D factorizations should be broken up into subclasses; one subclass corresponding to faithful representations of $\mathscr{G}(0, 1)$ and the other to faithful representations of $\mathscr{G}(0, 0)$. The type D'' factorizations are a special case of the trivial factorizations corresponding to Eq. (7.11).)

To find additional solutions of (7.10) one might introduce the trial function

$$B_m(x) = j(x) + \sum_{l=1}^{n} k_l(x)\, m^l, \qquad k_n(x) \neq 0, \quad m \in S,$$

where $n \geqslant 2$. However, the only possible solution in this case is the trivial $B_m(x) = f(m)$ considered earlier (see Infeld and Hull [1], p. 28).

One trial function which does lead to new results is

$$B_m(x) = \frac{h(x)}{m} + j(x) + mk(x), \qquad m \in S.$$

In fact, substituting this expression into (7.10) and equating powers of m on both sides of the resulting formula we can obtain the following new solutions:

Type (E) $h(x) = q$, $\quad j(x) = 0$, $\quad k(x) = a \cot a(x + p)$,

$$a_m = a^2m^2 - q^2/m^2;$$

Type (F) $h(x) = q$, $\quad j(x) = 0$, $\quad k(x) = 1/(x + p)$, $\quad a_m = -q^2/m^2$,

where a, p, q are constants.

The differential equations admitting type E factorizations take the form ($a = 1, p = 0$)

$$\left[-\frac{d^2}{dx^2} + \frac{m(m+1)}{\sin^2 x} + 2q \cot x\right] Y_\lambda(m, x) = \lambda Y_\lambda(m, x).$$

The solutions of this equation are closely related to the type A eigenfunctions. Indeed, transforming the equation to a new normal form by means of the substitutions

$$i\left(z + \frac{\pi}{2}\right) = \ln \tan \frac{x}{2}, \quad W = \sin^{-1/2} x \; Y_\lambda(m),$$

we obtain

$$\left[-\frac{d^2}{dz^2} - \frac{2iq \cos z + \lambda + \frac{1}{4}}{\sin^2 z}\right] W(z) = -(m + \tfrac{1}{2})^2 \, W(z).$$

In terms of the constants r, l given by $\lambda = -l^2 - r^2$, $q = irl$ this expression becomes

$$\left[-\frac{d^2}{dz^2} + \frac{2rl \cos z + r^2 + (l + \frac{1}{2})(l - \frac{1}{2})}{\sin^2 z}\right] W(z) = -(m + \tfrac{1}{2})^2 \, W(z) \qquad (7.16)$$

which can be identified with the type A equation (7.15). Thus, the eigenfunctions corresponding to type A and type E factorizations can be identified. (They are hypergeometric functions.) However, the type A recurrence relations raise and lower the l-value of these eigenfunctions whereas the type E recurrence relations raise and lower the m-value. This is the same kind of behavior exhibited in Section 6-2. There the *type E* operators were used to find realizations of irreducible representations of $\mathscr{T}_6$. The basis vectors for these realizations were hypergeometric functions and the representation theory yielded two kinds of differential recurrence relations. The first class of recurrence relations was associated with the representation theory of $sl(2)$ and we have seen that it is equivalent to the type A factorization. However, the second class of differential recurrence relations, (6.14), was not directly associated with a Lie algebra representation, but rather with a representation of the universal enveloping algebra of $\mathscr{T}_6$. By means of an appropriate change of variable it is not difficult to show that this latter class of recurrence formulas is equivalent to the formulas obtained from the type E factorizations.

Similarly the recurrence relations obtained from type F factorizations are equivalent to Eqs. (6.19) derived from a representation of the universal enveloping algebra of $\mathscr{T}_6$ by *type F* differential operators. Here again the type B and type F factorizations yield different recurrence formulas for the same eigenfunctions. (In this case the confluent hypergeometric functions.)

The eight factorization types A–F derived above constitute the complete list of factorization types given by Infeld and Hull. Indeed, it is a straightforward exercise to show that the key equation (7.10) has no solutions of the form

$$B_m(x) = \sum_{l=-n'}^{n} k_l(x)\, m^l,$$

where n, n' are finite nonnegative integers, other than the factorization types already listed. We have seen that each of the factorization types A–F corresponds to one of the differential operator *types A–F* studied in Chapters 3–6. Thus there is a complete equivalence between the factorization method as formulated by Infeld and Hull and the study of realizations of irreducible representations of the Lie algebras $\mathscr{G}(a, b)$ and $\mathscr{T}_6$ by differential operators in two complex variables. For this reason we will not delve deeper into the theory of the factorization method.

It should not be supposed, however, that the factorization types of Infeld and Hull constitute all possible factorizations. In fact, if $f(x)$ is an arbitrary analytic function and $\{a_m\}$, $m \in S = \{m_o, m_o \pm 1, m_o \pm 2, ...\}$, is a set of arbitrary complex constants, there is a solution $B_m(x)$ of (7.10) such that $B_{m_o}(x) = f(x)$. Such a solution (not unique) can be constructed recursively from (7.10). Thus there are an infinite number of factorizations in addition to those we have listed. However, for none of the new factorizations can the function $B_m(x)$ be expressed in the form (7.17), i.e., in terms of a finite number of powers of m. In view of this these new factorizations have not proved to be of much practical importance. Furthermore such factorizations seem to have no reasonable Lie algebraic interpretation.