CHAPTER **8**

Generalized Lie Derivatives

As stated earlier, the method we have used to associate special functions with a Lie algebra $\mathscr{G}$ consists of the three steps: (1) Determine an abstract representation ρ of $\mathscr{G}$. (2) Find a realization of ρ by generalized Lie derivatives acting on a space $\mathscr{A}$ of analytic functions. (3) Compute the multiplier representation of the local Lie group G induced by ρ on $\mathscr{A}$. The first of these steps constitutes a problem in the representation theory of Lie algebras, a subject which has undergone intensive investigation (Jacobson [1]). The third step can be handled by the use of Theorem 1.10. Thus, for low-dimensional Lie algebras at least, steps (1) and (3) can be carried through on the basis of existing results in the mathematical literature.

By constrast, step (2) is not a well-defined procedure treated in the literature. For the purposes of special function theory we need to classify "all" realizations of $\mathscr{G}$ by generalized Lie derivatives. That such a classification is important follows from our study of the Lie algebras $\mathscr{G}(a, b)$ and $\mathscr{T}_6$. We found realizations of these Lie algebras in terms of generalized Lie derivatives in two complex variables, operator *types A–F*. This choice of operators completely determined the possible special functions which could arise as basis vectors. There is no a priori guarantee, however, that there are not other realizations of these Lie algebras by generalized Lie derivatives in two variables which lead to new classes of special functions. (In fact it will be shown there do exist realizations of $\mathscr{G}(0, 1)$ and $\mathscr{G}(0, 0)$ by generalized Lie derivatives distinct from the ones we have studied, but that they lead to no new classes of special functions.)

To solve such problems we will develop the rudiments of a classification theory of generalized Lie derivatives. Significant results in the classification theory of ordinary Lie derivatives are well known (Lie [1], Vol. III). In particular, Lie himself found all possible realizations of complex Lie algebras by Lie derivatives in one and two complex variables. For more than two variables the results are incomplete. Here we shall extend Lie's methods slightly so as to apply to generalized Lie derivatives. Among the results which will be obtained from this analysis are a proof that the operator *types A–F* are the only realizations of $\mathscr{G}(a, b)$ and $\mathscr{T}_6$ by generalized Lie derivatives in two complex variables which lead to interesting special functions, and a table listing all Lie algebras which have a realization in terms of generalized Lie derivatives in one complex variable. In Chapter 9 the table will be used to derive new classes of special functions which are not of hypergeometric type.

The material of this chapter is more difficult than that of the rest of the book. In particular, fairly deep results from the theory of local Lie groups and the cohomology theory of Lie algebras will be employed. Therefore, those readers approaching the subject for the first time may prefer to omit Chapter 8 on a first reading.

8-1 Generalized Derivations

Here, notation and terminology for generalized Lie derivatives and multiplier representations will largely follow that which was introduced in Section 1-3. However, it will prove convenient to introduce an alternative definition of the generalized Lie derivative which has the advantage of being independent of local coordinates. Given a point $\mathbf{x}^o \in \mathbb{C}^n$, let $\mathscr{A}$ be the space of all functions analytic in some neighborhood of $\mathbf{x}^o$, i.e., the germs of functions at $\mathbf{x}^o$ (Gunning and Rossi [1], Chapter 2). (Without loss of generality we can assume $\mathbf{x}^o = \mathbf{0} = (0,..., 0)$.) $\mathscr{A}$ is a complex vector space, but it is also an associative algebra, since the product of two functions in $\mathscr{A}$ is a function in $\mathscr{A}$.

To expose the algebraic structure of the results in this chapter we introduce the concept of a realization of $\mathscr{G}$ by generalized derivations operating on an abstract associative algebra A. When $A = \mathscr{A}$ this concept is identical to the notion of a realization of $\mathscr{G}$ by generalized Lie derivatives on $\mathscr{A}$.

Definition Let A be an associative algebra over $\mathbb{C}$ with a multiplicative identity 1. The mapping D of A into A is said to be a **generalized derivation** (gd) on A if

(1) $D(af + bh) = aD(f) + bD(h)$,

(2) $D(fh) = fD(h) + D(f)\,h - D(1)\,fh$

for all $a, b \in \mathbb{C}$ and $f, h \in A$. If in addition $D(1) = 0$, where 0 is the additive identity of A, then D is an (ordinary) derivation on A.

If D is a gd on A, it is easy to verify that the operator $\tilde{D} = D - D(1)$ defined by $\tilde{D}(f) = D(f) - D(1)f, f \in A$, is a derivation on A. Thus, $D = \tilde{D} + D(1)$ and $\tilde{D}$ is uniquely defined by this relation. Conversely, if $\tilde{D}$ is a derivation on A and $h \in A$, the operator $D = \tilde{D} + h$, defined by $D(f) = \tilde{D}(f) + hf, f \in A$, is a gd on A. In fact $h = D(1)$. Note: We are using the same notation for the operator $D(1)$ and the element $D(1)$ in A. This should cause no confusion.

If D_1, D_2 are gd's on A the commutator $[D_1, D_2] = D_1D_2 - D_2D_1$ is a gd on A. Indeed, a straightforward computation gives $\widetilde{[D_1, D_2]} = \tilde{D}_1\tilde{D}_2 - \tilde{D}_2\tilde{D}_1$ and $[D_1, D_2](1) = \tilde{D}_1(D_2(1)) - \tilde{D}_2(D_1(1))$. For $a, b \in \mathbb{C}$ the operator $aD_1 + bD_2$ has an obvious definition as a gd on A. Thus, the gd's on A form a (possibly infinite-dimensional) Lie algebra.

In terms of the local coordinates $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ in a neighborhood of $\mathbf{0} \in \mathbb{C}^n$ a gd D on the algebra $\mathcal{A}$ can be expressed uniquely in the form

$$Df(\mathbf{x}) = \sum_{j=1}^{n} p_j(\mathbf{x}) \frac{\partial}{\partial x_j} f(\mathbf{x}) + p(\mathbf{x})f(\mathbf{x}), \qquad p_j, p \in \mathcal{A}, \tag{8.1}$$

for all $f \in \mathcal{A}$ (see Cohn [1], p. 17). Here,

$$\tilde{D} = \sum_{j=1}^{n} p_j \frac{\partial}{\partial x_j}, \qquad D(1) = p.$$

(The multiplicative identity of $\mathcal{A}$ is the constant function $1(\mathbf{x}) \equiv 1$ for all $\mathbf{x} \in \mathbb{C}^n$.) Thus, on the algebra $\mathcal{A}$, the definition of generalized derivations is equivalent to the definition of generalized Lie derivatives given in Section 1–3.

Definition A **realization** τ of a complex Lie algebra $\mathcal{G}$ by gd's on A is a mapping $\alpha \to \tau_\alpha$ where τ_α is a gd on A, such that

(1) $\tau_{a\alpha+b\beta} = a\tau_\alpha + b\tau_\beta$,

(2) $\tau_{[\alpha,\beta]} = [\tau_\alpha, \tau_\beta]$

for every $a, b \in \mathbb{C}, \alpha, \beta \in \mathcal{G}$. The τ is **transitive** ($A \neq \mathcal{A}$) if $\tilde{\tau}_\alpha(f) = 0$, for all $\alpha \in \mathcal{G}$ implies $f = a1$, $a \in \mathbb{C}$; or ($A = \mathcal{A}$) if $[\tilde{\tau}_\alpha(f)](0) = 0$, all $\alpha \in \mathcal{G}$ implies $[\mu(f)](0) = 0$ for all derivations μ of $\mathcal{A}$.

In the case $A = \mathcal{A}$, this definition is equivalent to the coordinate-dependent definition of a realization of $\mathcal{G}$ given in Section 1-3. Moreover, the following theorem holds.

Theorem 8.1 Let $\mathcal{G}$ be a complex Lie algebra and G the local Lie group such that $\mathcal{G} = L(G)$. There is a one-to-one correspondence between realizations τ of $\mathcal{G}$ by gd's on $\mathcal{A}$ and local multiplier representations T^ν of G on $\mathcal{A}$. The relationship between the multiplier $\nu(\mathbf{x}, g)$ and the gd's τ_α is given by

$$\tau_\alpha f(\mathbf{x}) = \frac{d}{dt}[\mathbf{T}^\nu(\exp \alpha t) f](\mathbf{x})\Big|_{t=0}, \qquad f \in \mathcal{A}.$$

τ is transitive if and only if T^ν is transitive.

PROOF (Rough sketch) If τ is an effective realization of $\mathcal{G}$ (or if T^ν is effective) this theorem is merely a reformulation of Theorems 1.9 and 1.10. Suppose then the set $\mathcal{N} = \{\alpha \in \mathcal{G}: \tau_\alpha f = 0 \text{ for all } f \in \mathcal{A}\}$ is a nonzero ideal of $\mathcal{G}$. We must show there exists exactly one multiplier representation T^ν of G whose infinitesimal representation is τ. Uniqueness follows from Theorem 1.9. To prove existence note that τ induces an effective realization τ' of the quotient algebra $\mathcal{G}/\mathcal{N}$. (The elements of $\mathcal{G}/\mathcal{N}$ can be denoted as α' for all $\alpha \in \mathcal{G}$. [We will sometimes write $\alpha' = \alpha + \mathcal{N}$.] Then $\alpha' = \beta'$ if and only if $\alpha - \beta \in \mathcal{N}$. Addition of elements and multiplication by a scalar are defined by $a\alpha' + b\beta' = (a\alpha + b\beta)'$, $a, b \in \mathbb{C}$, $\alpha, \beta \in \mathcal{G}$. The commutator $\langle ., . \rangle$ of $\mathcal{G}/\mathcal{N}$ is given by $\langle \alpha', \beta' \rangle = [\alpha, \beta]'$ where $[., .]$ is the commutator of $\mathcal{G}$. These operations on $\mathcal{G}/\mathcal{N}$ are well defined since $\mathcal{N}$ is an ideal of $\mathcal{G}$.) Here, $\tau'_{\alpha'} = \tau_\alpha$ for all $\alpha \in \mathcal{G}$.

According to Theorem 1.10, τ' induces an effective multiplier representation $T^{\nu'}$ of the local Lie quotient group G/N where N is a normal subgroup of G. [Here $\mathcal{G} = L(G)$, $\mathcal{N} = L(N)$, $\mathcal{G}/\mathcal{N} = L(G/N)$ (see Pontrjagin [1], Chapter 9).] Moreover, the canonical projection φ: $G \to G/N$ is a local analytic homomorphism. Thus the operators $\mathbf{T}^\nu(g)$ on $\mathcal{A}$ defined by

$$\mathbf{T}^\nu(g) = \mathbf{T}^{\nu'}(\varphi(g)), \qquad g \in G,$$

determine a local multiplier representation T^ν of G whose infinitesimal representation is τ. Q.E.D.

Lemma 8.1 Let τ be a realization of $\mathcal{G}$ by gd's on the algebra A. Then,

(1) $[\tilde{\tau}_\alpha, \tilde{\tau}_\beta] = \tilde{\tau}_{[\alpha,\beta]}$,

(2) $\tilde{\tau}_\alpha(\tau_\beta(1)) - \tilde{\tau}_\beta(\tau_\alpha(1)) = \tau_{[\alpha,\beta]}(1)$,

(3) $\tilde{\tau}_{a\alpha+b\beta} = a\tilde{\tau}_\alpha + b\tilde{\tau}_\beta$,

(4) $\tau_{a\alpha+b\beta}(1) = a\tau_\alpha(1) + b\tau_\beta(1)$

for all $a, b \in \mathcal{C}$, $\alpha, \beta \in \mathcal{G}$. In particular the $\tilde{\tau}_\alpha$ form a realization $\tilde{\tau}$ of $\mathcal{G}$ by derivations on A. Conversely, derivations $\tilde{\tau}_\alpha$ and elements $\tau_\alpha(1)$ of A defined for each $\alpha \in \mathcal{G}$ and satisfying relations (1)–(4) determine a realization τ of $\mathcal{G}$ by gd's on A, where $\tau_\alpha = \tilde{\tau}_\alpha + \tau_\alpha(1)$.

PROOF If τ is a realization of $\mathcal{G}$ every gd τ_α can be decomposed uniquely in the form $\tau_\alpha = \tilde{\tau}_\alpha + \tau_\alpha(1)$. If we carry out this decomposition on both sides of the identities

(1)′ $\tau_{a\alpha+b\beta} = a\tau_\alpha + b\tau_\beta$,

(2)′ $\tau_{[\alpha,\beta]} = [\tau_\alpha, \tau_\beta]$

we see that (1)′ and (2)′ are equivalent to conditions (1)–(4). Q.E.D.

8-2 Cohomology Classes of Realizations

Let τ^o be a realization of $\mathcal{G}$ by derivations on A, i.e., $\tau^o_\alpha(1) = 0$ for all $\alpha \in \mathcal{G}$. Denote by $A(\tau^o)$ the set of all realizations τ of $\mathcal{G}$ by gd's on A such that $\tilde{\tau} = \tau^o$. The $A(\tau^o)$ can be given the structure of a vector space over $\mathcal{C}$: If $\tau, \tau' \in A(\tau^o)$ and $a, b \in \mathcal{C}$, define the realization $a\tau \oplus b\tau' \in A(\tau^o)$ by

$$\widetilde{(a\tau \oplus b\tau')}_\alpha = \tau^o_\alpha$$
$$(a\tau \oplus b\tau')_\alpha(1) = a\tau_\alpha(1) + b\tau'_\alpha(1), \qquad \text{all} \quad \alpha \in \mathcal{G}. \tag{8.2}$$

The additive identity element in the vector space $A(\tau^o)$ is τ^o.

Clearly, to classify all possible realizations of $\mathcal{G}$ by gd's on $\mathcal{A}$ it is enough to find all of the vector spaces $\mathcal{A}(\tau^o)$ where τ^o runs through a complete list of possible realizations of $\mathcal{G}$ by ordinary Lie derivatives (derivations). The problem of classifying the realizations of $\mathcal{G}$ by derivations has been studied in detail by Lie and completely solved for Lie derivatives in one or two complex variables ($n = 1, 2$). For larger values of n the results are incomplete (Lie [1], Vol. III). Here, we shall fall back on Lie's results and assume we know all of the realizations τ^o of $\mathcal{G}$ by derivations on $\mathcal{A}$. (This is equivalent to the knowledge of all possible ways in which G can act as a local transformation group on an n-dimensional complex manifold.) To complete the classification of all possible realizations of $\mathcal{G}$ by gd's we need only compute the vector space $\mathcal{A}(\tau^o)$ for each τ^o. This is still a difficult problem since in general $\mathcal{A}(\tau^o)$ is infinite-dimensional. To simplify the computations involved in the

solution of this problem note that, for the purposes of special function theory, the elements of $\mathcal{A}(\tau^o)$ can be conveniently divided into equivalence classes. This can be seen as follows: Let $\varphi \in \mathcal{A}$ where $\varphi(\mathbf{0}) \neq 0$. Then, $\varphi^{-1} \in \mathcal{A}$. This function defines vector space isomorphisms $\boldsymbol{\varphi}$ and $\boldsymbol{\varphi}^{-1}$ of $\mathcal{A}$ onto $\mathcal{A}$ such that $\boldsymbol{\varphi}[f](\mathbf{x}) = \varphi(\mathbf{x})\, f(\mathbf{x})$ and $\boldsymbol{\varphi}^{-1}[f](\mathbf{x}) = \varphi^{-1}(\mathbf{x})\, f(\mathbf{x})$ for all $f \in \mathcal{A}$. Similarly φ induces an isomorphism $\boldsymbol{\varphi}'$ of the vector space $\mathcal{A}(\tau^o)$ onto itself. Here,

$$\boldsymbol{\varphi}'[\tau]_\alpha = \boldsymbol{\varphi}^{-1}\tau_\alpha\boldsymbol{\varphi} = \tilde{\tau}_\alpha + \tau_\alpha(1) + \varphi^{-1}\tilde{\tau}_\alpha(\varphi) \tag{8.3}$$

for all $\tau \in \mathcal{A}(\tau^o)$, $\alpha \in \mathcal{G}$ (see Lemma 4.4). The function $\varphi(\mathbf{x})$ can always be expressed in the form $\varphi(\mathbf{x}) = \exp(f(\mathbf{x})) = \sum_{k=0}^{\infty} (f(\mathbf{x}))^k/k!$ where $f \in \mathcal{A}$ (f is not unique). Then (8.3) becomes

$$\boldsymbol{\varphi}'[\tau]_\alpha = \tilde{\tau}_\alpha + \tau_\alpha(1) + \tilde{\tau}_\alpha(f). \tag{8.4}$$

In Chapters 4 and 5 it was shown by means of examples that the special function theory associated with the operators τ_α is completely equivalent to the special function theory associated with the operators $\boldsymbol{\varphi}'[\tau]_\alpha$. Clearly, the special functions corresponding to the two sets of operators are related by the multiplicative factor $\varphi(\mathbf{x})$. Using this discussion as motivation we make the general definition:

Definition Two realizations τ, τ' of $\mathcal{G}$ by gd's on A are said to be **cohomologous**, $\tau \approx \tau'$, if there is an $f \in A$ such that $\tilde{\tau}_\alpha = \tilde{\tau}'_\alpha$, $\tau'_\alpha(1) = \tau_\alpha(1) + \tilde{\tau}_\alpha(f)$, all $\alpha \in \mathcal{G}$.

By definition "$\approx$" is an equivalence relation on $A(\tau^o)$. Thus, $A(\tau^o)$ is divided into equivalence classes of cohomologous operators. Moreover, this equivalence relation respects the vector space structure of $A(\tau^o)$. The set $N(\tau^o) = \{\mu \in A(\tau^o)\colon \mu \approx \tau^o\}$, i.e., the equivalence class containing τ^o is clearly a subspace of $A(\tau^o)$. Furthermore, it is easy to show $\tau \approx \tau'$ for $\tau, \tau' \in A(\tau^o)$ if and only if $\tau = \tau' \oplus \mu$ for some $\mu \in N(\tau^o)$. This proves the existence of an isomorphism between the set of equivalence classes in $A(\tau^o)$ and the quotient space $A(\tau^o)/N(\tau^o)$. In particular the set of equivalence classes can be assigned a vector space structure.

As suggested by the term "cohomologous," the above result can be formulated in terms of the cohomology theory of Lie algebras. We follow Jacobson's presentation [1, Chapter 3]. Let τ^o be a realization of $\mathcal{G}$ by derivations on the algebra A. If $i \geqslant 1$, an **i-dimensional A-cochain** for $\mathcal{G}$ is an i-linear mapping f which associates to each i-tuple $(\alpha_1, \ldots, \alpha_i)$, $\alpha_j \in \mathcal{G}$, an element $f(\alpha_1, \ldots, \alpha_i)$ of A in such a way that f is skew-symmetric in its i arguments. A **0-dimensional A-cochain** is a map $\alpha \to h$ where $\alpha \in \mathcal{G}$ and h is a fixed element of A. The collection

$C^i(\mathscr{G}, A)$ of all i-cochains for A forms a complex vector space under the usual definitions of addition and scalar multiplication of functions.

The **coboundary operator** δ is a linear mapping of $C^i(\mathscr{G}, A)$ into $C^{i+1}(\mathscr{G}, A)$, $i \geqslant 0$, defined by

$$\delta f(\alpha_1, \ldots, \alpha_{i+1}) = \sum_{q=1}^{i+1} (-1)^{q-1} \tau^o_{\alpha_q} f(\alpha_1, \ldots, \alpha_q, \ldots, \alpha_{i+1})$$
$$+ \sum_{p<q} (-1)^{p+q} f([\alpha_p, \alpha_q], \alpha_1, \ldots, \hat{\alpha}_p, \ldots, \hat{\alpha}_q, \ldots, \alpha_{i+1}). \quad (8.5)$$

(Omit the arguments $\hat{\alpha}_p$, $\hat{\alpha}_q$.) For $f \in C^0$, $f\colon \alpha \to h$, we require $\delta f(\alpha) = \tau^o_\alpha h$. If $f \in C^i$ the cochain $\delta f \in C^{i+1}$ is called the **coboundary** of f. Note that

$$\delta f(\alpha_1, \alpha_2) = \tau^o_{\alpha_1} f(\alpha_2) - \tau^o_{\alpha_2} f(\alpha_1) - f([\alpha_1, \alpha_2]), \quad (8.6)$$

$$\delta f(\alpha_1, \alpha_2, \alpha_3) = \tau^o_{\alpha_1} f(\alpha_2, \alpha_3) - \tau^o_{\alpha_2} f(\alpha_1, \alpha_3)$$
$$+ \tau^o_{\alpha_3} f(\alpha_1, \alpha_2) - f([\alpha_1, \alpha_2], \alpha_3) + f([\alpha_1, \alpha_3], \alpha_2)$$
$$- f([\alpha_2, \alpha_3], \alpha_1). \quad (8.7)$$

An i-cochain f is a **cocycle** if $\delta f = 0$ and a **coboundary** if $f = \delta h$ for some $h \in C^{i-1}$. Clearly, the set $Z^i(\mathscr{G}, A)$ of all i-dimensional cocycles and the set $B^i(\mathscr{G}, A)$ of all i-dimensional coboundaries are subspaces of C^i. From the definition (8.5) it is a standard argument to show that $\delta^2 = 0$, i.e. $B^i \subseteq Z^i$. The factor space $H^i(\mathscr{G}, A) \equiv Z^i(\mathscr{G}, A)/B^i(\mathscr{G}, A)$ is called the **i-dimensional cohomology space of $\mathscr{G}$ relative to A**.

In the above discussion we have defined the cohomology spaces of $\mathscr{G}$ relative to an algebra A. However, the same construction goes through for a module, $\mathscr{M}$, and one can define cohomology spaces $H^i(\mathscr{G}, \mathscr{M})$ (See Jacobson [1], Chapter 3). We shall have occasion to make use of this remark later.

A comparison of our study of $A(\tau^o)$ and the cohomology theory for the pair $\mathscr{G}$, A is illuminating. $H^0(\mathscr{G}, A)$ can be identified with the space of all elements $h \in A$ such that $\tau^o_\alpha h = 0$, all $\alpha \in \mathscr{G}$. Thus, τ^o is transitive implies that $\dim H^0(\mathscr{G}, A) = 1$. From Lemma 8.1, $\tau \in A(\tau^o)$ if and only if $\tau_\alpha(1)$ is a 1-cocycle. In fact, this correspondence establishes an isomorphism between the vector space $A(\tau^o)$ and the space $Z^1(\mathscr{G}, A)$ of 1-cocycles. Two realizations $\tau, \tau' \in A(\tau^o)$ are cohomologous if and only if their corresponding 1-cocycles differ by a coboundary, i.e., $\tau \approx \tau'$ if and only if $\tau'_\alpha(1) = \tau_\alpha(1) + \delta f, f \in C^0(\mathscr{G}, A)$. The space $N(\tau^o)$ can be identified with the space of 1-coboundaries $B^1(\mathscr{G}, A)$.

Theorem 8.2 $A(\tau^o)/N(\tau^o) \equiv H^1(\mathscr{G}, A)$. The $H^1(\mathscr{G}, A)$ is isomorphic to the space of equivalence classes of realizations of $\mathscr{G}$ by generalized derivations on A.

We return now to the special case $A = \mathscr{A}$. Suppose τ^o is analytic in an open set U containing $\mathbf{0} \in \mathscr{C}^n$. According to Theorem 8.1 there is a one-to-one correspondence between elements of $\mathscr{A}(\tau^o)$ and elements of the set $X(\tau^o)$ consisting of all multiplier representations T^ν of G on $\mathscr{A}$ such that the action $\mathbf{x} \to \mathbf{x}g$, of G as a local transformation group on U is induced by τ^o. The operators $\mathbf{T}^\nu(g)$ are given by

$$[\mathbf{T}^\nu(g)f](\mathbf{x}) = \nu(\mathbf{x}, g) f(\mathbf{x}g), \qquad g \in G, \quad f \in \mathscr{A}, \quad \mathbf{x} \in U, \tag{8.8}$$

where ν is a multiplier (see Section 1-3). Put another way, there is one-to-one correspondence between elements of $\mathscr{A}(\tau^o)$ and complex functions ν on $U \times G$ such that

$$\begin{aligned} &\text{(a)} \quad \nu(\mathbf{x}, \mathbf{e}) = 1, \\ &\text{(b)} \quad \nu(\mathbf{x}, g_1 g_2) = \nu(\mathbf{x}, g_1)\, \nu(\mathbf{x}g_1, g_2) \end{aligned} \tag{8.9}$$

for $\mathbf{x}$ in a suitably small neighborhood of $\mathbf{0}$ and g_1, g_2 in a suitably small neighborhood of the identity element $\mathbf{e}$ of G. In particular the multiplier $\nu_o(\mathbf{x}, g) \equiv 1$ corresponds to τ^o.

We can use this one-to-one correspondence to induce a vector space structure and an equivalence relation on $X(\tau^o)$. Thus if the multiplier representations T^ν, $T^{\nu'}$ are associated with the realizations τ, τ', respectively, we require the **sum** $T^{\nu\nu'}$ to be the multiplier representation associated with the realization $\tau \oplus \tau'$ and the **scalar multiple** T^{ν^a} of T^ν by $a \in \mathscr{C}$ to be the multiplier representation associated with $a\tau$. Similarly we say T^ν is **cohomologous** to $T^{\nu'}$ ($T^\nu \sim T^{\nu'}$) if and only if $\tau \approx \tau'$. The result of this construction is the definition:

Definition The **sum** of T^ν, $T^{\nu'} \in X(\tau^o)$ is the multiplier representation $T^{\nu\nu'} \in X(\tau^o)$ where $\nu\nu'(\mathbf{x}, g) = \nu(\mathbf{x}, g)\, \nu'(\mathbf{x}, g)$; and the **scalar multiple** of T^ν by $a \in \mathscr{C}$ is the representation $T^{\nu^a} \in X(\tau^o)$ where $\nu^a(\mathbf{x}, g) = [\nu(\mathbf{x}, g)]^a$. The representations T^ν and $T^{\nu'}$ are **cohomologous** ($T^\nu \sim T^{\nu'}$) if $T^{\nu'} = \boldsymbol{\varphi}^{-1} T^\nu \boldsymbol{\varphi}$ for some $\boldsymbol{\varphi}$, i.e., if $\nu'(\mathbf{x}, g) = \nu(\mathbf{x}, g)\, \varphi(\mathbf{x}g)/\varphi(\mathbf{x})$ for some $\varphi \in \mathscr{A}$, $\varphi(\mathbf{0}) \neq 0$.

According to these definitions $X(\tau^o)$ is a vector space isomorphic to $\mathscr{A}(\tau^o)$. Moreover the factor space $X(\tau^o)/X^o(\tau^o)$ is isomorphic to $H^1(\mathscr{G}, \mathscr{A})$ where $X^o(\tau^o) = \{T^\nu \in X(\tau^o)\colon T^\nu \sim T^{\nu_o}\}$ is the cohomology class of the identity element of $X(\tau^o)$. These relationships can be used to derive more information about $H^1(\mathscr{G}, \mathscr{A})$.

At this point we make the assumption that τ^o is transitive or, what is the same thing, that the elements of $X(\tau^o)$ are locally transitive on U. Intransitive multiplier representations do not seem to be of much interest for the purposes of special function theory. In particular, all of the multiplier representations studied in this book and applied to the functions of hypergeometric type are locally transitive.

Let $K_{\mathbf{x}}$ be the **isotropy subgroup** of G with respect to the point $\mathbf{x} \in U$, i.e., $K_{\mathbf{x}} = \{g \in G: \mathbf{x}g = \mathbf{x}\}$. It is well known that the groups $K_{\mathbf{x}}, K_{\mathbf{y}}$ corresponding to any two points $\mathbf{x}, \mathbf{y}$ in a sufficiently small neighborhood of $\mathbf{0} \in U$ are conjugate subgroups of G (Pontrjagin [1], Chapter 9). There will be no loss of generality if we restrict ourselves to consideration of the isotropy subgroup $K = K_{\mathbf{0}}$. [Since G is locally transitive and K is closed in G the set U can be identified with an open set in the local homogeneous space G/K and the action of G on U as a local transformation group can be identified with the natural action of G on G/K by right multiplication (Pontrjagin [1], Chapter 9)].

Let $T^\nu \in X(\tau^o)$. Since G is locally transitive it is possible to choose an element $g_{\mathbf{x}} \in G$ for all $\mathbf{x}$ in a suitably small neighborhood of $\mathbf{0}$ such that $\mathbf{0}g_{\mathbf{x}} = \mathbf{x}$ and the function $\mathbf{x} \to g_{\mathbf{x}}$ is analytic in the coordinates of $\mathbf{x}$. From (8.9) there follows the relation

$$\nu(\mathbf{0}, k_1k_2) = \nu(\mathbf{0}, k_1)\,\nu(\mathbf{0}, k_2), \qquad k_1, k_2, k_1k_2 \in K.$$

Thus, $\mu(k) = \nu(\mathbf{0}, k)$ defines a local 1-dimensional representation of K. Furthermore,

$$\nu(\mathbf{0}, kg) = \nu(\mathbf{0}, k)\,\nu(\mathbf{0}, g), \qquad k \in K, \quad g, kg \in G.$$

This equation and (8.9) yield the useful relation

$$\nu(\mathbf{x}, g) = \frac{\nu(\mathbf{0}, g_{\mathbf{x}}g)}{\nu(\mathbf{0}, g_{\mathbf{x}})} = \frac{\mu(k(\mathbf{x}, g))\,\nu(\mathbf{0}, g_{\mathbf{x}g})}{\nu(\mathbf{0}, g_{\mathbf{x}})} \tag{8.10}$$

where $k(\mathbf{x}, g) \in K$ is defined by $g_{\mathbf{x}}g = k(\mathbf{x}, g)\,g_{\mathbf{x}g}$. If we set $\varphi(\mathbf{x}) = \nu(\mathbf{0}, g_{\mathbf{x}})$, then $\varphi \in \mathcal{A}$, $\varphi(\mathbf{0}) \neq 0$, and (8.10) becomes

$$\nu(\mathbf{x}, g) = \frac{\mu(k(\mathbf{x}, g))\,\varphi(\mathbf{x}g)}{\varphi(\mathbf{x})}. \tag{8.11}$$

Conversely, if $\varphi \in \mathcal{A}$, $\varphi(\mathbf{0}) \neq 0$, and μ is a local 1-dimensional representation of K, by reversing the above argument it is easy to show that the function ν defined by (8.11) satisfies conditions (8.9), hence, defines a multiplier representation T^ν of G.

Comparing (8.11) with the definition of cohomologous multiplier representations we find $T^{\nu} \sim T^{\nu'}$ if and only if $\mu(k) = \mu'(k)$ for all $k \in K$. Thus to determine all noncohomologous multiplier representations of G with given action on U, we need only determine the possible analytic 1-dimensional representations of K.

If μ is an analytic 1-dimensional representation of K, then μ maps all commutators in K into $1 \in \mathbb{C}$:

$$\mu(k_1 k_2 k_1^{-1} k_2^{-1}) = \mu(k_1)\, \mu(k_2)\, \mu(k_1)^{-1}\, \mu(k_2)^{-1} = 1.$$

Let K^* be the commutator subgroup of K, i.e., the closed normal subgroup generated by the commutators $k_1 k_2 k_1^{-1} k_2^{-1}$ of K. In this case $\mu(K^*) = 1$ and μ induces a representation μ^* of the abelian factor group K/K^*:

$$\mu^*(kK^*) = \mu(k), \qquad k \in K. \tag{8.12}$$

Suppose K/K^* has dimension s as a local Lie group. Then there are local coordinates $(x_1, \ldots, x_s)$ for K/K^* in a neighborhood of $\mathbf{0} \in \mathbb{C}^s$ such that the coordinates of $\mathbf{e}K^*$ are $(0, \ldots, 0)$, and if $k_1 K^*$, $k_2 K^*$ have coordinates $(a_1, \ldots, a_s)$, $(b_1, \ldots, b_s)$, respectively, then $(k_1 K^*)(k_2 K^*) = k_3 K^*$ has coordinates $(a_1 + b_1, \ldots, a_s + b_s)$. Since μ^* is a 1-dimensional analytic representation of an s-dimensional abelian group, there must exist complex constants $c_1, \ldots, c_s$ such that

$$\mu^*(kK^*) = \mu^*(x_1, \ldots, x_s) = \exp(x_1 c_1 + x_2 c_2 + \cdots + x_s c_s) \tag{8.13}$$

where $(x_1, \ldots, x_s)$ are the local coordinates of $kK^* \in K/K^*$. Conversely, if complex constants $c_1, \ldots, c_s$ are given and μ^* is defined on K/K^* by (8.13), we can define a representation μ of K by (8.12) and thus determine a multiplier representation T of G. There is a one-to-one correspondence between classes of cohomologous multiplier representations of G and complex s-tuples $(c_1, \ldots, c_s)$. Indeed, two multiplier representations are cohomologous if and only if they are associated with the same complex s-tuple.

The set of all complex s-tuples has a natural vector space structure which agrees with the induced structure of the space of cohomology classes. In fact, if T^{ν}, $T^{\nu'} \in X(\tau^o)$ are associated with $(c_1, \ldots, c_s)$, $(c_1', \ldots, c_s')$, respectively, then an easy computation shows $T^{\nu\nu'}$ is associated with the s-tuple $(c_1 + c_1', \ldots, c_s + c_s')$. Similarly, T^{ν^a} is associated with the s-tuple $(ac_1, \ldots, ac_s)$.

As a consequence of the above construction there follow the equalities

$$\dim H^1(\mathscr{G}, \mathscr{A}) = \dim X(\tau^o)/X^o(\tau^o) = \dim K - \dim K^* = s.$$

These equalities prove that the cohomology space $H^1(\mathscr{G}, \mathscr{A})$ is finite-dimensional and that each cohomology class is associated with a unique s-tuple $(c_1, \ldots, c_s)$. Such results suggest the possibility of a practical method for computing the cohomology classes.

We can use the isomorphism between $X(\tau^o)$ and $\mathscr{A}(\tau^o)$ to phrase our conclusions in terms of differential operators.

Theorem 8.3 Let τ^o be a transitive realization of $\mathscr{G}$ by derivations on $\mathscr{A}$ and let $\mathscr{K}$ be the isotropy subalgebra of $\mathscr{G}$ with respect to τ^o,

$$\mathscr{K} = \{\alpha \in \mathscr{G}: [\tau^0_\alpha f](\mathbf{0}) = 0, \qquad \text{all } f \in \mathscr{A}\}.$$

Then, if $\mathscr{K}^* = [\mathscr{K}, \mathscr{K}]$ is the derived algebra of $\mathscr{K}$, we have

$$\dim H^0(\mathscr{G}, \mathscr{A}) = 1,$$

$$\dim H^1(\mathscr{G}, \mathscr{A}) = \dim \mathscr{K} - \dim \mathscr{K}^*.$$

PROOF $\mathscr{K} = L(K)$, $\mathscr{K}^* = L(K^*)$.

Here $\mathscr{K}^*$ is the Lie algebra generated by the set of all commutators $[\alpha, \beta]$, $\alpha, \beta \in \mathscr{K}$. Theorem 8.3 is of fundamental importance for the computation of cohomology classes for realizations of $\mathscr{G}$. In fact, given a transitive realization τ of $\mathscr{G}$ on $\mathscr{A}$ we can read off the dimensions of $\mathscr{K}$ and $\mathscr{K}^*$ directly and thus determine the dimension of $H^1(\mathscr{G}, \mathscr{A})$. We will use these results in Section 8-3 to explicitly compute realizations τ in each of the cohomology classes of $\mathscr{A}(\tau^o)$.

Before leaving the abstract theory, however, we make several observations:

(1) Let τ be a realization of $\mathscr{G}$ by gd's on $\mathscr{A}$ and let $\mathscr{I}$ be the subalgebra of $\mathscr{G}$ defined by $\mathscr{I} = \{\alpha \in \mathscr{G}: \tau_\alpha f \equiv 0, \text{ all } f \in \mathscr{A}\}$. It is easy to show that $\mathscr{I}$ is an ideal in $\mathscr{G}$, $[\mathscr{G}, \mathscr{I}] \subset \mathscr{I}$. Thus, τ induces a realization τ' of the factor algebra $\mathscr{G}' = \mathscr{G}/\mathscr{I}$. ($\tau'_{\alpha+\mathscr{I}} = \tau_\alpha$ for all $\alpha \in \mathscr{G}$.) Now τ' is an effective realization of $\mathscr{G}'$ in the sense that if $\alpha + \mathscr{I} \in \mathscr{G}'$, $\alpha \notin \mathscr{I}$, then $\tau'_{\alpha+\mathscr{I}} \neq 0$. As a consequence of this fact there is no loss in generality if we study only those realizations of $\mathscr{G}$ which are effective. Thus, in the remainder of this chapter only the spaces of realizations $\mathscr{A}(\tau^o)$ of $\mathscr{G}$ by gd's such that at least one $\tau \in \mathscr{A}(\tau^o)$ is effective will be considered.

(2) Let $\tau \in \mathscr{A}(\tau^o)$ be an effective realization of $\mathscr{G}$ and consider the set $\mathscr{M} = \{\alpha \in \mathscr{G}: \tau^o_\alpha f \equiv 0, \text{ all } f \in \mathscr{A}\}$. Clearly, $\mathscr{M}$ is an abelian ideal in $\mathscr{G}$:

$$[\mathscr{G}, \mathscr{M}] \subset \mathscr{M}, \qquad [\mathscr{M}, \mathscr{M}] = \mathscr{O}.$$

If $\dim \mathscr{M} = m$, the set of functions $\tau_\alpha(1) \in \mathscr{A}$ defined for all $\alpha \in \mathscr{M}$ form an m-dimensional vector space $\mathscr{A}(\mathscr{M})$ in $\mathscr{A}$. (If $\dim \mathscr{A}(\mathscr{M}) < m$ there exists an $\alpha \in \mathscr{M}$ such that $\tau_\alpha = 0$, in which case τ is not effective.) $\mathscr{A}(\mathscr{M})$ can be given the structure of an abelian Lie algebra by defining the commutator of any two elements to be $0 \in \mathscr{A}(\mathscr{M})$. Then the Lie algebras $\mathscr{M}$ and $\mathscr{A}(\mathscr{M})$ are isomorphic and can be identified. $\mathscr{G}$ acts on $\mathscr{M}$ by the derivations τ^o,

$$\tau^o_\alpha(\tau_\beta(1)) = \tau_{[\alpha,\beta]}(1), \qquad \alpha \in \mathscr{G}, \quad \beta \in \mathscr{M}. \tag{8.14}$$

τ^o induces an effective realization $\tau^{o\prime}$ of the factor algebra $\mathscr{G}' = \mathscr{G}/\mathscr{M}$ on $\mathscr{M}$ obtained in an obvious manner from (8.14). Thus, given an effective realization of $\mathscr{G}$ by gd's on $\mathscr{A}$ we have constructed an effective realization of the factor algebra $\mathscr{G}'$ by **derivations** on $\mathscr{A}(\mathscr{M})$.

Conversely, let μ^o be an effective realization of a Lie algebra $\mathscr{G}'$ by derivations on an m-dimensional vector space $\mathscr{A}(\mathscr{M}) \subset \mathscr{A}$. We impose upon $\mathscr{A}(\mathscr{M})$ the structure of an abelian Lie algebra and identify it with the abstract m-dimensional Lie algebra $\mathscr{M}$. By reversing the argument in the preceding paragraph and using the theory of extensions of Lie algebras (Seminaire Sophus Lie [1], Cartan and Eilenberg [1]), we can find a Lie algebra $\mathscr{G}$ containing $\mathscr{M}$ as an ideal and an effective realization τ of $\mathscr{G}$ by gd's on $\mathscr{A}$ (neither $\mathscr{G}$ nor τ is unique) such that $\mathscr{G}/\mathscr{M} \cong \mathscr{G}'$ and τ induces the realization μ^o of $\mathscr{G}'$ on $\mathscr{M}$. In fact, it can be shown that the possible extensions $\mathscr{G}$ of $\mathscr{M}$ by $\mathscr{G}'$ are in one-to-one correspondence with the elements of the space $H^2(\mathscr{G}', \mathscr{A}(\mathscr{M}))$ where the action of $\mathscr{G}'$ on $\mathscr{A}(\mathscr{M})$ is given by μ^o.

In the case where $\mathscr{G}'$ is semisimple we can apply the Whitehead lemmas (Jacobson [1], Chapter 3) which state $H^i(\mathscr{G}', \mathscr{A}(\mathscr{M})) = 0$, $i = 0, 1, 2, \ldots$. Thus, $H^2(\mathscr{G}', \mathscr{A}(\mathscr{M}))$ consists of the zero element alone and there is only one possible extension of $\mathscr{M}$ by $\mathscr{G}'$. As is well known, that extension is $\mathscr{G} = \mathscr{G}' \oplus \mathscr{M}$, the split extension of $\mathscr{G}'$ by $\mathscr{M}$ (Jacobson [1], Chapter 1; Seminaire Sophus Lie [1]).

Theorem 8.4 Let $\mathscr{G}$ be a Lie algebra and τ a transitive effective realization of $\mathscr{G}$ by gd's on $\mathscr{A}$. Let $\mathscr{M} = \{\alpha \in \mathscr{G} : \tilde{\tau}_\alpha \equiv 0\}$, $\mathscr{G}' = \mathscr{G}/\mathscr{M}$ as defined above. If $\mathscr{G}'$ is semisimple then $\mathscr{G} \cong \mathscr{G}' \oplus \mathscr{M}$.

The above results constitute a "bootstrap" method for constructing realizations τ of a Lie algebra $\mathscr{G}$ by gd's from a knowledge of the realizations μ^o of lower-dimensional Lie algebras $\mathscr{G}'$ by derivations. This method will prove useful in the next section. Note that the Whitehead lemmas are not applicable to the computation of the cohomology spaces $H^i(\mathscr{G}, \mathscr{A})$, $i = 0, 1, 2, \ldots$, because $\mathscr{A}$ is infinite-dimensional.

8-3 Computation of the Cohomology Classes

The abstract theory presented in the last section will now be applied to explicitly construct the possible realizations of a Lie algebra. Let τ^o be a transitive realization of $\mathscr{G}$ by derivations on $\mathscr{A}$ with isotropy subalgebra $\mathscr{K}$,

$$\mathscr{K} = \{\alpha \in \mathscr{G}: [\tau^o_\alpha f](\mathbf{0}) = 0, \qquad \text{all } f \in \mathscr{A}\}.$$

If $r = \dim \mathscr{G}$ we can choose a basis $\alpha_1, \ldots, \alpha_r$ for $\mathscr{G}$ and define the **structure constants** C^l_{ij} with respect to this basis by

$$[\alpha_i, \alpha_j] = \sum_{l=1}^{r} C^l_{ij}\alpha_l, \qquad 1 \leqslant i, j \leqslant r. \tag{8.15}$$

As is well known (Jacobson [1], Chapter 1), the skew-symmetry of the commutator and the Jacobi equality imply the relations

$$C^l_{ij} = C^l_{ji}, \qquad \sum_{p=1}^{r} (C^p_{ij}C^b_{pl} + C^p_{li}C^b_{pj} + C^p_{jl}C^b_{pi}) = 0 \tag{8.16}$$

for $1 \leqslant i, j, l, b \leqslant r$.

According to (8.15) the Lie derivatives Y_j defined by $Y_j = \tau^o_{\alpha_j}$, $j = 1, \ldots, r$, satisfy the relations

$$Y_jY_k - Y_kY_j = \sum_{p=1}^{r} C^p_{jk}Y_p, \qquad 1 \leqslant j, k \leqslant r. \tag{8.17}$$

In terms of local coordinates in $U \subset \mathscr{C}^n$ the Y_j can be expressed in the form

$$Y_j = \sum_{l=1}^{n} P^l_j(\mathbf{x}) \frac{\partial}{\partial x_l}, \qquad 1 \leqslant j \leqslant r, \quad r \geqslant n,$$

where $P^l_j \in \mathscr{A}$. Since τ^o is transitive the $r \times n$ matrix $(P^l_j(\mathbf{0}))$ has rank n.

If $\tau \in \mathscr{A}(\tau^o)$ the functions $f_j \in \mathscr{A}$ defined by $f_j = \tau_{\alpha_j}(1)$, $1 \leqslant j \leqslant r$, satisfy the relations

$$Y_j(f_k) - Y_k(f_j) = \sum_{p=1}^{r} C^p_{jk}f_p, \qquad 1 \leqslant j, k \leqslant r. \tag{8.18}$$

Conversely, any set $\{f_j\}$, $1 \leqslant j \leqslant r$, of elements of $\mathscr{A}$ that satisfies (8.18) uniquely defines a member τ of $\mathscr{A}(\tau^o)$. In fact $\tau_{\alpha_j} = Y_j + f_j$. Thus, the problem of determining all members of $\mathscr{A}(\tau^o)$ is the same as the problem of determining all solutions $f_1, \ldots, f_r$ of (8.18).

If $\tau, \tau' \in \mathcal{A}(\tau^o)$, then $\tau \approx \tau'$ if and only if

$$f'_j = f_j + Y_j(f), \qquad 1 \leqslant j \leqslant r, \tag{8.19}$$

for some $f \in \mathcal{A}$, where $f_j = \tau_{\alpha_j}(1)$, $f'_j = \tau'_{\alpha_j}(1)$. To find a realization in each cohomology class of $\mathcal{A}(\tau^o)$ we need to find all solutions $\{f_j\}$ of (8.18) and then from among these solutions pick out one element in each equivalence class (8.19). $\mathcal{A}(\tau^o)$ is in general infinite-dimensional so this seems to be a difficult problem. However, use of Theorem 8.3 simplifies the computation considerably.

For simplicity assume there exists an n-dimensional subalgebra $\mathcal{H}$ of $\mathcal{G}$ such that $\mathcal{G} = \mathcal{H} + \mathcal{K}$ (direct sum as a vector space), where $\mathcal{K}$ is the isotropy subalgebra of $\mathcal{G}$ with respect to τ^o. It is not necessary that $\mathcal{H}$ be an ideal in $\mathcal{G}$. If this condition is satisfied, we say τ^o is **regular**. Clearly, if τ^o is regular the restriction of τ^o to the subalgebra $\mathcal{H}$ yields a transitive realization of $\mathcal{H}$ on $\mathcal{A}$ with isotropy subalgebra zero.

Regular realizations are of frequent occurrence. In fact, an examination of Lie's tables (Lie [1], Vol. III, pp. 4–5 and 71–73) listing the local transformation groups operating effectively on spaces of one and two complex variables shows that **all** of these realizations satisfy this condition. Furthermore, most of the realizations τ^o studied in this book also satisfy it.

To recapitulate, we are concerned with an r-dimensional complex Lie algebra $\mathcal{G}$ and a regular realization τ^o of $\mathcal{G}$ by derivations on the space $\mathcal{A}$ of analytic functions of n complex variables. $\mathcal{G} = \mathcal{H} + \mathcal{K}$ where $\mathcal{H}$ is an n-dimensional subalgebra and $\mathcal{K}$ is the isotropy subalgebra of $\mathcal{G}$ with respect to τ^o. Let $\dim \mathcal{K} = k$, $\dim \mathcal{K}^* = s$ where $\mathcal{K}^*$ is the derived subalgebra of $\mathcal{K}$. Clearly $k = r - n \geqslant 0$. We can choose a basis $\alpha_1, \ldots, \alpha_r$ for $\mathcal{G}$ such that $\alpha_1, \ldots, \alpha_n$ form a basis for $\mathcal{H}$ and $\alpha_{n+1}, \ldots, \alpha_r$ form a basis for $\mathcal{K}$. In this case the Lie derivatives $Y_1, \ldots, Y_n$ are linearly independent in a neighborhood of $\mathbf{0} \in \mathbb{C}^n$ while $[Y_j f](\mathbf{0}) = 0$ for all $f \in \mathcal{A}$ and $j = n+1, \ldots, r$. Since $\tau^o/\mathcal{H}$ has isotropy subalgebra zero we know from Theorem 8.3 that $\tau/\mathcal{H} \approx \tau^o/\mathcal{H}$ for any $\tau \in \mathcal{A}(\tau^o)$. (This follows from the fact that $H^1(\mathcal{H}, \mathcal{A}) = 0$.) Thus any element of $\mathcal{A}(\tau^o)$ is cohomologous to a realization τ for which $f_j = \tau_{\alpha_j}(1) = 0$, $1 \leqslant j \leqslant n$. Furthermore, if $\tau, \tau' \in \mathcal{A}(\tau^o)$ with $\tau_{\alpha_j}(1) = \tau'_{\alpha_j}(1) = 0$, $1 \leqslant j \leqslant n$, then $\tau \approx \tau'$ if and only if $\tau_\alpha = \tau'_\alpha$ for all $\alpha \in \mathcal{K}$. Applying Theorem 8.3 again we have:

Lemma 8.2 If τ^o is regular, there is an isomorphism between the $(k - s)$-dimensional space of cohomology classes of $\mathcal{A}(\tau^o)$ and the space of solutions $(f_1, f_2, \ldots, f_r)$ of Eqs. (8.18) such that $f_1 = f_2 = \cdots = f_n = 0$.

The most general solution of this kind depends linearly on $k - s$ constants of integration and the choice of these constants determines the cohomology class of the solution.

As an example we will find a realization in each cohomology class of the space $\mathcal{A}(\tau^o)$ where τ^o is the realization of $sl(2)$ by differential operators

$$\tau^o_{\alpha_1} = \frac{d}{dz}, \qquad \tau^o_{\alpha_2} = z\frac{d}{dz}, \qquad \tau^o_{\alpha_3} = z^2\frac{d}{dz}. \tag{8.20}$$

Here,

$$[\alpha_2, \alpha_1] = -\alpha_1, \qquad [\alpha_2, \alpha_3] = \alpha_3, \qquad [\alpha_3, \alpha_1] = -2\alpha_2.$$

The realization $\tau \in \mathcal{A}(\tau^o)$ takes the form

$$\tau_{\alpha_1} = \frac{d}{dz} + f_1(z), \qquad \tau_{\alpha_2} = z\frac{d}{dz} + f_2(z), \qquad \tau_{\alpha_3} = z^2\frac{d}{dz} + f_3(z)$$

where

$$\frac{df_2}{dz} - z\frac{df_1}{dz} = f_1, \qquad \frac{df_3}{dz} - z^2\frac{df_1}{dz} = 2f_2, \qquad z\frac{df_3}{dz} - z^2\frac{df_2}{dz} = f_3. \tag{8.21}$$

Clearly, $\mathcal{K}$ is the 2-dimensional subalgebra of $sl(2)$ with basis $\{\alpha_2, \alpha_3\}$ and $\mathcal{K}^*$ has basis $\{\alpha_3\}$. For $\mathcal{H}$ we can choose the subalgebra with basis $\{\alpha_1\}$. Then, Theorem 8.3 implies $\dim H^1(sl(2), \mathcal{A}) = 2 - 1 = 1$. According to Lemma 8.2, τ is cohomologous to a realization for which $f_1(z) \equiv 0$. Thus, Eqs. (8.21) reduce to

$$\frac{df_2}{dz} = 0, \qquad 2zf_2 = f_3,$$

with solution

$$\tau_{\alpha_1} = \frac{d}{dz}, \qquad \tau_{\alpha_2} = z\frac{d}{dz} + c, \qquad \tau_{\alpha_3} = z^2\frac{d}{dz} + 2cz, \qquad c \in \mathcal{C}. \tag{8.22}$$

The value of the constant c determines the cohomology class in which τ lies. All other solutions of (8.21) are cohomologous to a solution (8.22) and no two solutions of the form (8.22) are cohomologous unless they are identical. The general solution of (8.21) is

$$\tau_{\alpha_1} = \frac{d}{dz} + \frac{df}{dz}, \qquad \tau_{\alpha_2} = z\frac{d}{dz} + c + z\frac{df}{dz}, \qquad \tau_{\alpha_3} = z^2\frac{d}{dz} + 2cz + z^2\frac{df}{dz},$$

where f is an arbitrary element of $\mathcal{A}$.

The above example was so elementary that the cohomology classes could easily have been computed without recourse to Theorem 8.3 and Lemma 8.2. However, for higher-dimensional Lie algebras these results assume considerable practical importance. Nonregular realizations of a Lie algebra do not obey Lemma 8.2 and it is much more difficult to compute explicitly the cohomology classes of such realizations. An example of a nonregular realization is $\tilde{\zeta}_2$ (Section 8-4).

In Sections 2-7 and 2-8 we computed realizations of $sl(2)$ that seem to violate the analyticity assumptions made in this chapter. In particular the differential operators

$$\tau_{\alpha_1} = \frac{d}{dz} - \frac{\rho}{z}, \qquad \tau_{\alpha_2} = z\frac{d}{dz} + \lambda, \qquad \tau_{\alpha_3} = z^2\frac{d}{dz} + (2\lambda + \rho)\, z \tag{8.23}$$

form a realization of $sl(2)$, where λ, ρ are complex constants. If $\rho \neq 0$ this realization has a singularity at $z = 0$. Thus, $\tau \notin \mathcal{A}(\tau^o)$ for $\rho \neq 0$. However, if τ is transformed to the realization $\tau' = \boldsymbol{\varphi}^{-1}\tau\boldsymbol{\varphi}$ where φ is the function $\varphi(z) = z^\rho$, then

$$\tau'_{\alpha_1} = \frac{d}{dz}, \qquad \tau'_{\alpha_2} = z\frac{d}{dz} + c, \qquad \tau'_{\alpha_3} = z^2\frac{d}{dz} + 2cz, \qquad c = \lambda + \rho. \tag{8.24}$$

Clearly, τ' is an element of $\mathcal{A}(\tau^o)$ in the normalized form (8.22). (Note: Equation (4.82) remains valid even if $\varphi(\mathbf{0}) = 0$ or $\varphi \notin \mathcal{A}$.) In Chapter 5 it was shown that the realizations τ for $\rho \neq 0$ could be used to construct certain infinite-dimensional irreducible representations of $sl(2)$. The basis functions for these representations were $h_k(z) = z^k$, $k = 0, \pm 1, \pm 2, \ldots$. However, it is easy to show that these same representations could be obtained from the realizations $\tau' = \boldsymbol{\varphi}^{-1}\tau\boldsymbol{\varphi}$ operating on the space generated by the functions $h'_k(z) = \varphi^{-1}(z)\, h_k(z) = z^{k-\rho}$, $k = 0, \pm 1, \pm 2, \ldots$. Thus, it was not necessary to use the singular operators in Chapter 5; we could have obtained the same results in special function theory by using the analytic operators τ' and the basis $\{h'_k\}$. The singular operators were used simply because the basis $\{h_k\}$ is more convenient for computational purposes. There is thus no loss of generality in restricting ourselves to elements of $\mathcal{A}(\tau^o)$.

These remarks hold true for all of the realizations of the Lie algebras $\mathcal{G}(a, b)$ and $\mathcal{T}_6$ by differential operators as computed in Chapters 3–6. In each case where the realization τ has a singularity at the point $\mathbf{x}^o = \mathbf{0}$ we can find a function $\varphi(\mathbf{x})$ such that the realization $\tau' = \boldsymbol{\varphi}^{-1}\tau\boldsymbol{\varphi}$ is an element of $\mathcal{A}(\tilde{\tau}')$. There is no loss of generality for special function theory involved in the restriction to elements of $\mathcal{A}(\tilde{\tau}')$. Note, however, that in the construction of irreducible representations of $\mathcal{G}$ in terms of a reali-

zation τ we do not necessarily assume that the basis space for this representation is a subspace of $\mathcal{A}$. The elements of the basis space may have a singularity at $\mathbf{x}^o = \mathbf{0}$.

8-4 Tables of Cohomology Classes

As a first application of our results we list all transitive effective realizations of Lie algebras by gd's acting on functions of one complex variable, i.e., $n = 1$. To determine this list it is sufficient to use the methods indicated in observation (2), Section 8–2. Lie has proved that, to within an analytic change of variable, the only transitive effective realizations of a Lie algebra by ordinary derivations in one complex variable are (Lie [1], Vol. III, pp. 4–5)

$$\begin{aligned} &(1)\quad \frac{d}{dz};\\ &(2)\quad \frac{d}{dz},\quad z\frac{d}{dz};\\ &(3)\quad \frac{d}{dz},\quad z\frac{d}{dz},\quad z^2\frac{d}{dz}. \end{aligned} \tag{8.25}$$

In each of the three possible cases the differential operators corresponding to a basis of the associated Lie algebra are given. In particular, (3) is the realization of $sl(2)$ studied in the last section.

For $n = 1$ the only possibilities for the factor algebras $\mathcal{G}'$ defined in observation (2) are the three Lie algebras determined by (8.25). Corresponding to each of these three cases we must find all finite-dimensional subspaces $\mathcal{M}$ of $\mathcal{A}$ which are invariant under the action of $\mathcal{G}'$ on $\mathcal{A}$ as given by (8.25). (Such a classification is most easily obtained by finding a basis for $\mathcal{M}$ such that the matrix of d/dz is in Jordan canonical form.) Considering each such subspace $\mathcal{M}$ as an abelian Lie algebra we must then determine up to isomorphism all possible Lie algebras $\mathcal{G} \supset \mathcal{M}$ such that $\mathcal{G}/\mathcal{M} \cong \mathcal{G}'$. The list of all Lie algebras $\mathcal{G}$ so obtained exhausts the list of Lie algebras which have a transitive effective realization in terms of gd's in one complex variable. The above computation is straightforward but tedious so we shall merely list the results.

For each Lie algebra in the list we give an effective realization τ. All other realizations in the set $\mathcal{A}(\tilde{\tau})$ can easily be obtained from τ by using the methods presented in Section 8-2. For each realization the values of $r = \dim \mathcal{G}$, $k = \dim \mathcal{K}$, $s = \dim \mathcal{K}^*$ will be given. Recall $\dim H^1(\mathcal{G}, \mathcal{A}) = k - s$.

Theorem 8.5 If τ' is a transitive effective realization of a finite-dimensional Lie algebra by gd's in one complex variable, then either $\tau' \in \mathcal{A}(\tilde{\tau})$ for some τ in the following list or τ' can be obtained from such a realization by an analytic change of variable. The list is:

$$\frac{d}{dz}, \quad z^{l_i}e^{a_i z}; \qquad i = 1, 2, \ldots, q, \qquad l_i = 0, 1, 2, \ldots, m_i,$$

$$a_i \in \mathcal{C}, \quad a_1(a_1 - 1) = 0,$$

$$a_i \neq a_j \quad \text{for} \quad i \neq j. \qquad m_i, q \text{ nonnegative integers}$$

$$r = q + 1 + \sum_{i=1}^{q} m_i, \qquad k = r - 1, \quad s = 0. \tag{8.26}$$

$$\frac{d}{dz}, \quad z\frac{d}{dz}; \qquad r = 2, \quad k = 1, \quad s = 0. \tag{8.27}$$

$$\begin{gathered} \frac{d}{dz}, \quad z\frac{d}{dz}, \quad z^l; \qquad l = 0, 1, \ldots, q, \quad q \text{ an integer}, \\ r = q + 3, \quad k = q + 2, \quad s = q. \end{gathered} \tag{8.28}$$

$$\frac{d}{dz}, \quad z\frac{d}{dz}, \quad z^2\frac{d}{dz}; \qquad r = 3, \quad k = 2, \quad s = 1. \tag{8.29}$$

$$\frac{d}{dz}, \quad z\frac{d}{dz}, \quad z^2\frac{d}{dz}, \quad 1; \qquad r = 4, \quad k = 3, \quad s = 1. \tag{8.30}$$

All of the realizations of the Lie algebras $\mathcal{G}(a, b)$ obtained in Section 2–8 are special cases of this theorem. Thus, the realizations of $\mathcal{G}(0, 0)$ are obtained from (8.26) by setting $q = 3$, $a_1 = 0$, $a_2 = 1$, $a_3 = -1$, $m_1 = m_2 = m_3 = 0$:

$$\frac{d}{dz}, \quad 1, \quad e^z, \quad e^{-z}; \qquad k - s = 3.$$

(Introduce the new variable $z' = e^z$.) The realizations of $\mathcal{G}(0, 1)$ follow from (8.28), where $q = 1$:

$$\frac{d}{dz}, \quad z\frac{d}{dz}, \quad 1, \quad z; \qquad k = 3, \quad s = 1.$$

The realizations of $\mathcal{G}(1, 0)$ are obtained from (8.30):

$$\frac{d}{dz}, \quad z\frac{d}{dz}, \quad z^2\frac{d}{dz}, \quad 1; \qquad k = 3, \quad s = 1.$$

There are no realizations of $\mathscr{G}(a, b)$ for $n = 1$ other than those given here. This proves that the restrictive assumptions made in Section 2–8 were not necessary; indeed, the most general realization of $\mathscr{G}(a, b)$ is cohomologous to a realization derived in Section 2–8. (Note that there is no realization of $\mathscr{T}_6$ by gd's in one complex variable.)

The algebraic difficulties involved in computing all transitive realizations of Lie algebras by gd's in two complex variables are great enough so as to make it impractical to write down such a list. However, it is not difficult to find all possible such realizations of a given Lie algebra $\mathscr{G}$. Lie has computed a list of all possible realizations of Lie algebras by **ordinary** derivations for the case $n = 2$ (Lie [1], Vol. III, pp. 71–73). To find all realizations of a given Lie algebra $\mathscr{G}$ by gd's it is necessary to classify the possible abelian ideals $\mathscr{M}$ of $\mathscr{G}$. (The ideals $\mathscr{M}$ and $\mathscr{M}'$ are identified if there is an automorphism of $\mathscr{G}$ which maps $\mathscr{M}$ onto $\mathscr{M}'$.) For each such $\mathscr{M}$, Lie's tables can be used to determine the possible realizations of the factor algebra $\mathscr{G}' \cong \mathscr{G}/\mathscr{M}$ by ordinary derivations. Each of these realizations of $\mathscr{G}'$ can then be extended to give realizations of $\mathscr{G}$. According to observation (2) at the end of Section 8-2, every realization of $\mathscr{G}$ can be obtained in this way.

In Chapters 3–6 it was shown that the functions of hypergeometric type were associated with realizations of the Lie algebras $sl(2)$, $\mathscr{G}(0, 1)$, $\mathscr{T}_3$, $\mathscr{T}_6$ by the differential operator *types A–F*. It was not clear why those particular realizations were singled out for attention or whether there were other realizations in terms of gd's in two complex variables. We shall resolve this problem by listing all transitive effective realizations of these Lie algebras for $n = 2$. The results are:

(I) $sl(2)$

$$\mu_1\colon\quad \frac{\partial}{\partial z_1},\quad e^{z_1}\frac{\partial}{\partial z_2} + e^{z_1}\tan z_2\frac{\partial}{\partial z_1},\quad -e^{-z_1}\frac{\partial}{\partial z_2} + e^{-z_1}\tan z_2\frac{\partial}{\partial z_1};$$

$$r = 3,\quad k = 1,\quad s = 0.$$

$$\mu_2\colon\quad \frac{\partial}{\partial z_1},\quad e^{z_1}\frac{\partial}{\partial z_2} - e^{z_1}\frac{\partial}{\partial z_1},\quad e^{-z_1}\frac{\partial}{\partial z_2} + e^{-z_1}\frac{\partial}{\partial z_1};$$

$$r = 3,\quad k = 1,\quad s = 0.$$

Every transitive realization of $sl(2)$ by gd's in two complex variables is, to within a change of coordinates, either an element of $\mathscr{A}(\mu_1)$ or an element of $\mathscr{A}(\mu_2)$. The *type A* and *type B* operators correspond to a selection of an element in each cohomology class in $\mathscr{A}(\mu_1)$ and $\mathscr{A}(\mu_2)$, respectively.

(II) $\mathscr{G}(0,1)$

$$\zeta_1\colon\ \frac{\partial}{\partial z_1},\quad e^{z_1}\frac{\partial}{\partial z_2}-\frac{e^{z_1}z_2}{2},\quad -e^{-z_1}\frac{\partial}{\partial z_2}-\frac{e^{-z_1}z_2}{2},\quad 1;$$

$$r=4,\quad k=2,\quad s=0,$$

$$\zeta_2\colon\ z_2\frac{\partial}{\partial z_2}-z_1\frac{\partial}{\partial z_1},\quad \frac{\partial}{\partial z_1}+z_2,\quad \frac{\partial}{\partial z_2},\quad 1;$$

$$r=4,\quad k=2,\quad s=0,$$

$$\zeta_3\colon\ z_1\frac{\partial}{\partial z_1},\quad z_1\frac{\partial}{\partial z_2},\quad \frac{\partial}{\partial z_1},\quad \frac{\partial}{\partial z_2};$$

$$r=4,\quad k=2,\quad s=1,$$

$$\zeta_4\colon\ \frac{\partial}{\partial z_1},\quad e^{z_1}z_2,\quad e^{-z_1}\frac{\partial}{\partial z_2},\quad 1;$$

$$r=4,\quad k=2,\quad s=0.$$

Every transitive effective realization of $\mathscr{G}(0, 1)$ by gd's in two complex variables is, to within change of coordinates, an element of one of the spaces $\mathscr{A}(\tilde{\zeta}_i)$, $i = 1,\ldots, 4$. The *type* D' and C' operators correspond to a selection of an element in each cohomology class in $\mathscr{A}(\tilde{\zeta}_1)$ and $\mathscr{A}(\tilde{\zeta}_2)$, respectively. The realizations in $\mathscr{A}(\tilde{\zeta}_3)$ and $\mathscr{A}(\tilde{\zeta}_4)$ have not occurred in this book up to now. However, as the reader can verify, they do not lead to any new results in special function theory.

(III) $\mathscr{T}_3$

$$\xi_1\colon\ \frac{\partial}{\partial z_1},\quad e^{z_1}\frac{\partial}{\partial z_2},\quad e^{-z_1}\frac{\partial}{\partial z_2};\qquad r=3,\quad k=1,\quad s=0,$$

$$\xi_2\colon\ z_2\frac{\partial}{\partial z_2}-z_1\frac{\partial}{\partial z_1},\quad \frac{\partial}{\partial z_1},\quad \frac{\partial}{\partial z_2};\qquad r=3,\quad k=1,\quad s=0,$$

$$\xi_3\colon\ \frac{\partial}{\partial z_1},\quad e^{z_1}\frac{\partial}{\partial z_2},\quad e^{-z_1};\qquad r=3,\quad k=1,\quad s=0.$$

Every transitive effective realization of $\mathscr{T}_3$ by gd's in two variables is, to within a change of coordinates, an element of $\mathscr{A}(\tilde{\xi}_i)$ for $i = 1, 2$, or 3. The *type* D'' and C'' operators correspond to a selection of an element in each cohomology class in $\mathscr{A}(\xi_1)$ and $\mathscr{A}(\xi_2)$, respectively. Realizations in $\mathscr{A}(\tilde{\xi}_3)$ did not occur in Chapter 3, but they lead to no new special functions.

(IV) $\mathscr{T}_6$

$$\rho_1\colon\ \frac{\partial}{\partial z_1},\quad e^{z_1}\frac{\partial}{\partial z_2}+e^{z_1}\tan z_2\frac{\partial}{\partial z_1},\quad -e^{-z_1}\frac{\partial}{\partial z_2}+e^{-z_1}\tan z_2\frac{\partial}{\partial z_1},$$

$$e^{z_1}\cos z_2,\quad -\sin z_2,\quad e^{-z_1}\cos z_2;\quad r=6,\quad k=4,\quad s=2,$$

$$\rho_2\colon\ \frac{\partial}{\partial z_1},\quad e^{z_1}\frac{\partial}{\partial z_2}-e^{z_1}\frac{\partial}{\partial z_1},\quad e^{-z_1}\frac{\partial}{\partial z_2}+e^{-z_1}\frac{\partial}{\partial z_1},$$

$$-e^{z_1+z_2},\quad e^{z_2},\quad e^{-z_1+z_2};\quad r=6,\quad k=4,\quad s=2.$$

Every transitive effective realization of $\mathscr{T}_6$ by gd's in two complex variables is, to within a change of coordinates, an element of $\mathscr{A}(\tilde{\rho}_1)$ or $\mathscr{A}(\tilde{\rho}_2)$. The *type E* and *F* operators correspond to a selection of an element in each cohomology class of $\mathscr{A}(\tilde{\rho}_1)$ and $\mathscr{A}(\tilde{\rho}_2)$, respectively.