CHAPTER 2

The Schrödinger and Heat Equations

2.1 Separation of Variables for the Schrödinger Equation $(i\partial_t + \partial_{xx})\Psi(t,x)=0$

In the quantum-mechanical study of a nonrelativistic system in two-dimensional space-time, consisting of a particle (mass m) subject to a potential $V(x)$, it is postulated that the state of the system at time t is completely determined by a state function $\Psi(t,x)$ which is a solution of the *time-dependent Schrödinger equation*

$$i\hbar\partial_t\Psi = -\frac{\hbar^2}{2m}\partial_{xx}\Psi + V(x)\Psi, \tag{1.1}$$

where $\hbar = h/2\pi$ and h is Planck's constant [70]. (The constants $\hbar$ and $\hbar^2/2m$, although very important in physics, are simply a nuisance in this book, so we will henceforth choose units such that $\hbar=\hbar^2/2m=1$.) Among the most important Schrödinger equations are those for which the potential function $V(x)$ takes one of the forms in Table 5. For systems (1)–(4) the variable x ranges over the real line, while for (5)–(7) we assume x is nonnegative. (These latter equations arise from Schrödinger equations in higher-dimensional space-time which separate in polar or spherical coordinates. In (5)–(7) $x=r$, the radial coordinate [70].) Anderson *et al.* [3] and Boyer [18] have classified all Schrödinger equations (1.1) that admit nontrivial symmetry algebras. (Clearly all Schrödinger equations admit the two-dimensional complex symmetry algebra with basis ∂_t and $E=1$. By "nontrivial" we mean that the symmetry algebra is at least three dimensional.) They have shown that the only such equations are those with potentials (1)–(7). These potentials can be characterized in terms of symmetry groups.

ENCYCLOPEDIA OF MATHEMATICS and Its Applications, Gian-Carlo Rota (ed.).
Vol. 4: Willard Miller, Jr., Symmetry and Separation of Variables

Table 5 Potentials $V(x)$ with Nontrivial Symmetries

$V(x)$	Name of system
(1) 0	Free particle
(2) $kx^2,\quad k>0$	Harmonic oscillator
(3) $-kx^2,\quad k>0$	Repulsive oscillator
(4) $ax,\quad a\neq 0$	Free fall (linear potential)
(5) $a/x^2,\quad a\neq 0$	Radial free particle
(6) $a/x^2+kx^2,\quad a\neq 0,\ k>0$	Radial harmonic oscillator
(7) $a/x^2-kx^2,\quad a\neq 0,\ k>0$	Radial repulsive oscillator

In the next three sections we shall study these seven equations and uncover the surprising relations between them and the connection with separation of variables.

We write the free-particle Schrödinger equation in the form

$$Q\Psi=0,\qquad Q=i\partial_t+\partial_{xx}. \tag{1.2}$$

To compute the symmetry algebra of this equation, we follow the method described in Section 1.1. That is, we find all linear differential operators

$$L=a(t,x)\partial_x+b(t,x)\partial_t+c(t,x),$$

a, b, c, analytic in a suitable region $\mathfrak{D}$ of the x–t plane, such that $L\Psi$ satisfies (1.2) whenever Ψ, analytic in $\mathfrak{D}$, satisfies (1.2). A necessary and sufficient condition for L to belong to the symmetry algebra is

$$[L,Q]=R_L(t,x)Q \tag{1.3}$$

for some function R_L analytic in $\mathfrak{D}$. By equating coefficients of ∂_{xx}, ∂_t, ∂_x, and 1 on both sides of (1.3), we obtain a system of differential equations for a, b, c, and R. Details of the straightforward computation can be found in [3, 15, 18]. The final result is that the symmetry operators L form a six-dimensional complex Lie algebra $\mathcal{G}_2^c$ with basis

$$\begin{aligned}&K_2=-t^2\partial_t-tx\partial_x-t/2+ix^2/4, \qquad K_1=-t\partial_x+ix/2,\\ &K_0=i,\qquad K_{-1}=\partial_x,\qquad K_{-2}=\partial_t,\qquad K^0=x\partial_x+2t\partial_t+1/2\end{aligned} \tag{1.4}$$

and commutation relations

$$\begin{aligned}&[K^0,K_j]=jK_j\quad (j=\pm 2,\pm 1,0),\qquad [K_{-1},K_1]=\tfrac{1}{2}K_0,\\ &[K_{-1},K_2]=K_1,\qquad [K_{-2},K_1]=-K_{-1},\qquad [K_{-2},K_2]=-K^0.\end{aligned} \tag{1.5}$$

The reader should now be able to appreciate expression (1.3), since it has enabled us to compute symmetry operators for (1.2) that are not im-

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mediately obvious. Furthermore, some of the operators are not purely differential but involve multipliers. The geometrical significance of K_0, K_{-1}, and K_{-2} is obvious and K^0 is the generator for a dilatation symmetry $\Psi(t,x)\to\Psi(\alpha^2 t,\alpha x)$. However, K_1 is the generator of a Galilean transformation, not an obvious symmetry, and the geometrical interpretation of K_2 is unknown to the writer. Furthermore, K^0 and K_2 do not commute with Q even though they map solutions to solutions, so they correspond to operators L in (1.3) with $R_L\not\equiv 0$.

Since x and t are real variables and since we wish to exponentiate the symmetry operators (1.4) to obtain group symmetries, we restrict ourselves to the *real* six-dimensional Lie algebra $\mathcal{G}_2$ with basis (1.4). (Note that we cannot throw out the identity operator K_0, since K_0 occurs as the commutator $2[K_{-1},K_1]$.) A second useful basis for $\mathcal{G}_2$ is $\{C_j,L_k,E\}$ where

$$\begin{aligned} &C_1=K_{-1},\quad C_2=K_1,\quad L_3=K_{-2}-K_2,\\ &L_1=K^0,\quad L_2=K_{-2}+K_2,\quad E=K_0. \end{aligned}\tag{1.6}$$

The commutation relations become

$$\begin{aligned} &[L_1,L_2]=-2L_3, \qquad [L_3,L_1]=2L_2, \qquad [L_2,L_3]=2L_1,\\ &[C_1,C_2]=\tfrac{1}{2}E, \qquad [L_3,C_1]=C_2, \qquad [L_3,C_2]=-C_1,\\ &[L_2,C_1]=[C_2,L_1]=-C_2, \qquad [L_1,C_1]=[L_2,C_2]=-C_1 \end{aligned}\tag{1.7}$$

where E generates the center of $\mathcal{G}_2$.

To explain the structure of $\mathcal{G}_2$ we recall some facts about the group $SL(2,R)$ of all real 2×2 matrices A with determinant $+1$.

$$A=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix},\qquad \alpha\delta-\gamma\beta=1,\qquad \alpha,\beta,\gamma,\delta\in R. \tag{1.8}$$

As is well known [46, 82], the Lie algebra $sl(2,R)$ of $SL(2,R)$ consists of all 2×2 real matrices $\mathcal{A}$ with trace zero,

$$A=\begin{pmatrix}a & b\\ c & -a\end{pmatrix},\qquad a,b,c\in R. \tag{1.9}$$

This Lie algebra is three dimensional and the matrices

$$\mathcal{L}_1=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \mathcal{L}_2=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \mathcal{L}_3=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \tag{1.10}$$

form a basis with commutation relations

$$[\mathcal{L}_1,\mathcal{L}_2]=-2\mathcal{L}_3,\qquad [\mathcal{L}_3,\mathcal{L}_1]=2\mathcal{L}_2,\qquad [\mathcal{L}_2,\mathcal{L}_3]=2\mathcal{L}_1. \tag{1.11}$$

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It follows immediately that the symmetry operators L_k form a basis for a subalgebra of $\mathcal{G}_2$ isomorphic to $sl(2,R)$.

Furthermore, the operators C_1, C_2, E form a basis for a subalgebra of $\mathcal{G}_2$ isomorphic to the Weyl algebra $\mathcal{W}_1$. The Weyl group W_1 consists of all real 3×3 matrices

$$B(u,v,\rho)=\begin{bmatrix} 1 & v & 2\rho+uv/2 \\ 0 & 1 & u \\ 0 & 0 & 1 \end{bmatrix}, \qquad u,v,\rho\in R, \tag{1.12}$$

with group multiplication

$$B(u,v,\rho)B(u',v',\rho')=B(u+u',v+v',\rho+\rho'+(vu'-uv')/4). \tag{1.13}$$

The Lie algebra $\mathcal{W}_1$ has basis

$$\mathcal{C}_1=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathcal{C}_2=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathcal{E}=\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with commutation relations

$$[\mathcal{C}_1,\mathcal{C}_2]=\tfrac{1}{2}\mathcal{E}, \qquad [\mathcal{C}_k,\mathcal{E}]=0. \tag{1.14}$$

Using standard results from Lie theory (Theorem A.3), we can exponentiate the differential operators of $\mathcal{G}_2$ to obtain a local Lie group G_2 of symmetry operators. The action of the Weyl group W_1 is given by operators

$$\mathbf{T}(u,v,\rho)=\exp([\rho+uv/4]E)\exp(uC_2)\exp(vC_1)$$

where

$$\mathbf{T}(u,v,\rho)\Phi(t,x)=\exp[i\rho+i(uv+2ux-u^2t)/4]\Phi(t,x+v-ut) \tag{1.15}$$

and Φ belongs to the space $\mathcal{F}$ of analytic functions with domain $\mathcal{D}$. The group multiplication property is given by (1.13). The action of $SL(2,R)$ is given by

$$\mathbf{T}(A)\Phi(t,x)=\exp\left[i\left(\frac{x^2\beta/4}{\delta+t\beta}\right)\right](\delta+t\beta)^{-1/2}\Phi\left(\frac{\gamma+t\alpha}{\delta+t\beta},\frac{x}{\delta+t\beta}\right) \tag{1.16}$$

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where $A\in SL(2,R)$ is represented in the form (1.8). Here,

$$\mathbf{T}\begin{pmatrix}1&\beta\\0&1\end{pmatrix}=\exp(\beta K_2),\qquad \mathbf{T}\begin{pmatrix}1&0\\ \gamma&1\end{pmatrix}=\exp(\gamma K_{-2}),$$

$$\mathbf{T}\begin{pmatrix}e^{\alpha}&0\\0&e^{-\alpha}\end{pmatrix}=\exp(\alpha K^0),\qquad \mathbf{T}\begin{pmatrix}\cos\theta&-\sin\theta\\ \sin\theta&\cos\theta\end{pmatrix}=\exp(\theta L_3),\tag{1.17}$$

$$\mathbf{T}\begin{pmatrix}\cosh\varphi&\sinh\varphi\\ \sinh\varphi&\cosh\varphi\end{pmatrix}=\exp(\varphi L_2).$$

Now $SL(2,R)$ acts on W_1 via the adjoint representation

$$\mathbf{T}(A^{-1})\mathbf{T}(u,v,\rho)\mathbf{T}(A)=\mathbf{T}(u\delta+v\beta,u\gamma+v\alpha,\rho),\tag{1.18}$$

so the full symmetry group G_2, the *Schrödinger group* in two-dimensional space-time, is obtained as a semidirect product of $SL(2,R)$ and W_1 [18, 59]:

$$\begin{gathered} g=(A,\mathbf{w})\in G_2,\quad A\in SL(2,R),\\ \mathbf{w}=(u,v,\rho)\in W_1,\qquad \mathbf{T}(g)=\mathbf{T}(A)\mathbf{T}(\mathbf{w}),\\ \mathbf{T}(g)\mathbf{T}(g')=\mathbf{T}(AA')\{\mathbf{T}(A'^{-1})\mathbf{T}(\mathbf{w})\mathbf{T}(A')\}\mathbf{T}(\mathbf{w}')=\mathbf{T}(gg'). \end{gathered}\tag{1.19}$$

It follows from our general theory that $\mathbf{T}(g)$ maps solutions Ψ of (1.2) into solutions $\mathbf{T}(g)\Psi$. However, G_2 is only a local symmetry group, for not only do we have the domain problem in defining $\mathbf{T}(g)\Phi$, as discussed in Section 1.1, but also expression (1.16) makes no sense when $\delta+t\beta=0$. Expression (1.16) follows from the exponentiation of Lie derivatives only if $|t\beta/\delta|<1$. For $|t\beta/\delta|>1$ this expression still defines a symmetry, but one that is not directly obtainable from the symmetry algebra.

The Schrödinger group G_2 acts on the Lie algebra $\mathcal{G}_2$ of symmetry operators K via the adjoint representation

$$K\rightarrow K^{g}=\mathbf{T}(g)K\mathbf{T}(g^{-1})$$

and this action splits $\mathcal{G}_2$ into G_2 orbits. For our purposes the operator $K_0=i$, which generates the center $\{K_0\}$ of $\mathcal{G}_2$, is trivial, so we merely determine the orbit structure of the factor space $\mathcal{G}_2'=\mathcal{G}_2/\{K_0\}$. The results are as follows. Let

$$K=a_2K_2+a_1K_1+a_0K^0+a_{-1}K_{-1}+a_{-2}K_{-2}$$

be a nonzero element of $\mathcal{G}_2'$ and set $\alpha=a_2a_{-2}+a_0^2$. It is straightforward to show that α is invariant under the adjoint representation. In Table 6 we give a complete set of orbit representatives. That is, K lies on the same G_2

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orbit as a real multiple of exactly one of the five operators in the list.

$$\begin{aligned} &\text{Case 1} \quad (\alpha<0) \quad K_{-2}-K_2=L_3; \\ &\text{Case 2} \quad (\alpha>0) \quad K^0; \\ &\text{Case 3} \quad (\alpha=0) \quad K_2+K_{-1}, K_{-2}, K_{-1}. \end{aligned} \tag{1.20}$$

Note that there are five orbits.

Since K_{-2} and K_{-1} commute, they can be simultaneously diagonalized. Furthermore, $K_{-2}\Psi=iK^2_{-1}\Psi$ for all solutions Ψ of $Q\Psi=0$. Thus we associate the same coordinate system $\{t,x\}$ with both of these orbits and end up with only four separable coordinate systems.

One can also compute the second-order symmetries of $Q\Psi=0$ and show that the free-particle Schrödinger equation is class I. However, all separable coordinate systems for the equation turn out to be associated with orbits of first-order symmetries. This is related to the fact that the Schrödinger equation is only first order in the variable t.

For this equation it is useful (and necessary) to consider R-separable solutions. To explain this concept we choose a nonzero analytic function $R(t,x)=\exp(i\mathfrak{R}(t,x))$ and write $\Psi=R\Phi$ where Ψ satisfies the Schrödinger equation $Q\Psi=0$. Writing the differential equation in terms of Φ, we find $Q'\Phi=0$ where $Q'=R^{-1}QR$ is the transformed differential operator. Now suppose the new equation $Q'\Phi=0$ admits separable solutions $\Phi_\lambda=U_\lambda(u)V_\lambda(v)$ in terms of a $\{u,v\}$ coordinate system. If $R=a(u)b(v)$—that is, if R factors in the $\{u,v\}$ coordinates—then $\Psi_\lambda=a(u)U_\lambda(u)b(v)V_\lambda(v)$ is a separable solution of $Q\Psi_\lambda=0$ and we have obtained nothing new. However, if $R(u,v)$ does not factor, then we have obtained a new family of R-separable solutions $\Psi_\lambda=\exp(iR(u,v))U_\lambda(u)V_\lambda(v)$. Thus R-separability is a generalization of ordinary separability. R-separable solutions of one equation $Q\Psi=0$ correspond to ordinary separable solutions of an equivalent equation $Q'\Phi=0$, $Q'=R^{-1}QR$.

We have not introduced the notion of R-separability earlier because the equations studied in Chapter 1 admit no R-separable solutions that are not already separable in the ordinary sense. However, the situation changes for the Schrödinger operators. The existence of R-separable solutions is clearly related to the existence of symmetry operators K that do not commute with Q, even though they map solutions into solutions.

In [59], Kalnins and the author have computed all coordinate systems that permit R-separation of variables for equation (1.2) and have shown that the associated R-separated solutions $\Psi_\lambda=\exp(iR(u,v))U_\lambda(u)V_\lambda(v)$ can be characterized as eigenfunctions of some $K\in\mathcal{G}_2$, $K\Psi_\lambda=i\lambda\Psi_\lambda$, $Q\Phi_\lambda=0$. The association between orbits in $\mathcal{G}_2'$ and separable coordinates is given in Table 6.

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Table 6 R-Separable Coordinates for the Equation $(i\partial_t + \partial_{xx})\Psi(t,x)=0$

Operator	Coordinates $\{u,v\}$	Multiplier $R=e^{i\mathfrak{R}}$	Separated Solutions
1 K_{-1}, K_{-2}	$x=u$	$\mathfrak{R}=0$	Product of exponentials
2a $K_{-2}-K_1$	$x=u+v^2/2$	$\mathfrak{R}=uv/2$	Airy times exponential function
2b K_2+K_{-1}	$x=uv+1/2v$	$\mathfrak{R}=(u^2v-u/v)/4$	Airy times exponential functions
3a K^0	$x=u\sqrt{v}$	$\mathfrak{R}=0$	Parabolic cylinder times exponential function
3b K_2+K_{-2}	$x=u\lvert 1-v^2\rvert^{1/2}$	$\mathfrak{R}=\pm u^2v/4$ $(+ \textit{if} \lvert v\rvert>1$, $-\textit{if} \lvert v\rvert<1)$	Parabolic cylinder times exponential function
4 K_2-K_{-2}	$x=u(1+v^2)^{1/2}$	$\mathfrak{R}=u^2v/4$	Hermite times exponential function

For all coordinate systems $\{u,v\}$ in Table 6, $v=t$. As stated earlier, there are only four types of separable coordinates and these are associated with four nontrivial G_2 orbits in $\mathcal{G}_2'$. (Here we are identifying the two orbits with commuting representatives K_{-1}, K_{-2}.) However, the table contains six entries, and each of the six separable systems appears to be distinct from the rest. The explanation for this relates to our definition of equivalent coordinate systems. We regard two systems as equivalent if one system can be mapped into the other by a G_2 transformation $\mathbf{T}(g)$. However, such transformations, particularly (1.16), can sometimes have a rather complicated form, so that two equivalent systems will appear very different. Since the operator K_2 has a rather obscure physical significance, it is difficult to interpret the physical or geometrical relationship between two systems that are related by the exponential of this operator.

However, there is a five-parameter subgroup of G_2 whose physical significance is well understood [73]. This is the Galilei group plus dilations with Lie algebra basis $\{K_{-2}, K_{\pm 1}, K_0, K^0\}$. If we regard systems from the point of view of equivalence under the Galilei group plus dilatations, we find that G_2 orbits 2 and 3 each split into two Galilean–dilatation orbits. This accounts for the six systems listed in Table 6. (However, the classification is based more on significance for separation of variables than accuracy for Galilean–dilatation orbits. Indeed, $2a$ splits into two Galilean–dilatation orbits $K_{-2}\pm K_1$ and 2b splits into $K_2 \pm K_{-1}$. These subcases yield coordinates that differ only in the sign of a parameter, and we choose not to distinguish between them.)

We can describe the equivalences on orbits 2 and 3 in terms of the operator $J=\exp[(\pi/4)(K_2-K_{-2})]=\exp(-(\pi/4)L_3)$.

$$J\Phi(t,x)=\frac{2^{1/4}}{(1+t)^{1/2}}\exp\left(\frac{ix^2/4}{1+t}\right)\Phi\left(\frac{t-1}{t+1},\frac{x\sqrt{2}}{t+1}\right) \qquad (1.21)$$

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Note that $J^2=\exp[(\pi/2)(K_2-K_{-2})]=\exp((-\pi/2)L_3)$, and

$$J^2\Phi(t,x)=\frac{\exp(ix^2/4t)}{\sqrt{t}}\Phi\left(\frac{-1}{t},\frac{x}{t}\right),\qquad J^8\Phi=-\Phi,\qquad J^{16}\Phi=\Phi. \quad (1.22)$$

A direct computation yields

$$J(K_{-2}+K_2)J^{-1}=K^0,\qquad J^2(K_1-K_{-2})J^{-2}=K_{-1}+K_2, \quad (1.23)$$

which proves the G_2 equivalence of systems 2a, 2b and 3a, 3b.

We now show that the operators (1.4) can be interpreted as a Lie algebra of skew-Hermitian operators on the Hilbert space $L_2(R)$ of complex-valued Lebesgue square-integrable functions on the real line (Chapter 1, (5.2)). To do this we consider the operators (1.4) formally restricted to the solution space of (1.2). Then we can replace ∂_t by $i\,\partial_{xx}$ in these expressions and consider t as a fixed parameter. It is easy to show that the resulting operators restricted to the domain $\mathfrak{D}\subset L_2(R)$ of infinitely differentiable functions with compact support are skew-symmetric. Moreover, each of these operators, when multiplied by i, has a unique self-adjoint extension. Indeed, the operators (1.4) are real linear combinations of

$$\begin{gathered}\mathcal{K}_2=ix^2/4,\qquad \mathcal{K}_1=ix/2,\\ \mathcal{K}_{-1}=\partial_x,\qquad \mathcal{K}_{-2}=i\,\partial_{xx},\qquad \mathcal{K}_0=i,\qquad \mathcal{K}^0=x\,\partial_x+\tfrac{1}{2},\end{gathered} \quad (1.24)$$

and $i\mathcal{K}_j, i\mathcal{K}^0$ have unique self-adjoint extensions. When the parameter t is set equal to zero, K_j becomes $\mathcal{K}_j$ and K^0 becomes $\mathcal{K}^0$. It follows that the script operators also satisfy commutation relations (1.5).

From spectral theory [111, Chapter VIII], we know that to each skew-Hermitian $\mathcal{K}\in\mathcal{G}_2$ there corresponds a one-parameter group $U(\alpha)=\exp(\alpha\mathcal{K})$ of unitary operators on $L_2(R)$. This group in turn acts on $\mathcal{G}_2$ via $\mathcal{K}\to U(\alpha)\mathcal{K}U(-\alpha)$. In particular, the following result is of importance in quantum mechanics:

$$\exp(a\mathcal{K}_{-2})f(x)=\text{l.i.m.}(4\pi ia)^{-1/2}\int_{-\infty}^{\infty}\exp\left[-(x-y)^2/4ia\right]f(y)\,dy, \quad (1.25)$$

$$f\in L_2(R), a\neq 0.$$

(Here $(ia)^{1/2}=e^{i\pi/4}|a|^{1/2}$ for $a>0$ and $e^{-i\pi/4}|a|^{1/2}$ for $a<0$. See [67, p. 493] for a proof of (1.25).) We can verify that

$$\begin{gathered}\exp(t\mathcal{K}_{-2})\mathcal{K}_j\exp(-t\mathcal{K}_{-2})=K_j,\\ \exp(t\mathcal{K}_{-2})\mathcal{K}^0\exp(-t\mathcal{K}_{-2})=K^0.\end{gathered} \quad (1.26)$$

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(A formal proof is easily obtained from the commutation relations (1.5), although a rigorous proof specifying domains is somewhat more difficult.)

Now if $f \in L_2(R)$ and f belongs to the domain of the self-adjoint operator $\mathcal{K}_{-2}$, then $\Psi(t,x) = \exp(t\mathcal{K}_{-2})f(x)$ satisfies $\partial_t\Psi = \mathcal{K}_{-2}\Psi$ or $i\partial_t\Psi = -\partial_{xx}\Psi$ for almost every t, and $\Psi(0,x) = f(x)$. We see that $\exp(t\mathcal{K}_{-2})$ is the operator of time translation in quantum mechanics [70, 109]. Moreover, this operator is unitary, so

$$\int_{-\infty}^{\infty} \Psi_1(t,x)\overline{\Psi}_2(t,x)\,dx = \langle \exp(t\mathcal{K}_{-2})f_1, \exp(t\mathcal{K}_{-2})f_2\rangle$$

$$= \langle f_1, f_2\rangle = \int_{-\infty}^{\infty} f_1(x)\bar{f}_2(x)\,dx, \qquad (1.27)$$

independent of t. We have introduced a Hilbert space structure on the solutions of (1.2) that agrees exactly with the usual Hilbert space of states corresponding to a free-particle system. Moreover, the mappings (1.26) relating the time-zero (script) operators to the time-t (italic) operators are the usual transformations relating the Heisenberg and Schrödinger pictures in quantum theory [109, 118]. It is also easy to show that the unitary operators $\exp(\alpha K) = \exp(t\mathcal{K}_{-2})\exp(\alpha\mathcal{K})\exp(-t\mathcal{K}_{-2})$ map a Hilbert space solution Ψ of (1.2) into $\Phi = \exp(\alpha K)\Psi$, which also satisfies (1.2). Thus the unitary operators $\exp(\alpha K)$ are symmetries of (1.2).

Later we will see that the $\mathcal{K}$ operators generate a global unitary representation of a covering group $\tilde{G}_2$ of G_2, although not of G_2 itself. Assuming this for the moment, let $\mathbf{U}(g), g \in \tilde{G}_2$, be the corresponding unitary operators, and set $\mathbf{T}(g) = \exp(t\mathcal{K}_{-2})\mathbf{U}(g)\exp(-t\mathcal{K}_{-2})$. Again it is easy to demonstrate that the $\mathbf{T}(g)$ are unitary symmetries of (1.2) and that the associated infinitesimal operators are $K = \exp(t\mathcal{K}_{-2})\mathcal{K}\exp(-t\mathcal{K}_{-2})$.

Now consider the operator $\mathcal{L}_3 = \mathcal{K}_{-2} - \mathcal{K}_2 = i\partial_{xx} - ix^2/4 \in \mathcal{G}_2$. If $f \in L_2(R)$, then $\Psi(t,x) = \exp(t\mathcal{L}_3)f(x)$ satisfies $\partial_t\Psi = \mathcal{L}_3\Psi$ or $i\partial_t\Psi = -\partial_{xx}\Psi + x^2\Psi/4$ and $\Psi(0,x) = f(x)$. Similarly, the unitary operators $\mathbf{V}(g) = \exp(t\mathcal{L}_3)\mathbf{U}(g)\exp(-t\mathcal{L}_3)$ are symmetries of this equation, the Schrödinger equation for the harmonic oscillator, (2) in Table 5. (Here we have normalized k to the value $\frac{1}{4}$.) One can verify that the associated infinitesimal operators $\exp(t\mathcal{L}_3)\mathcal{K}\exp(-t\mathcal{L}_3)$ can be expressed as first-order differential operators in x and t. (In particular these operators will be real linear combinations of the basis operators (1.24) with coefficients that depend on t. Considering these operators as acting on the solution space of the harmonic oscillator Schrödinger equation, we can replace $i\partial_{xx}$ by $\partial_t + ix^2/4$ wherever it occurs.) Conversely, if K' is a first-order symmetry operator for the time-dependent harmonic oscillator Schrödinger equation, we can show that at time $t=0$, K' reduces to a real linear combination of the operators (1.24). It follows that the symmetry algebras of the equations with potentials (1) and (2) in Table 5 are both isomorphic to $\mathcal{G}_2$ with basis

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(1.24). For the free-particle equation the symmetries are $K = \exp(t\mathcal{K}_{-2})\mathcal{K}\exp(-t\mathcal{K}_{-2})$, while for the harmonic oscillator equation the symmetries are $K' = \exp(t\mathcal{L}_3)\mathcal{K}\exp(-t\mathcal{L}_3)$. In each case the $\mathcal{K}$ operators are identical. Moreover, for fixed $\mathcal{K}$, the operators K and K' are unitary equivalent, $K' = A(t)KA(t)^{-1}$, although the unitary operator $A(t) = \exp(t\mathcal{L}_3)\exp(-t\mathcal{K}_{-2})$ depends on t.

Continuing in this manner, we consider the operator $\mathcal{L}_2 = \mathcal{K}_{-2} + \mathcal{K}_2 = i\partial_{xx} - ix^2/4 \in \mathcal{G}_2$. If $f \in L_2(R)$, then $\Psi(t,x) = \exp(t\mathcal{L}_2)f(x)$ satisfies $\partial_t\Psi = \mathcal{L}_2\Psi$ or $i\partial_t\Psi = -\partial_{xx}\Psi - x^2\Psi/4$ and $\Psi(0,x) = f(x)$. The operators $\mathbf{W}(g) = \exp(t\mathcal{L}_2)\mathbf{U}(g)\exp(-t\mathcal{L}_2)$ form the unitary symmetry group of this equation, repulsive harmonic oscillator potential ((3) in Table 5), and the associated infinitesimal operators $\exp(t\mathcal{L}_2)\mathcal{K}\exp(-t\mathcal{L}_2)$ are first order in x and t. Finally, we consider the operator $\mathcal{W} = \mathcal{K}_{-2} - \mathcal{K}_1 = i\partial_{xx} - ix/2 \in \mathcal{G}_2$. If $f \in L_2(R)$, then $\Psi(t,x) = \exp(t\mathcal{W})f(x)$ satisfies $\partial_t\Psi = \mathcal{W}\Psi$ or $i\partial_t\Psi = -\partial_{xx}\Psi + x\Psi/2$ and $\Psi(0,x) = f(x)$. The unitary operators $\mathbf{X}(g) = \exp(t\mathcal{W})\mathbf{U}(g)\exp(-t\mathcal{W})$ are symmetries of this Schrödinger equation corresponding to a linear potential and the infinitesimal operators $\exp(t\mathcal{W})\mathcal{K}\exp(-t\mathcal{W})$ are first order in x and t.

Note further from (1.20) that the operators $\mathcal{K}_{-2}$, $\mathcal{L}_3$, $\mathcal{L}_2$, $\mathcal{K}_{-2} - \mathcal{K}_1$ corresponding to the free-particle, attractive and repulsive harmonic oscillator, and linear potential Hamiltonians lie on the same G_2 orbits as the four representatives $\mathcal{K}_{-2}$, $\mathcal{L}_3$, $\mathcal{K}^0$, and $\mathcal{K}_2 + \mathcal{K}_{-1}$, respectively. Thus, these four Hamiltonians correspond exactly to the four systems of coordinates in which equation (1.2) separates. We see that these Hamiltonians form a complete set of orbit representatives in $\mathcal{G}_2$.

It now follows that the Schrödinger equations (1)–(4) in Table 5 have isomorphic symmetry algebras. In each case if we compute the symmetry operators at time $t=0$, we obtain the Lie algebra $\mathcal{G}_2$ with basis (1.24). Although we first obtained this symmetry algebra through a study of the Schrödinger equation (1), we could equally have obtained it by studying (2), (3), or (4). Moreover, we see from the preceding paragraphs how to construct the (time-dependent) unitary operators on $L_2(R)$ that map a solution of any one of these equations to a solution of another equation. The four equations can and should be studied as a unit.

The connection between orbits and separation of variables can now be made clear. Suppose $\Psi(t,x)$ is a solution of the free-particle equation

$$i\partial_t\Psi = -\partial_{xx}\Psi. \tag{1.28}$$

This equation clearly separates in the variables $\{t,x\}$ and these variables are "naturally" associated with (1.28). Now we have seen that the operator $A(t) = \exp(t\mathcal{L}_3)\exp(-t\mathcal{K}_{-2}) = \exp(-t\mathcal{K}_{-2})\exp(tL_3)$ maps Ψ to a solution $\Phi(t,x) = A(t)\Psi(t,x)$ of the harmonic oscillator equation

$$i\partial_t\Phi = -\partial_{xx}\Phi - x^2\Phi/4. \tag{1.29}$$

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Explicitly,

$$\Phi(t,x)=(\cos t)^{-1/2}\exp(-ix^2\tan(t)/4)\Psi(\tan t, x/\cos t).$$

Now equation (1.29) "naturally" separates in the variables $\{t,x\}$, so we can find solutions Ψ of (1.28) in the form

$$\Psi(t,x)=(1+v^2)^{-1/4}\exp(iu^2v/4)\Phi(\tan^{-1}v,u), \qquad x=u(1+v^2)^{1/2}, t=v. \tag{1.30}$$

Since (1.29) separates in $\{\tan^{-1}v,u\}$, hence $\{v,u\}$, it follows that equation (1.28) R-separates in the coordinates $\{v,u\}$ where the multiplier $R=e^{i\mathcal{R}}$ is given by $\mathcal{R}=iu^2v/4$. (The factor $(1+v^2)^{-1/4}$ can be absorbed in the separated solution.) Thus we have explained the existence of the coordinates 4 in Table 6, associated with the operator $\mathcal{K}_{-2}-\mathcal{K}_2$. In a similar manner we can associate a "natural" coordinate system with each of our four Hamiltonians, thus exhausting the possible R-separable coordinate systems inequivalent with respect to G_2.

Note that if two operators lie on the same G_2 orbit, then the first operator is unitary equivalent to a real constant times the second operator. Thus two suitable normalized operators on the same orbit have the same spectrum. In particular, if $\mathcal{K},\mathcal{K}'\in\mathcal{G}_2$ with $\mathcal{K}'=\mathbf{U}(g)\mathcal{K}\mathbf{U}(g^{-1})$ and the self-adjoint operator $i\mathcal{K}$ has a complete set of (possibly generalized) eigenvectors $f_\lambda(x)$ with

$$i\mathcal{K}f_\lambda=\lambda f_\lambda, \qquad \langle f_\lambda,f_\mu\rangle=\delta_{\lambda,\mu} \tag{1.31}$$

where

$$\langle h_1,h_2\rangle=\int_{-\infty}^{\infty}h_1(x)\bar{h}_2(x)\,dx, \qquad h_j\in L_2(R), \tag{1.32}$$

then for $f'_\lambda=\mathbf{U}(g)f_\lambda$ we have

$$i\mathcal{K}'f'_\lambda=\lambda f'_\lambda, \qquad \langle f'_\lambda,f'_\mu\rangle=\delta_{\lambda,\mu} \tag{1.33}$$

and the f'_λ form a complete set of eigenvectors for $i\mathcal{K}'$ [77]. These remarks imply that if we wish to compute the spectrum corresponding to each operator $\mathcal{K}\in\mathcal{G}_2$, it is enough to determine the spectra of the four Hamiltonians listed earlier. Moreover, we may be able to choose another operator $\mathcal{K}$ on the same G_2 orbit as a given Hamiltonian such that the spectral decomposition of $\mathcal{K}$ is especially easy. The spectral decomposition of the Hamiltonian and the corresponding eigenfunction expansions then follow from those of $\mathcal{K}$ by application of a group operator $\mathbf{U}(g)$.

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As a special case of these remarks consider the operator $\mathcal{H}_{-2}=i\partial_{xx}$. If $\{f_\lambda\}$ is the basis of generalized eigenvectors for some operator $\mathcal{H}\in\mathcal{G}_2$, then $\{\Psi_\lambda=\exp(t\mathcal{H}_{-2})f_\lambda\}$ is the basis of generalized eigenvectors for $K=\exp(t\mathcal{H}_{-2})\mathcal{H}\exp(-t\mathcal{H}_{-2})$ and the Ψ_λ satisfy the free-particle Schrödinger equation (1.28). Similar remarks hold for the other Hamiltonians.

We begin our explicit computations by determining the spectral resolution of the operator $\mathcal{L}_3=\mathcal{H}_{-2}-\mathcal{H}_2$. The results are well known [141]. The eigenfunction equation is

$$i\mathcal{L}_3 f=\lambda f, \qquad (-\partial_{xx}+x^2/4)f=\lambda f,$$

and the normalized eigenfunctions are

$$f_n^{(4)}(x)=[n!(2\pi)^{1/2}2^n]^{-1/2}\exp(-x^2/4)H_n(x/\sqrt{2}\,), \qquad \lambda_n=n+\tfrac{1}{2}, \tag{1.34}$$
$$n=0,1,2,\ldots,\langle f_n^{(4)},f_m^{(4)}\rangle=\delta_{nm},$$

where $H_n(x)$ is a Hermite polynomial, (B.12). The $\{f_n^{(4)}\}$ form an ON basis for $L_2(R)$.

From (1.34) we see that

$$\exp(2\pi\mathcal{L}_3)f_n^{(4)}=\exp\left[-2\pi i\left(n+\tfrac{1}{2}\right)\right]f_n^{(4)}=-f_n^{(4)}$$

so $\exp(2\pi\mathcal{L}_3)=-E$ where E is the identity operator on $L_2(R)$. However, from (1.17) if the operators $\exp(\alpha\mathcal{H})$ generate a global unitary representation of G_2 on $L_2(R)$, we should have $\exp(2\pi\mathcal{L}_3)=E$. In fact, it can be shown that the $\mathcal{H}$ operators exponentiate to a global irreducible representation of the simply connected covering group $\tilde{G}_2$ of G_2.

To describe this covering group we first consider the topology of the group manifold $SL(2,R)$, (1.8).

$$A=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in SL(2,R), \qquad \alpha\delta-\beta\gamma=1.$$

Setting

$$2a=(\alpha+\delta)+i(\gamma-\beta), \qquad 2b=(-\alpha+\delta)+i(\gamma+\beta). \tag{1.35}$$

we see that the complex numbers a,b satisfy the identity

$$|a|^2-|b|^2=1. \tag{1.36}$$

Conversely, if $a=a_1+ia_2$, $b=b_1+ib_2$, and a,b satisfy (1.36), then relations (1.35) can be uniquely inverted to yield an $A\in SL(2,R)$ with parameters $\alpha=a_1-b_1$, $\beta=-a_2+b_2$, $\gamma=a_2+b_2$, $\delta=a_1+b_1$. It follows from (1.36) that

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topologically $SL(2,R)$ can be identified with the hyperboloid

$$a_1^2+a_2^2-b_1^2-b_2^2=1.$$

Another parametrization of $SL(2,R)$ is due to Bargmann [10]. He sets

$$\mu=b/a, \qquad \omega=\arg a, \qquad -\pi<\omega\leqslant\pi (\mathrm{mod}\, 2\pi). \tag{1.37}$$

It follows from (1.36) that $|\mu|<1$. Furthermore,

$$a=e^{i\omega}(1-|\mu|^2)^{-1/2}, \qquad b=e^{i\omega}\mu(1-|\mu|^2)^{-1/2}. \tag{1.38}$$

We can now write $A\equiv(\mu,\omega)$, $|\mu|<1$, $-\pi<\omega\leqslant\pi$, and parametrize $SL(2,R)$ in terms of μ and ω. The group product can be expressed as follows. If $A=(\mu,\omega)$, $A'=(\mu',\omega')$, then $AA'=(\mu'',\omega'')$ where

$$\begin{aligned}\mu''&=(\mu+\mu' e^{-2i\omega})(1+\bar{\mu}\mu' e^{-2i\omega})^{-1},\\ \omega''&=\omega+\omega'+(1/2i)\ln\left[(1+\bar{\mu}\mu' e^{-2i\omega})(1+\mu\bar{\mu}' e^{2i\omega})^{-1}\right],\end{aligned} \tag{1.39}$$

$\ln z$ is defined by its principal value ($\ln re^{i\theta}=\ln r+i\theta$, $r>0$, $-\pi<\theta\leqslant\pi$), and ω' is defined mod 2π. It is easy to check that μ,ω are appropriate Lie group parameters [115]. We now have a topological characterization of $SL(2,R)$ as the product of the open unit disk $|\mu|<1$ and the circle $-\pi<\omega\leqslant\pi$, mod 2π.

The universal covering group $\widetilde{SL}(2,R)$ of $SL(2,R)$ is the Lie group with elements

$$\widetilde{SL}(2,R)=\{\{\mu,\omega\}: |\mu|<1, -\infty<\omega<\infty\}.$$

Here, distinct values of ω correspond to distinct group elements. Group multiplication is defined by (1.39) except that ω'' is no longer defined mod 2π. There is a homomorphism of $\widetilde{SL}(2,R)$ onto $SL(2,R)$ given by $\{\mu,\omega\}\to(\mu,\omega)$ and the elements $\{0,2\pi n\}$, $n=0,\pm1,\pm2,\ldots,$of $\widetilde{SL}(2,R)$ are exactly those which map onto the identity element $(0,0)$ of $SL(2,R)$.

Finally, it is easy to verify that any element of $\widetilde{SL}(2,R)$ can be factored in the form

$$\{\mu,\omega\}=\{0,-\theta/2\}\{r,0\}\{0,\omega+\theta/2\}, \qquad \mu=re^{i\theta}, \tag{1.40}$$

and if $r>0$, $-\pi<\theta\leqslant\pi$, this factorization is unique.

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It is now straightforward to show that the $\mathcal{H}$ operators exponentiate to a global unitary irreducible representation of the simply connected covering group $\tilde{G}_2$ of G_2. Indeed, from the known recurrence formulas for the Hermite polynomials one can check that the operators $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ acting on the $f^{(4)}$ basis define a reducible representation of $sl(2,R)$ belonging to the discrete series. We will work out these recurrence formulas in Section 2.2.) The value of the Casimir operator is $\frac{1}{4}(\mathcal{L}_1{}^2+\mathcal{L}_2{}^2-\mathcal{L}_3{}^2)=-\frac{3}{16}$. As first shown by Bargmann ([10]; see also [115]), this Lie algebra representation extends to a global unitary reducible representation of $\widetilde{SL}(2,R)$. Similarly, the operators $\mathcal{C}_1,\mathcal{C}_2,\mathcal{L}_3$ acting on the $f^{(4)}$ basis define the irreducible representation $(\lambda,l)=(-\frac{1}{2},1)$ of the Lie algebra of the harmonic oscillator group S [82]. Again this Lie algebra representation is known to generate a global unitary irreducible representation of S [80, 86]. Since from (1.40) we see that every operator from $\widetilde{SL}(2,R)$ can be written in the form $\exp(-(\theta/2)\mathcal{L}_3)\exp(-\tau\mathcal{L}_1)\exp\{[(\theta/2)+\omega]\mathcal{L}_3\}$ with $2\tau=\ln[(1+r)/(1-r)]$ where $\exp(\theta\mathcal{L}_3)$ also belongs to S, and since $\mathcal{L}$ is a first-order operator whose exponential is easily determined, we can check that the identity (1.18) holds in general. (That is, we replace $\mathbf{T}(A)$ by $\exp(-(\theta/2)\mathcal{L}_3)\exp(-\tau\mathcal{L}_1)\exp\{[(\theta/2)+\omega]\mathcal{L}_3\}$ and use (1.35), (1.37), (1.38), (1.40) to express $\alpha,\beta,\gamma,\delta$ in terms of θ,τ,ω on the right-hand side of (1.18).) Then expressions (1.19) define $\tilde{G}_2$ as a semidirect product of $\widetilde{SL}(2,R)$ and W_1. Thus our representation of $\mathcal{G}_2$ extends to a global unitary representation $\mathbf{U}$ of $\tilde{G}_2$ that is irreducible since $\mathbf{U}|S$ is already irreducible.

The unitary operators $\mathbf{U}(g)$ on $L_2(R)$ are easily computed. The operators

$$\mathbf{U}(u,v,\rho)=\exp([\rho+uv/4]\mathcal{E})\exp(u\mathcal{C}_2)\exp(v\mathcal{C}_1)$$

defining a representation of W_1 take the form

$$\mathbf{U}(u,v,\rho)h(x)=\exp[i(\rho+uv/4+ux/2)]h(x+v), \qquad h\in L_2(R). \tag{1.41}$$

The operators $\mathbf{U}\{\mu,\omega\}$, $\{\mu,\omega\}\in\widetilde{SL}(2,R)$, are more complicated. Here $\exp(a\mathcal{H}_{-2})$ is given by (1.25) and it is elementary to show

$$\exp(b\mathcal{H}^0)h(x)=\exp(b/2)h(e^b x), \qquad \exp(c\mathcal{H}_2)h(x)=\exp(icx^2/4)h(x). \tag{1.42}$$

Relations (1.17), (1.39) imply

$$\exp(\varphi\mathcal{L}_2)=\exp(\tanh(\varphi)\mathcal{H}_2)\exp(\sinh(\varphi)\cosh(\varphi)\mathcal{H}_{-2})\exp(-\ln\cosh(\varphi)\mathcal{H}^0),$$

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so (1.25) and (1.42) yield

$$\exp(\varphi\mathcal{L}_2)h(x)=\frac{\exp\left[(ix^2/4)\tanh\varphi\right]}{(4\pi i\sinh\varphi)^{1/2}}\text{l.i.m.}\int_{-\infty}^{\infty}\exp\left[\frac{-(x-y\cosh\varphi)^2}{4i\sinh\varphi\cosh\varphi}\right]h(y)\,dy, \tag{1.43}$$

$$\varphi\neq 0.$$

A similar computation for $\exp(\theta\mathcal{L}_3)$ gives

$$\exp(\theta\mathcal{L}_3)h(x)=\frac{\exp\left[(ix^2/4)\cot\theta\right]}{(4\pi i\sin\theta)^{1/2}}\text{l.i.m.}\int_{-\infty}^{\infty}\exp\left[\frac{-(y^2\cos\theta-2xy)}{4i\sin\theta}\right]h(y)\,dy, \tag{1.44a}$$

$$0<|\theta|<\pi,$$

$$\exp(2\pi\mathcal{L}_3)h(x)=-h(x). \tag{1.44b}$$

The general group operator $\mathbf{U}(g)$ can be obtained from these results.

From (1.25) we see that the ON basis functions $f_n^{(4)}(x)$ map to the ON basis functions $F_n^{(4)}(t,x)=\exp(t\mathcal{K}_{-2})f_n^{(4)}(x)$ or

$$F_n^{(4)}(t,x)=\left\{n!2^n\left[2\pi(1+t^2)\right]^{1/2}\right\}^{-1/2}\exp\left(\frac{i}{4}\frac{x^2t}{1+t^2}-\frac{x^2}{4(1+t^2)}-i\left(n+\frac{1}{2}\right)\tan^{-1}t\right)H_n\left\{x/\left[2(1+t^2)\right]^{1/2}\right\},\quad n=0,1,2,\ldots, \tag{1.45}$$

which are solutions of (1.28). This result can be derived from (1.30) or 4, Table 6. Indeed we know that variables $\{u,v\}$ R-separate in the integral expression for (1.45) where $u=x/(1+t^2)^{1/2}$, $v=t$, and $\Re=iu^2v/4$. Applying the standard methods discussed in Chapter 1, we obtain expressions (1.45).

Next we study the spectral theory for the orbit containing the operators $\mathcal{K}_{-2}+\mathcal{K}_2$ (repulsive oscillator) and $\mathcal{K}^0$. Since the spectral analysis for $\mathcal{K}^0$ is more elementary, we study it first. (The corresponding results for

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$\mathcal{K}_{-2}+\mathcal{K}_2$ will then follow by application of the unitary operator $\mathcal{J} = \exp((-\pi/4)\mathcal{L}_3)$, (1.21), (1.23).) The eigenfunction equation is

$$i\mathcal{K}^0 f=\lambda f, \qquad \mathcal{K}^0 = x\,\partial_x+\tfrac{1}{2}.$$

The spectral resolution for this operator is well known [128]. It is obtained by considering $L_2(R)$ as the direct sum $L_2(R-)\oplus L_2(R+)$ of square-integrable functions on the negative and positive reals, respectively, and taking the Mellin transform of each component. Then $i\mathcal{K}^0$ transforms into multiplication by the transform variable. The spectrum is continuous and covers the real axis with multiplicity two. The generalized eigenfunctions are

$$\begin{aligned} &f_\lambda^{(3)\pm}(x)=(2\pi)^{-1/2}x_{\pm}^{-i\lambda-\frac{1}{2}}, \qquad -\infty<\lambda<\infty,\\ &\langle f_\lambda^{(3)\pm},f_\mu^{(3)\pm}\rangle=\delta(\lambda-\mu), \qquad \langle f_\lambda^{(3)\pm},f_\mu^{(3)\mp}\rangle=0, \end{aligned} \tag{1.46}$$

where

$$x_+^\alpha=\begin{cases} x^\alpha & \text{if } x>0,\\ 0 & \text{if } x<0;\end{cases} \qquad x_-^\alpha=\begin{cases} 0 & \text{if } x>0,\\ (-x)^\alpha & \text{if } x<0.\end{cases}$$

From (1.25) we find $F_\lambda^{(3)\pm}(t,x)=\exp(t\mathcal{K}_{-2})f_\lambda^{(3)\pm}(x)$ where

$$\begin{aligned} F_\lambda^{(3)\pm}(t,x)=\exp\left(-\frac{x^2}{8it}+\frac{\pi\lambda}{4}+\frac{i\pi}{8}\right)&\frac{(2t)^{-i\lambda/2+\frac{1}{4}}}{(8\pi^2 it)^{1/2}}\Gamma\left(\frac{1}{2}-i\lambda\right)\\ &\times D_{i\lambda-\frac{1}{2}}\left(\mp\frac{xe^{-i\pi/4}}{(2t)^{1/2}}\right), \qquad t>0, \end{aligned} \tag{1.47}$$

$\Gamma(z)$ is a gamma function (Appendix B, Section 1) and $D_\nu(z)$ is a parabolic cylinder function (Appendix B, Section 4). (This follows from (1.25) by displacement of the integration contour from the positive real axis to a ray making an angle of $\pi/4$ with the real axis. We also use the fact that, from 3a in Table 6, we have pure separation of variables in the coordinates $u=x/\sqrt{t}$, $v=t$.) Also we have

$$F_\lambda^{(3)+}(t,x)=\overline{F_{-\lambda}^{(3)+}(-t,x)}, \qquad F_\lambda^{(3)-}(t,x)=F_\lambda^{(3)+}(t,-x). \tag{1.48}$$

It follows immediately from (1.46) that the $\{F_\lambda^{(3)\pm}\}$ form a basis for $L_2(R)$ with orthogonality relations

$$\langle F_\lambda^{(3)\pm},F_\mu^{(3)\pm}\rangle=\delta(\lambda-\mu), \qquad \langle F_\lambda^{(3)\pm},F_\mu^{(3)\mp}\rangle=0 \tag{1.49}$$

for each fixed t. Application of these orthogonality and completeness relations to expand an arbitrary $h\in L_2(R)$ yields the Hilbert space version of Cherry's theorem [28, 37], which is an expansion in terms of parabolic cylinder functions. Note that our expansion is simply related to the spectral resolution of the operator

$$K^0=2t\,\partial_t+x\,\partial_x+\tfrac{1}{2}=2it\,\partial_{xx}+x\,\partial_x+\tfrac{1}{2}.$$

The next orbit we consider contains the operators $\mathcal{K}_{-2}-\mathcal{K}_1$ (linear potential) and $\mathcal{K}_2+\mathcal{K}_{-1}$. Since the spectral analysis for the second operator is simpler, we study it. (The corresponding results for $\mathcal{K}_{-2}-\mathcal{K}_1$ will follow upon application of the unitary operator $\mathcal{J}^2=\exp[(-\pi/2)\mathcal{L}_3]$, (1.21)–(1.23).) The eigenfunction equation is

$$i(\mathcal{K}_2+\mathcal{K}_{-1})f=\lambda f,\qquad \mathcal{K}_2+\mathcal{K}_{-1}=ix^2/4+\partial_x.$$

The spectral resolution is easily obtained from the Fourier integral theorem. The spectrum is continuous and covers the real axis, and a basis of generalized eigenfunctions is

$$\begin{aligned}&f_\lambda^{(2)}(x)=(2\pi)^{-1/2}\exp\left[-i(\lambda x+x^3/12)\right],\qquad -\infty<\lambda<\infty,\\&\langle f_\lambda^{(2)},f_\mu^{(2)}\rangle=\delta(\lambda-\mu).\end{aligned}\tag{1.50}$$

We find that

$$F_\lambda^{(2)}(t,x)=\exp\left[-\frac{i}{4}\left(\pi+\frac{1}{8v^2}-u^2v+\frac{u}{v}+\frac{4\lambda}{v}\right)\right]2^{1/6}\mathrm{Ai}\left[2^{2/3}\left(\frac{u}{2}+\lambda\right)\right],\tag{1.51}$$

$$x=uv+(2v)^{-1},\ t=v,$$

where $\mathrm{Ai}(z)$ is an Airy function

$$\mathrm{Ai}(z)=\pi^{-1}(z/3)^{1/2}K_{1/3}(2z^{3/2}/3),\qquad |\arg z|<2\pi/3.\tag{1.52}$$

As usual, we derive (1.51) by R-separation of variables in (1.25). The $\{F_\lambda^{(2)}\}$ are basis functions for the operator $K_2+K_{-1}=-it^2\partial_{xx}+(1-tx)\partial_x-t/2+ix^2/4$.

Finally, for the operator $\mathcal{K}_{-1}=\partial_x$ a complete set of generalized eigenfunctions is

$$\begin{aligned}&f_\lambda^{(1)}(x)=(2\pi)^{-1/2}e^{-i\lambda x},\qquad -\infty<\lambda<\infty,\\&i\mathcal{K}_{-1}f_\lambda^{(1)}=\lambda f_\lambda^{(1)},\qquad \langle f_\lambda^{(1)},f_\mu^{(1)}\rangle=\delta(\lambda-\mu).\end{aligned}\tag{1.53}$$

Furthermore,

$$F_\lambda^{(1)}(t,x)=\exp(t\mathcal{K}_{-2})f_\lambda^{(1)}(x)=(2\pi)^{-1/2}\exp\left[i(\lambda^2t-\lambda x)\right].\tag{1.54}$$

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If $\{f_\lambda(x)\}$ is a basis of (generalized) eigenfunctions of some $\mathcal{K}\in\mathcal{G}_2$ and $F_\lambda(t,x)=\exp(t\mathcal{K}_{-2})f_\lambda(x)$, then $F_\lambda(\tau,x)=\exp([\tau-t]\mathcal{K}_{-2})F_\lambda(t,x)$ and we have the Hilbert space expansions

$$
\begin{aligned}
k(t,x-y)&=\int F_\lambda(t,x)\bar{f}_\lambda(y)\,d\lambda,\\
k(\tau-t,x-y)&=\int F_\lambda(\tau,x)\bar{F}_\lambda(t,y)\,d\lambda
\end{aligned}
\tag{1.55}
$$

where the integration domain is the spectrum of $i\mathcal{K}$ and

$$k(t,x)=(4\pi it)^{-\frac{1}{2}}\exp(-x^2/4it)$$

is the kernel of the integral operator $\exp(t\mathcal{K}_{-2})$. These expansions are known as continuous generating functions [35, 136].

Now we compute the overlap functions $\langle f_\lambda^{(i)},f_\mu^{(j)}\rangle$ which allow us to expand eigenfunctions $f_\lambda^{(i)}$ in terms of eigenfunctions $f_\mu^{(j)}$. Since $\langle \mathbf{U}(g)f_\lambda^{(i)},\mathbf{U}(g)f_\mu^{(j)}\rangle=\langle f_\lambda^{(i)},f_\mu^{(j)}\rangle$, the same expressions allow us to expand eigenfunctions $\mathbf{U}(g)f_\lambda^{(i)}$ in terms of eigenfunctions $\mathbf{U}(g)f_\mu^{(j)}$. In particular, for $\mathbf{U}(g)=\exp(t\mathcal{K}_{-2})$ we have $\langle F_\lambda^{(i)},F_\mu^{(j)}\rangle=\langle f_\lambda^{(i)},f_\mu^{(j)}\rangle$ for fixed t, and this permits us to expand one basis of solutions for the free-particle Schrödinger equation in terms of another basis.

We give here some overlaps of special interest.

$$
\begin{aligned}
\langle f_n^{(4)},f_\lambda^{(3)\pm}\rangle&=(2)^{(3n/2)+i\lambda-\frac{1}{2}}\frac{\Gamma\left(i\lambda/2+\frac{1}{4}+n/2\right)}{2\pi(2^n n!)^{1/2}}\\
&\quad\times\begin{Bmatrix}+1\\(-1)^n\end{Bmatrix}{}_2F_1\left[\left.\begin{matrix}-n/2,\frac{1}{2}-n/2\\ \frac{3}{4}-i\lambda/2-n/2\end{matrix}\right|\frac{1}{2}\right].
\end{aligned}
\tag{1.56}
$$

For the calculation of the overlaps $\langle f_n^{(4)},f_\lambda^{(2)}\rangle$ it is convenient to give a generating function rather than an explicit expression. The result is

$$
\begin{aligned}
&2^{2/3}\exp\left\{-i\left[\tfrac{1}{6}+\lambda+(2y)^{1/2}\right]\right\}\operatorname{Ai}\left\{2^{2/3}\left[\tfrac{1}{4}-i\lambda-i(2y)^{1/2}\right]\right\}\\
&\quad=\sum_{n=0}^{\infty}\frac{(\sqrt{2}\,iy)^n}{(n!)^{1/2}}\langle f_n^{(4)},f_\lambda^{(2)}\rangle.
\end{aligned}
\tag{1.57}
$$

This expression follows from the form of the generating function for Hermite polynomials that we will derive in Section 2.2.

$$\langle f_n^{(4)},f_\lambda^{(1)}\rangle=\left[n!(-2)^n\pi\right]^{-1/2}\exp(-\lambda^2)H_n\left[(2\lambda)^{1/2}\right],\tag{1.58}$$

$$\langle f_\lambda^{(2)},f_\mu^{(1)}\rangle=2^{2/3}\operatorname{Ai}\left[2^{2/3}(\mu-\lambda)\right].\tag{1.59}$$

Additional overlaps can be found in [59].

The computation of mixed-basis matrix elements $\langle \mathbf{U}(g)f_\lambda^{(i)}, f_\mu^{(j)}\rangle$ permits the derivation of many more expansions relating solutions of the Schrödinger equations. For example, we have

$$\langle \exp(t\mathcal{K}_{-2})f_n^{(4)}, f_\mu^{(3)+}\rangle = \langle f_n^{(4)}, \exp(-t\mathcal{K}_{-2})f_\mu^{(3)+}\rangle$$

$$= \frac{(2)^{(3n/2)+i\mu-\frac{1}{2}}(1+it)^{i\mu/2}}{(2\pi)^{3/4}(2^n n!)^{1/2}(1-it)^{n/2+\frac{1}{4}}} \exp\left[-i\left(n+\tfrac{1}{2}\right)\tan^{-1}t\right]$$

$$\cdot\Gamma\left(\frac{i\mu}{2}+\frac{1}{4}+\frac{n}{2}\right) {}_2F_1\left[\begin{matrix} -n/2, \frac{1}{2}-n/2 \\ \frac{3}{4}-i\mu/2-n/2 \end{matrix}\middle| \frac{1-it}{2}\right], \tag{1.60}$$

with a similar result for $f_\mu^{(3)-}$. This expression allows us to expand Hermite polynomials as an integral over parabolic cylinder functions and parabolic cylinder functions in series of Hermite polynomials.

The matrix elements $\langle \mathbf{U}(g)f_n^{(4)}, f_m^{(4)}\rangle = \langle \mathbf{T}(g)F_n^{(4)}, F_m^{(4)}\rangle$ are easily computed and of great interest. However, in Section 2.2 we will apply Weisner's method to the complex heat equation and derive expansions for Hermite polynomials that yield these matrix elements as special cases.

It is also of great interest to compute the matrix elements with respect to the basis $\{f_\lambda^{(3)\pm}\}$ of generalized eigenvectors for $\mathcal{K}^0$. In this case the addition theorem for the matrix elements becomes an integral. In [128] Vilenkin has computed these matrix elements for the subgroup of $\tilde{G}_2$ whose Lie algebra has basis $\{\mathcal{K}_1, \mathcal{K}_{-1}, \mathcal{K}_0, \mathcal{K}^0\}$. The group operators are $\mathbf{U}(a,b,c,\tau)$ where

$$\mathbf{U}(a,b,c,\tau)h(x) = \exp(a\mathcal{K}_1)\exp(c\mathcal{K}_0)\exp(\tau\mathcal{K}^0)\exp(b\mathcal{K}_{-1})h(x)$$

$$= \exp(\tau/2 + iax/2 + ic)h(e^\tau x + b), \tag{1.61}$$

$$a,b,c,\tau \in R,\ h \in L_2(R).$$

(The rule for group multiplication can easily be determined from (1.61).) Vilenkin shows that the matrix elements of the operator $\mathbf{U}(a,b,c,\tau)$ in the $\{f_\lambda^{(3)\pm}\}$ basis can be expressed in terms of confluent hypergeometric functions ${}_1F_1$ and that the resulting addition theorems yield many interesting integral identities for these functions. Furthermore, just as in the analogous case for the group $E(1,1)$ (Section 1.5), we can allow the parameter a in (1.61) to become complex and derive more general integral identities. For these results see Vilenkin ([128]; also [81]).

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2.2 The Heat Equation $(\partial_t - \partial_{xx})\Phi = 0$

The heat equation in two-dimensional space-time (with units appropriately normalized) is

$$Q\Phi = 0, \qquad Q = \partial_t - \partial_{xx}, \tag{2.1}$$

where t, x are the real time and space variables, respectively [107]. Clearly, this equation can be obtained from the Schrödinger equation by replacing t in (1.2) by $-it$, so the symmetry algebras of these equations are closely related. Indeed a simple computation shows that the symmetry algebra of (2.1) is six dimensional, with basis

$$\begin{aligned} &H_2 = t^2\partial_t + tx\,\partial_x + t/2 + x^2/4, \qquad H_1 = t\,\partial_x + x/2, \\ &H_0 = 1, \qquad H_{-1} = \partial_x, \qquad H_{-2} = \partial_t, \qquad H^0 = x\,\partial_x + 2t\,\partial_t + \tfrac{1}{2} \end{aligned} \tag{2.2}$$

and commutation relations (H_0 commutes with everything),

$$\begin{aligned} &[H^0, H_j] = jH_j, \quad j = \pm 2, \pm 1, 0, \qquad [H_1, H_2] = [H_{-1}, H_{-2}] = 0, \\ &[H_{-1}, H_2] = H_1, \qquad [H_{-1}, H_1] = \tfrac{1}{2}H_0, \\ &[H_{-2}, H_1] = H_{-1}, \qquad [H_{-2}, H_2] = H^0 \end{aligned} \tag{2.3}$$

We denote by $\mathcal{G}_2'$ the real Lie algebra with basis (2.2).

As usual we can exponentiate the elements of $\mathcal{G}_2'$ to obtain a local Lie group G_2' of operators acting on the space $\mathcal{F}$ of functions $\Psi(t,x)$ analytic in some given domain $\mathcal{D}$ of the x–t plane. The operators H_{-1}, H_1, H_0 form a basis for the Weyl algebra $\mathcal{W}_1$ and the corresponding action of the Weyl group W_1 is given by operators

$$\mathrm{T}(u,v,\rho) = \exp([\rho + uv/4]H_0)\exp(uH_1)\exp(vH_{-1}) \tag{2.4}$$

with multiplication

$$\mathrm{T}(u,v,\rho)\mathrm{T}(u',v',\rho') = \mathrm{T}(u+u', v+v', \rho+\rho' + (vu' - uv')/4) \tag{2.5}$$

where

$$\mathrm{T}(u,v,\rho)\Psi(t,x) = \exp[\rho + (uv + 2ux + u^2t)/4]\Psi(t, x+v+ut), \qquad \Psi \in \mathcal{F}.$$

The operators H_2, H_{-2}, H^0 form a basis for a subalgebra isomorphic to

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$sl(2,R)$ and the corresponding action of $SL(2,R)$ is given by operators

$$\mathbf{T}(A)\Psi(t,x) = \exp\left(-\frac{x^2\beta/4}{\delta+t\beta}\right)(\delta+t\beta)^{-1/2}\Psi\left(\frac{\gamma+t\alpha}{\delta+t\beta},\frac{x}{\delta+t\beta}\right) \tag{2.6}$$

where

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,R).$$

Here,

$$\begin{aligned} &\mathbf{T}\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} = \exp(-\beta H_2), \qquad \mathbf{T}\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} = \exp(\gamma H_{-2}), \\ &\mathbf{T}\begin{pmatrix} e^{\alpha} & 0 \\ 0 & e^{-\alpha} \end{pmatrix} = \exp(\alpha H^0). \end{aligned} \tag{2.7}$$

The group $SL(2,R)$ acts on W_1 via the adjoint representation

$$\mathbf{T}^{-1}(A)\mathbf{T}(u,v,\rho)\mathbf{T}(A) = \mathbf{T}(u\delta - v\beta, v\alpha - u\gamma, \rho). \tag{2.8}$$

We can now define the symmetry group G_2' as a semidirect product of $SL(2,R)$ and W_1:

$$\begin{aligned} &g = (A,\mathbf{w}) \in G_2', \quad A \in SL(2,R), \quad \mathbf{w} = (u,v,\rho) \in W_1, \\ &\mathbf{T}(g) = \mathbf{T}(A)\mathbf{T}(\mathbf{w}), \\ &\mathbf{T}(g)\mathbf{T}(g') = \mathbf{T}(AA')\left[\mathbf{T}^{-1}(A')\mathbf{T}(\mathbf{w})\mathbf{T}(A')\right]\mathbf{T}(\mathbf{w}') = \mathbf{T}(gg'). \end{aligned} \tag{2.9}$$

Clearly, the operators $\mathbf{T}(g)$ map solutions of (2.1) into solutions. Furthermore, G_2' acts on the Lie algebra $\mathcal{G}_2'$ of differential operators H via the adjoint representation

$$H \to H^g = \mathbf{T}(g)H\mathbf{T}^{-1}(g)$$

and this action splits $\mathcal{G}_2'$ into G_2' orbits.

It is straightforward to show that there are five orbits in $\mathcal{G}_2'/\{H_0\}$ under the adjoint representation (just as in Section 2.1 we ignore the center of $\mathcal{G}_2'$) with corresponding orbit representatives $H^0, H_2 + H_{-2}, H_{-2} + H_1, H_{-2}, H_{-1}$. Since $H_{-2} = (H_{-1})^2$ for solutions of the heat equation, only four R-separable coordinate systems are associated with the five orbits. The results are presented in Table 7. (See [59] for more details.) For each system $\{u,v\}$ we have $t = v$.

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Table 7 R-separable Coordinates for the Equation $(\partial_t - \partial_{xx})\Phi(t,x)=0$

Operator H	Coordinates $\{u,v\}$	Multiplier $R=e^{\mathfrak{R}}$	Separated solutions
1 H_{-1}, H_{-2}	$x=u$	$\mathfrak{R}=0$	Product of exponentials
2 $H_{-2}+H_1$	$x=u+v^2/2$	$\mathfrak{R}=-uv/2$	Airy times exponential function
3 H^0	$x=u\sqrt{v}$	$\mathfrak{R}=0$	Hermite times exponential function
4 H_2+H_{-2}	$x=u(1+v^2)^{1/2}$	$\mathfrak{R}=u^2v/4$	Parabolic cylinder times exponential function

The eigenfunctions of H^0 are of special interest. From Table 7, the eigenfunctions separate in the variables $u=x/\sqrt{t}$, $v=t$. Moreover, the solutions $\Phi_n(t,x)$ of the heat equation that satisfy $H^0\Phi_n=(n+\frac{1}{2})\Phi_n$, $n=0,1,2,\ldots$, are the *heat polynomials*

$$\Phi_n(t,x)=\left(i\sqrt{t}\,/2\right)^n H_n\left(ix/2\sqrt{t}\,\right). \tag{2.10}$$

(These functions are easily seen to be polynomials in t and x.) Rosenbloom and Widder [113] have presented a complete theory of the expansion of solutions of the heat equation in terms of heat polynomials.

Although the symmetries (2.6) are not very well known, there is a special case that has been of great importance in the theory of the heat equation. If in (2.6) we set

$$A_0=2^{-1/2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad A_0^8=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we find the symmetries

$$\begin{aligned} \mathbf{T}(A_0)\Psi(t,x) &= \exp\left(\frac{-x^2/4}{1+t}\right)\left(\frac{\sqrt{2}}{1+t}\right)^{1/2}\Psi\left(\frac{t-1}{t+1}, \frac{(2x)^{1/2}}{t+1}\right), \\ \mathbf{T}(A_0^2)\Psi(t,x) &= \exp\left(\frac{-x^2}{4t}\right)t^{-1/2}\Psi\left(\frac{-1}{t}, \frac{x}{t}\right). \end{aligned} \tag{2.11}$$

The symmetry $\mathbf{T}(A_0^2)$ is called the *Appell transform* [4, 13]. We have embedded this transform in a Lie symmetry group.

It is well known that if $f(x)$ is a bounded continuous function defined on the real line, then there is exactly one solution $\Psi(t,x)$ of the heat equation (2.1), bounded and continuous in (t,x) for all $x\in R$ and $t\geqslant 0$ and continuously differentiable in t, twice continuously differentiable in x for all $x\in R$ and $t>0$, such that $\Psi(0,x)=f(x)$ [107]. This solution is given by the expression

$$\Psi(t,x)=(4\pi t)^{-1/2}\int_{-\infty}^{\infty}\exp\left[-(x-y)^2/4t\right]f(y)\,dy=I^t(f). \tag{2.12}$$

Moreover,

$$\Psi(t,x) = \left[4\pi(t-\tau)\right]^{-1/2}\int_{-\infty}^{\infty}\exp\left[-(x-y)^2/4(t-\tau)\right]\Psi(\tau,y)\,dy, \qquad t>\tau, \tag{2.13}$$

which shows how to obtain the solution Ψ at time t given a knowledge of Ψ at an *earlier* time $\tau < t$.

Although some expansion theorems for solutions of (2.1) can be obtained through use of the time-independent form

$$(\Psi,\Phi) = \int_{-\infty}^{\infty}\Psi(t,x)\bar{\Phi}(-t,x)\,dx$$

where Ψ,Φ are solutions of the heat equation (see [113]), the operators (2.4)–(2.6) are not all unitary. There appears to be no convenient Hilbert space structure for this problem. Nonetheless, in analogy with our work for the Schrödinger equations, we can find another model of the group action that is very useful. To obtain this model we consider the operators (2.2) restricted to the solution space of the heat equation. Then we can replace ∂_t by ∂_{xx} in expressions (2.2) and consider $t \geqslant 0$ as a fixed parameter. We now think of the H operators as the symmetry operators at a fixed time t. At time $t=0$ these operators become

$$\begin{aligned} &\mathcal{H}_2 = x^2/4, \qquad \mathcal{H}_1 = x/2, \qquad \mathcal{H}_0 = 1,\\ &\mathcal{H}_{-1} = \partial_x, \qquad \mathcal{H}_{-2} = \partial_{xx}, \qquad \mathcal{H}^0 = x\partial_x + \tfrac{1}{2} \end{aligned} \tag{2.14}$$

and when restricted, say, to the space $\mathfrak{F}_0$ of infinitely differentiable functions $f(x)$ on R with compact support, the $\mathcal{H}$ operators satisfy the usual commutation relations (2.3).

A deeper understanding of this procedure follows from the observation that (2.12) has the interpretation

$$\Psi(t,x) = I^t(f) = \exp(t\partial_{xx})f(x) = \exp(t\mathcal{H}_{-2})f(x), \qquad f\in\mathfrak{F}_0, t>0, \tag{2.15}$$

in analogy with (1.25). Then with integration by parts, we can check that

$$H\exp(t\mathcal{H}_{-2}) = \exp(t\mathcal{H}_{-2})\mathcal{H} \tag{2.16}$$

where $H\in\mathcal{G}_2'$ and $\mathcal{H}$ is obtained from H by setting $t=0$. (Precisely, if $\Psi(t,x) = I^t(f)$, then $H\Psi(t,x) = I^t(\mathcal{H}f)$.) Note that (2.16) is the counterpart of (1.26), except that here we avoid use of the unbounded operator $\exp(-t\mathcal{H}_{-2})$, $t>0$. The theory leading to (2.16) appears to have been first studied by Hida [49].

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Similarly we can derive results of the form

$$\exp(aH)\exp(t\mathcal{H}_{-2})=\exp(t\mathcal{H}_{-2})\exp(a\mathcal{H}). \tag{2.17}$$

Just as in the preceding section, we can show that the equations

$$\partial_t\Psi(t,x)=\left(\partial_{xx}+ax\,\partial_x+b\,\partial_x+cx^2+dx+e\right)\Psi(t,x),\qquad a,\dots,e\in R \tag{2.18}$$

all have symmetry algebras isomorphic to $\mathcal{G}_2'$ and that in fact these equations are all equivalent.

We present an example, due to Rosencrans [114], which exhibits this equivalence and shows how to make use of it to solve the Cauchy problem for each of the equations (2.18). (We shall present a formal argument. The rigorous validity of the result can easily be checked.)

We wish to determine the bounded solution $\Phi(t,x)$ of the heat equation with linear drift

$$\partial_t\Phi=\partial_{xx}\Phi-kx\,\partial_x\Phi,\qquad k>0, \tag{2.19}$$

for all $t>0$ such that $\Phi(0,x)=f(x)$ where $f(x)$ is bounded and continuous on the real line. Now (2.19) reads $\partial_t\Phi=(\mathcal{H}_{-2}-k\mathcal{H}^0+(k/2)\mathcal{H}_0)\Phi, \Phi(0,x)=f(x)$, or

$$\Phi(t,x)=\exp\left[i\left(\mathcal{H}_{-2}-k\mathcal{H}^0+(k/2)\mathcal{H}_0\right)\right]f(x).$$

Since the $\mathcal{H}$ operators satisfy the same commutation relations as the H operators, we can use expressions (2.7) and group multiplication in $SL(2,R)$ to evaluate products of exponentials of the operators $\mathcal{H}_{-2}, \mathcal{H}^0, \mathcal{H}_0$. We find

$$\begin{aligned}\exp&\{t[\mathcal{H}_{-2}-k\mathcal{H}^0+(k/2)\mathcal{H}_0]\}\\&=\exp[(tk/2)\mathcal{H}_0]\exp(-tk\mathcal{H}^0)\exp\{[(1-e^{-2kt})/2k]\mathcal{H}_{-2}\}\\&=\exp[(tk/2)\mathcal{H}_0]\exp[(e^{2kt}-1)/2k\mathcal{H}_{-2}]\exp(-tk\mathcal{H}^0).\end{aligned} \tag{2.20}$$

From (2.12), (2.15), and the easily verified relation

$$\exp(-tk\mathcal{H}^0)h(x)=\exp(-tk/2)h(\exp(-tk)x)$$

we obtain

$$\Phi(t,x)=\left\{\frac{2\pi}{k}\left[1-\exp(-2kt)\right]\right\}^{-1/2}\int_{-\infty}^{\infty}\exp\left\{\frac{k\left[\exp(-tk)x-y\right]^2}{2\left[1-\exp(-2kt)\right]}\right\}f(y)\,dy \tag{2.21}$$

as the solution to the Cauchy problem for (2.19).

Next we study the complex heat equation, that is, equation (2.1) with the variables t, x complex. It is easy to show that the symmetry algebra $\mathcal{G}_2^c$ of this equation is six dimensional with basis (2.2) and commutation relations (2.3). However, now the Lie algebra consists of all *complex* linear combinations of the basis elements. We can exponentiate the elements of $\mathcal{G}_2^c$ to obtain a local Lie group $\mathcal{G}_2^c$ of operators acting on the space $\mathcal{F}$ of functions $\Psi(t,x)$ analytic in some given domain $\mathcal{D}$ in the complex x–t plane. The group action is given by (2.4)–(2.9) where the parameters u, v, ρ are allowed to take arbitrary complex values and the matrices $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ now range over the group $SL(2, \mathbb{C})$ of all complex matrices with determinant $+1$: $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, $\alpha\delta - \beta\gamma = 1$. As usual, the operators $\mathbf{T}(g), g \in G_2$, map solutions of the complex heat equation into solutions. Furthermore, G_2^c acts on the Lie algebra $\mathcal{G}_2^c$ of Lie derivatives H via the adjoint representation

$$H \to H^g = \mathbf{T}(g) H \mathbf{T}^{-1}(g)$$

and splits $\mathcal{G}_2^c$ (as well as $\mathcal{G}_2^c/\{H_0\}$) into G_2^c orbits. It is straightforward to show that there are exactly four orbits in $\mathcal{G}_2^c/\{H_0\}$ with orbit representatives $H^0, H_{-2} + H_1, H_{-2}, H_{-1}$. (The distinct orbits in $\mathcal{G}_2'/\{H_0\}$ with representatives H^0 and $H_2 + H_{-2}$ become equivalent when the group G_2' is extended to G_2^c by complexification.) Since $H_{-2} = (H_{-1})^2$ when acting on solutions of the complex heat equation, there are only three R-separable coordinate systems associated with the four orbits. (It can be shown that these are the only R-separable systems admitted by the complex heat equation. Here an admissible coordinate system $\{u, v\}$ must be such that $u(t,x)$, $v(t,x)$ are complex analytic functions of (t,x) with nonzero Jacobian. Two separable systems are equivalent if one system can be mapped into the other by an element of G_2^c.) The results appear in Table 8, where $t = v$ for each separable system $\{u, v\}$.

Table 8 R-Separable Coordinates for the Complex Heat Equation

	Operator H	Coordinates $\{u,v\}$	Multiplier $R = e^{\mathcal{R}}$	Separated solutions
1	H_{-1}, H_{-2}	$x = u$	$\mathcal{R} = 0$	Product of exponentials
2	$H_{-2} + H_1$	$x = u + v^2/2$	$\mathcal{R} = -uv/2$	Airy times exponential function
3	H^0	$x = u\sqrt{v}$	$\mathcal{R} = 0$	Hermite times exponential function

Note that the complex heat equation is the complexification of both the real heat equation and the free-particle Schrödinger equation. The effect of the complexification in terms of separation of variables is that orbits 1 and 2 in Tables 6 and 7 correspond to orbits 1 and 2 in Table 8, while orbits 3 and 4 in Tables 6 and 7 collapse to the single orbit 3 in Table 8.

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To derive identities relating separated solutions of the complex heat equation we can use Weisner's method and expand arbitrary analytic solutions in terms of the Hermite functions (orbit 3, Table 8). This was carried out in detail by Weisner [135] and the Lie algebraic aspects are covered in [82], so here we discuss only a few of the features of these expansions.

As suggested by (2.10) and 3, Table 8, for Hermite polynomial solutions of (2.1) the coordinates $\{s,z\}$ are appropriate, where

$$s=-i\sqrt{t}\,/2, \qquad z=ix/2\sqrt{t}\,. \tag{2.22}$$

In terms of these coordinates the operators (2.2) become

$$\begin{gathered} H_2=-2s^2\left(z\,\partial_z+s\,\partial_s+1-2z^2\right), \qquad H_1=s(-\partial_z+2z), \qquad H_0=1, \\ H_{-1}=\frac{s^{-1}}{4}\,\partial_z, \qquad H_{-2}=\frac{s^{-2}}{8}(z\,\partial_z-s\,\partial_s), \qquad H^0=s\,\partial_s+\frac{1}{2}, \end{gathered} \tag{2.23}$$

and the heat equation reads

$$(\partial_{zz}-2z\,\partial_z+2s\,\partial_s)\Phi(z,s)=0. \tag{2.24}$$

Consider the solutions Φ of (2.24) which are eigenfunctions of H^0:

$$H^0\Phi=\left(n+\tfrac{1}{2}\right)\Phi\Rightarrow\Phi=f_n(z)s^n.$$

Substituting these solutions into (2.24) and comparing the resulting ordinary differential equation in z with (B.10), we find that the functions

$$\Phi_n(z,s)=H_n(z)s^n, \qquad \tilde{\Phi}_n(z,s)=e^{z^2}H_{-n-1}(iz)s^n \tag{2.25}$$

form a basis of simultaneous solutions where the *Hermite functions* $H_n(z)$ are defined by

$$H_n(z)=2^{n/2}\exp(z^2/2)D_n(\sqrt{2}\,z), \qquad n\in\mathbb{C}, \tag{2.26}$$

and $D_n(z)$ is a parabolic cylinder function. If $n=0,1,2,\ldots,$ then $H_n(z)$ is the *Hermite polynomial* (B.12).

To understand the significance of the polynomial solutions, let us consider the system of equations

$$H_{-1}\Phi=0, \qquad Q\Phi=0,$$

which has the solution $\Phi\equiv1$, unique to within a multiplicative constant. We will use this elementary solution and our knowledge of the symmetry

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algebra $\mathcal{G}_2^c$ to construct other solutions. If $\Phi(z,s)$ is in analytic function of (z,s), we know from standard Lie theory that

$$\exp(\alpha H_1)\Phi(z,s) = \exp(2\alpha zs - \alpha^2 s^2)\Phi(z-\alpha s, s) = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!}(H_1)^n \Phi(z,s).$$

Furthermore, if Φ is a solution of the complex heat equation, then $\exp(\alpha H_1)\Phi$ is a solution (provided it is well defined). Putting our solution $\Phi \equiv 1$ in this expression, we find

$$\exp(2\alpha zs - \alpha^2 s^2) = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!}\Phi_n(z,s),$$
$$\Phi_0 = 1, \quad \Phi_n = (H_1)^n \Phi_0, \quad n = 1,2,\ldots. \tag{2.27}$$

Now consider the action of the symmetry operators H_j on the Φ_n. An elementary induction argument based on $[H_{-1}, H_1] = \frac{1}{2}H_0$ shows $[H_{-1},(H_1)^n] = (n/2)(H_1)^{n-1}$, $n = 1,2,\ldots$. Applying both sides of this identity to Φ_0, we obtain

$$H_{-1}\Phi_n = (n/2)\Phi_{n-1}, \qquad n = 1,2,\ldots. \tag{2.28}$$

(This expression makes sense for $n = 0$ if we define $\Phi_n \equiv 0$ for $n < 0$.) By definition of Φ_n we also have

$$H_1\Phi_n = \Phi_{n+1}, \qquad n = 0,1,\ldots, \tag{2.29}$$

and, from $[H^0, H_1] = H_1$,

$$H^0\Phi_n = \left(n + \tfrac{1}{2}\right)\Phi_n, \qquad \Phi_n = f_n(z)s^n. \tag{2.30}$$

It follows from (2.30) that the $f_n(z)$ are expressible in terms of Hermite functions. Indeed, comparing (2.28), (2.29) with the recurrence relations (B.13), we find

$$\Phi_n(z,s) = H_n(z)s^n, \qquad n = 0,1,2,\ldots, \tag{2.31}$$

the Hermite polynomial solutions. Substituting (2.31) into (2.27) and setting $s = 1$, we obtain the fundamental generating function for Hermite polynomials

$$\exp(2\alpha z - \alpha^2) = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} H_n(z). \tag{2.32}$$

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In addition to the recurrence formulas (2.28), (2.29) for the Hermite polynomials we can use the commutation relations to derive

$$H_{-2}\Phi_n=(n/4)(n-1)\Phi_{n-2}, \qquad H_2\Phi_n=\Phi_{n+2}, \qquad H_0\Phi_n=\Phi_n, \qquad n=0,1,2,\dots. \tag{2.33}$$

We can obtain many other identities obeyed by Hermite polynomials if we apply the group operators $\mathbf{T}(g), g\in G_2^c$ (see (2.9) to a basis element Φ_m and expand the result in terms of the $\{\Phi_n\}$ basis:

$$\mathbf{T}(g)\Phi_m(z,s)=\sum_{n=0}^{\infty} T_{nm}(g)\Phi_n(z,s), \qquad m=0,1,2,\dots. \tag{2.34}$$

This procedure is practical provided we can compute the matrix elements $T_{nm}(g)$. However, for g close to the group identity element, these matrix elements can be computed directly from the Lie algebra relations (2.28)–(2.30) and (2.33).

To perform the computation it is convenient to construct a simpler model of the Lie algebra representation. We choose $f_n(w)=w^n$ and

$$\begin{aligned} H_{-1}&=\frac{1}{2}\frac{d}{dw}, \qquad H_1=w, \qquad H_2=w^2,\\ H^0&=w\frac{d}{dw}+\frac{1}{2}, \qquad H_0=1, \qquad H_{-2}=\frac{1}{4}\frac{d^2}{dw^2}. \end{aligned} \tag{2.35}$$

These operators satisfy the commutation relations of $\mathcal{G}_2^c$ and their action on the basis functions $f_n(w)$ agrees with the action (2.28)–(2.30), (2.33) on the Φ_n basis. In terms of this model we define the matrix elements $T_{nm}(\alpha,\beta)$, $R_{nm}(\alpha,\beta)$:

$$\exp(\alpha H_1)\exp(\beta H_2)f_m(w)=\sum_{n=0}^{\infty} T_{nm}(\alpha,\beta)f_n(w), \tag{2.36a}$$

$$\exp(\alpha H_1)\exp(\beta H_{-1})f_m(w)=\sum_{n=0}^{\infty} R_{nm}(\alpha,\beta)f_n(w); \tag{2.36b}$$

that is, we apply the group operators $\exp(\alpha H)\exp(\beta H')$ to a basis function w^m and expand the resulting analytic functions in power series about $w=0$. The matrix elements are model independent. We will compute them using the simple model (2.35), then apply the results to the heat equation. Elementary Lie theory yields

$$\exp(\alpha H_1)f(w)=\exp(\alpha w)f(w), \qquad \exp(\beta H_2)f(w)=\exp(\beta w^2)f(w),$$
$$\exp(\beta H_{-1})f(w)=f(w+\beta/2).$$

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Thus (2.36) becomes

$$\exp(\alpha w + \beta w^2) = \sum_{n=0}^{\infty} T_{nm}(\alpha, \beta) w^{n-m}, \tag{2.37a}$$

$$\exp(\alpha w)(w + \beta/2)^m = \sum_{n=0}^{\infty} R_{nm}(\alpha, \beta) w^n. \tag{2.37b}$$

These are well-known generating functions (2.32), (7.30), which yield

$$T_{nm}(\alpha,\beta) = \begin{cases} \dfrac{(-i\sqrt{\beta})^{n-m}}{(n-m)!} H_{n-m}\left(\dfrac{i\alpha}{2\sqrt{\beta}}\right), & n \geqslant m; \\ 0, & n < m; \end{cases} \tag{2.38}$$

$$R_{nm}(\alpha,\beta) = \left(\frac{\beta}{2}\right)^{m-n} L_n^{(m-n)}\left(\frac{-\alpha\beta}{2}\right),$$

where $L_n^{(\alpha)}(z)$ is a Laguerre polynomial (see (B.9i)).

Now we exponentiate the operators (2.23). In addition to (2.27) we obtain

$$\exp(\alpha H_2)\Phi(z,s) = (1+4\alpha s^2)^{-1/2} \exp\left(\frac{4\alpha z^2 s^2}{1+4\alpha s^2}\right) \Phi\left[\frac{z}{(1+4\alpha s^2)^{1/2}}, \right.$$

$$\left. \times \frac{s}{(1+4\alpha s^2)^{1/2}}\right], \qquad |4\alpha s^2| < 1,$$

$$\exp(\beta H_{-2})\Phi(z,s) = \Phi\left[\frac{z}{(1-\beta/4s^2)^{1/2}}, s\left(1-\frac{\beta}{4s^2}\right)^{1/2}\right], \qquad |\beta/4s^2| < 1,$$

$$\exp(\gamma H_{-1})\Phi(z,s) = \Phi(z+\gamma/4s, s),$$

$$\exp(\delta H^0)\Phi(z,s) = \exp(\delta/2)\Phi(z, e^{\delta}s),$$

$$\exp(\varphi H_0)\Phi(z,s) = e^{\varphi}\Phi(z,s). \tag{2.39}$$

Substituting (2.38) and (2.39) into (2.36), we find (after simplification)

$$(1-s^2)^{-(m+1)/2} \exp\left[\frac{2zs\alpha - (z^2+\alpha^2)s^2}{1-s^2}\right] H_m\left(\frac{z-s\alpha}{(1-s^2)^{1/2}}\right)$$

$$= \sum_{n=0}^{\infty} \frac{(s/2)^n}{n!} H_n(\alpha) H_{m+n}(z), \qquad |s| < 1, \tag{2.40a}$$

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$$\exp(-s^2\alpha^2-2zs\alpha)H_m(z+s\alpha-\beta/s)s^m$$
$$=\sum_{n=0}^{\infty}(-\beta)^{m-n}L_n^{(m-n)}(-\alpha\beta)H_n(z)s^n. \tag{2.40b}$$

Expression (2.40a) is a generalization of Mehler's theorem [37, p. 194], to which it reduces in the case in which $m=0$ $(H_0(z)=1)$,

$$(1-s^2)^{-1/2}\exp\left[\frac{2zs\alpha-(z^2+\alpha^2)s^2}{1-s^2}\right]=\sum_{n=0}^{\infty}\frac{(s/2)^n}{n!}H_n(\alpha)H_n(z), \quad |s|<1. \tag{2.41}$$

For $\beta=0$, $s=1$, expression (2.40b) simplifies to

$$\exp(-\alpha^2-2z\alpha)H_m(z+\alpha)=\sum_{n=0}^{\infty}\frac{(-\alpha)^n}{n!}H_{m+n}(z) \tag{2.42}$$

and for $\alpha=0$, $s=1$ it becomes

$$H_m(z-\beta)=\sum_{n=0}^{m}\binom{m}{n}\beta^{m-n}H_n(z) \tag{2.43}$$

where $\binom{m}{n}$ is a binomial coefficient (see (B.1)). By computing additional matrix elements $T_{mn}(g)$ of G_2^c, we could derive further generating functions for the Hermite polynomials [135, 82].

We now discuss the Hermite function, nonpolynomial solutions of the complex heat equation, that is, the eigenfunctions $\Phi_n(z,s)$, (2.25), with $n\in\mathcal{C}$, $n\neq 0,1,2,\ldots$. In particular, we will study the eigenfunctions

$$\Phi_\lambda(z,s)=H_\lambda(z)s^\lambda, \qquad \lambda\in\mathcal{C}, \tag{2.44}$$

where λ is not an integer. The Φ_λ satisfy

$$H^0\Phi_\lambda=\left(\lambda+\tfrac{1}{2}\right)\Phi_\lambda, \qquad Q\Phi_\lambda=0. \tag{2.45}$$

The commutation relations $[H^0,H_j]=jH_j, j=0,\pm1,\pm2$ imply that the operators H_j map a solution of (2.45) corresponding to eigenvalue λ into a solution corresponding to eigenvalue $\lambda+j$. Indeed, using the fundamental recurrence formulas (B.13) it is straightforward to show

$$\begin{aligned}&H_{-1}\Phi_\lambda=\tfrac{\lambda}{2}\Phi_{\lambda-1}, \qquad H_{-2}\Phi_\lambda=\tfrac{\lambda}{4}(\lambda-1)\Phi_{\lambda-2},\\ &H_1\Phi_\lambda=\Phi_{\lambda+1}, \qquad H_2\Phi_\lambda=\Phi_{\lambda+2}, \qquad H_0\Phi_\lambda=\Phi_\lambda.\end{aligned} \tag{2.46}$$

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These relations are the same as (2.28), (2.29), (2.33) except here λ is not an integer. By applying the operators H_j to a given Φ_λ we can thus obtain an infinite ladder of solutions $\Phi_{\lambda+n}$ where n runs over all integers.

For a study of the transformation properties of these solutions under G_2^c it is convenient to consider the operators $\exp(\alpha H_1)\exp(\beta H_2), \exp(\alpha H_1)$ $\exp(\beta H_{-1})$, and $\exp(\alpha H_2)\exp(\beta H_{-2})$. (We can obtain the general group action as a product of three such operators and the trivial $\exp(\gamma H^0)$ $\exp(\delta H_0)$.) The matrix elements are defined by

$$\exp(\alpha H_1)\exp(\beta H_2)\Phi_{\lambda+m} = \sum_{n=-\infty}^{\infty} \hat{T}_{nm}(\alpha,\beta)\Phi_{\lambda+n}, \tag{2.47a}$$

$$\exp(\alpha H_1)\exp(\beta H_{-1})\Phi_{\lambda+m} = \sum_{n=-\infty}^{\infty} \hat{R}_{nm}(\alpha,\beta)\Phi_{\lambda+n}, \tag{2.47b}$$

$$\exp(\alpha H_2)\exp(\beta H_{-2})\Phi_{\lambda+m} = \sum_{n=-\infty}^{\infty} \hat{S}_{nm}(\alpha,\beta)\Phi_{\lambda+n}. \tag{2.47c}$$

From (2.46) it is easy to see that the $T_{nm}(\alpha,\beta)$ are identical with the matrix elements $T_{nm}(\alpha,\beta)$, (2.38), except that now m and n may take negative integral values. Thus, (2.47a) becomes (2.40a) and H_m replaced by $H_{\lambda+m}$, H_{m+n} replaced by $H_{\lambda+m+n}$, where m can take on all integer values. This is a further generalization of Mehler's theorem.

To compute the matrix elements $\hat{R}_{nm}(\alpha,\beta)$ we choose a simpler model of some of the relations (2.46). Indeed, the choices $h_{\lambda+m}|(w)=w^m$, $m=0,\pm 1,\pm 2,\ldots,$

$$H_1 = w, \qquad H_{-1} = \frac{1}{2}\frac{d}{dw} + \frac{\lambda}{2w}, \qquad H_0 = 1$$

satisfy $[H_1, H_{-1}] = -\frac{1}{2}H_0$ and

$$H_1 h_{\lambda+m} = h_{\lambda+m+1}, \qquad H_{-1}h_{\lambda+m} = \left(\frac{\lambda+m}{2}\right)h_{\lambda+m-1},$$

in agreement with (2.46). In this model

$$\begin{aligned}\exp(\alpha H_1)\exp(\beta H_{-1})h_{\lambda+m}(w) &= \exp(\alpha w)(1+\beta/2w)^{\lambda+m}w^m\\ &= \sum_{n=-\infty}^{\infty}\hat{R}_{nm}(\alpha,\beta)w^n.\end{aligned} \tag{2.48}$$

Computing the coefficient of w^n, we find

$$\hat{R}_{nm}(\alpha,\beta) = \left(\frac{\beta}{2}\right)^{m-n} L_{\lambda+n}^{(m-n)}\left(\frac{-\alpha\beta}{2}\right) \tag{2.49}$$

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where $L_\lambda^{(\nu)}(z)$ is a generalized Laguerre function,

$$L_\lambda^{(\nu)}(z)=\frac{\Gamma(\nu+\lambda+1)}{\Gamma(\nu+1)\Gamma(\lambda+1)}\,{}_1F_1\left(\begin{matrix}-\lambda\\ \nu+1\end{matrix}\middle|\,z\right), \tag{2.50}$$

proportional to a general ${}_1F_1$. Thus, (2.47b) becomes

$$\begin{aligned}&\exp(-s^2\alpha^2-2zs\alpha)H_{\lambda+m}(z+s\alpha-\beta/s)s^m\\ &=\sum_{n=-\infty}^{\infty}(-\beta)^{m-n}L_{\lambda+n}^{(m-n)}(-\alpha\beta)H_{\lambda+n}(z)s^n, \qquad (2.51)\\ &\qquad m=0,\pm1,\pm2,\ldots.\end{aligned}$$

To compute the matrix elements $\hat{S}_{nm}(\alpha,\beta)$ we choose another model:

$$h_{\lambda+m}(w)=\Gamma\left(\frac{\lambda+m+2}{2}\right)w^m, \qquad H_2=\frac{w^3}{2}\frac{d}{dw}+\frac{(\lambda+2)}{2}w^2,$$

$$H_{-2}=\frac{1}{2w}\frac{d}{dw}+\frac{\lambda-1}{2w^2}, \qquad H^0=w\frac{d}{dw}+\lambda+\frac{1}{2}.$$

With these operators

$$\begin{aligned}&\exp(\alpha H_2)\exp(\beta H_{-2})h_{\lambda+m}(w)\\ &=\Gamma\left(\frac{\lambda+m+2}{2}\right)w^m(1-\alpha w^2)^{-(\lambda+m+2)/2}\\ &\quad\times\left(1+\frac{\beta}{(1-\alpha\beta)w^2}\right)^{(\lambda+m-1)/2}(1-\alpha\beta)^{(\lambda+m-1)/2}\\ &=\sum_{n=-\infty}^{\infty}\hat{S}_{nm}(\alpha,\beta)\Gamma\left(\frac{\lambda+n+2}{2}\right)w^n,\\ &\qquad |\alpha w^2|<1, \quad |\beta|<|(1-\alpha\beta)w^2|, \qquad (2.52)\end{aligned}$$

so

$$\begin{aligned}\hat{S}_{nm}(\alpha,\beta)&=\frac{(1-\alpha\beta)^{(\lambda+m-1)/2}\alpha^{(n-m)/2}}{\Gamma((n-m+2)/2)}\\ &\quad\times{}_2F_1\left[\begin{matrix}\frac{\lambda+n+2}{2},\ \frac{1-\lambda-m}{2}\\ \frac{n-m+2}{2}\end{matrix}\middle|\,\frac{-\alpha\beta}{1-\alpha\beta}\right] \quad \text{if } n-m \text{ even},\\ \hat{S}_{nm}(\alpha,\beta)&=0 \quad \text{if } n-m \text{ odd}. \qquad (2.53)\end{aligned}$$

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(We use the fact that ${}_2F_1(a,b;\, c;\, z)/\Gamma(c)$ is an entire function of c to make sense of these expressions for $m>n$.) Thus (2.47c) becomes

$$(1+4\alpha s^2)^{-(\lambda+m+1)/2}\left(1-\alpha\beta-\frac{\beta}{4s^2}\right)^{(\lambda+m)/2}\exp\left(\frac{4\alpha z^2 s^2}{1+4\alpha s^2}\right)$$
$$\times H_{\lambda+m}\left[z\left[(1+4\alpha s^2)\left(1-\alpha\beta-\frac{\beta}{4s^2}\right)\right]^{-1/2}\right]s^m$$
$$=\sum_{n=-\infty}^{\infty}\hat{S}_{nm}(\alpha,\beta)H_{\lambda+n}(z)s^n,\qquad \left|\frac{\beta}{(1-\alpha\beta)}\right|<|4s^2|<|\alpha|^{-1}. \tag{2.54}$$

Next we present a simple application of the general Weisner method to a case where the expansion coefficients in a Hermite polynomial basis cannot be computed from the Lie algebra alone. Consider the function $\Psi(z,s)=\exp(-4H_{-2})\Phi_\lambda(z,s)$ where Φ_λ is given by (2.25), (2.26) with $\lambda=n\in\not{C}$ and $|s|<1$. Then

$$\Psi(z,s)=H_\lambda\left[\frac{sz}{(1+s^2)^{1/2}}\right](1+s^2)^{\lambda/2}=\sum_{j=0}^{\infty}f_j(z)s^j,\qquad |s|<1. \tag{2.55}$$

(Note that this expansion is not a special case of (2.54). Since $Q\Psi=0$, it follows that $Q(f_j(z)s^j)=0$ for each j, hence $f_j(z)$ is a linear combination of the H^0 basis functions Φ_j and $\tilde{\Phi}_j$; see (2.25). However, since $H_\lambda(w)$ is an entire function of w, it follows easily from (2.55) that the highest power of z occurring in $f_j(z)$ is z^j. Thus $f_j(z)=c_jH_j(z)$. Setting $z=w^{-1}, s=wv$ in (2.55) and letting w go to zero, we obtain

$$H_\lambda(v)=\sum_{j=0}^{\infty}c_j(2v)^j.$$

However, the special case of (2.51) with $\beta=-v, s=1, m=0$ yields

$$H_\lambda(z+v)=\sum_{j=0}^{\infty}\binom{\lambda}{j}H_{\lambda-j}(z)(2v)^j. \tag{2.56}$$

Thus $c_j=\binom{\lambda}{j}H_{\lambda-j}(0)$. This result suggests the existence of a more general generating function. Indeed, consideration of $\exp(4wH_{-1}-4H_{-2})\Phi_\lambda$ leads to the generating function

$$(1+s^2)^{\lambda/2}H_\lambda\left[\frac{w+zs}{(1+s^2)^{1/2}}\right]=\sum_{j=0}^{\infty}\binom{\lambda}{j}H_{\lambda-j}(w)H_j(z)s^n,\qquad |s|<1. \tag{2.57}$$

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For derivations of this and many other generating functions for Hermite polynomials, including some from the Airy basis, see [135].

2.3 Separation of Variables for the Schrödinger Equation $(i\partial_t + \partial_{xx} - a/x^2)\Psi = 0$

Next we apply the methods discussed in Section 2.1 to the radial Schrödinger equation for a free particle

$$i\partial_t\Psi = -\partial_{xx}\Psi + \frac{a}{x^2}\Psi. \tag{3.1}$$

Here a is a nonzero real constant, t is real, and $x>0$. As mentioned in the discussion following Table 5, this equation arises for certain values of $a>0$ from free-particle Schrödinger equations for higher-dimensional spaces in which spherical coordinates have been introduced and all of the angular variables separated out (e.g., [70, p. 108]). Here $x=r$, the radial coordinate. We shall show that a group-theoretic study of (3.1) leads naturally to the Schrödinger equations for the radial harmonic oscillator and radial repulsive oscillator. Thus, our analysis of equations (1.2) and (3.1) will incorporate all seven potentials listed in Table 5.

A straightforward computation shows that the complex symmetry algebra of (3.1) is three dimensional, with basis

$$K_{-2}=\partial_t, \qquad K_2 = -t^2\partial_t - tx\partial_x - \frac{t}{2} + \frac{ix^2}{4}, \qquad K^0 = 2t\partial_t + x\partial_x + \tfrac{1}{2} \tag{3.2}$$

and commutation relations

$$[K^0, K_{\pm 2}] = \pm 2K_{\pm 2}, \qquad [K_2, K_{-2}] = K^0.$$

For the alternate basis $\{L_j\}$ where

$$L_1 = K^0, \qquad L_2 = K_{-2} + K_2, \qquad L_3 = K_{-2} - K_2$$

we have the relations

$$[L_1, L_2] = -2L_3, \qquad [L_3, L_1] = 2L_2, \qquad [L_3, L_2] = -2L_1. \tag{3.3}$$

Comparing these results with (1.4), (1.5), (1.7), we see that the real Lie algebra generated by the basis elements is $sl(2,R)$ and that the corresponding local group action of $SL(2,R)$ on functions $\Phi(t,x)$ is given by the operators $\mathbf{T}(A)$, (1.16). The explicit relations between the group and Lie algebra operators follow from (1.17). (Note, however, that in (1.16) we must require $x>0$.)

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The group $SL(2,R)$ acts on $sl(2,R)$ via the adjoint representation and splits the Lie algebra into orbits. Let

$$K = a_2K_2 + a_{-2}K_{-2} + a_0K^0 \in sl(2,R)$$

and set $\alpha = a_2a_{-2} + a_0^2$. It is straightforward to check that α is invariant under the adjoint action and that K lies on the same $SL(2,R)$ orbit as exactly one of these three operators:

$$\begin{aligned} &\text{Case 1} \quad (\alpha<0) \quad K_{-2} - K_2 = L_3; \\ &\text{Case 2} \quad (\alpha>0) \quad K^0; \\ &\text{Case 3} \quad (\alpha=0) \quad K_2. \end{aligned} \tag{3.4}$$

Thus, there are three orbits.

The computation of all R-separable coordinate systems for (3.1) is easily carried out, due to the fact that an R-separable system must also be R-separable for the free-particle equation (set $a=0$ in (3.1)). Thus the possible systems are those listed in Table 6, subject to the additional requirement that they are still R-separable when the potential a/x^2 is added to the free-particle Hamiltonian. We find that only orbit 2 in Table 6 is lost. The results are presented in Table 9, where as usual we have $t = v$.

Table 9 R-Separable Coordinates for the Equation $(i\partial_t + \partial_{xx} - a/x^2)\Psi(t,x) = 0$

Operator K	Coordinates $\{u,v\}$	Multiplier $R = e^{i\mathfrak{R}}$	Separated solutions
1a K_{-2}	$x = u$	$\mathfrak{R} = 0$	Bessel times exponential function
1b K_2	$x = uv$	$\mathfrak{R} = u^2v/4$	Bessel times exponential function
2a K^0	$x = u\sqrt{v}$	$\mathfrak{R} = 0$	Whittaker times exponential function
2b $K_2 + K_{-2}$	$x = u[\pm(1-v^2)]^{1/2}$	$\mathfrak{R} = \pm u^2v/4$	Whittaker times exponential function
3 $K_2 - K_{-2}$	$x = u(1+v^2)^{1/2}$	$\mathfrak{R} = u^2v/4$	Laguerre times exponential function

Note that we have listed two coordinate systems on each of the orbits 1 and 2 even though the systems are $SL(2,R)$ equivalent. These systems are inequivalent with respect to the subgroup of "obvious symmetries" generated by time translation and dilatations. The exact relationship is

$$J(K_2 + K_{-2})J^{-1} = K^0, \qquad J^2K_{-2}J^{-2} = -K_2, \tag{3.5}$$

where J and J^2 are given by (1.21), (1.22).

In analogy with our argument in Section 2.1 we can interpret the operators (3.2) as a Lie algebra of skew-symmetric operators on the Hilbert

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space $L_2(R+)$ of complex-valued Lebesgue square-integrable functions on the positive real line, $0<x<\infty$. This is accomplished by considering t as a fixed parameter and replacing ∂_t with $i\partial_{xx}-ia/x^2$ in expressions (3.2). The resulting operators, when multiplied by i and restricted to the domain of infinitely differentiable functions with compact support in $R+$, are, via Weyl's theorem [122, p. 297], seen to be essentially self-adjoint provided $a \geqslant 3/4$. In the remainder of this section we assume that the constant a satisfies this inequality. We see then that the operators are real linear combinations of the skew-symmetric operators

$$\mathcal{K}_{-2}=i\partial_{xx}-ia/x^2, \qquad \mathcal{K}_2=ix^2/4, \qquad \mathcal{K}^0=x\partial_x+\tfrac{1}{2} \tag{3.6}$$

to which they reduce when $t=0$. Similarly, the skew-symmetric operators

$$\begin{aligned} &\mathcal{L}_1=\mathcal{K}^0=x\partial_x+\tfrac{1}{2}, \qquad \mathcal{L}_2=\mathcal{K}_{-2}+\mathcal{K}_2=i\partial_{xx}-ia/x^2+ix^2/4,\\ &\mathcal{L}_3=\mathcal{K}_{-2}-\mathcal{K}_2=i\partial_{xx}-ia/x^2-ix^2/4, \end{aligned} \tag{3.7}$$

satisfy relations (3.3) and the L_j reduce to $\mathcal{L}_j$ when $t=0$.

In analogy with (1.26) we find

$$\begin{aligned} \exp(t\mathcal{K}_{-2})\mathcal{K}_j\exp(-t\mathcal{K}_{-2})&=K_j,\\ \exp(t\mathcal{K}_{-2})\mathcal{L}_j\exp(-t\mathcal{K}_{-2})&=L_j, \end{aligned} \tag{3.8}$$

where $\exp(t\mathcal{K}_{-2})$ is a unitary operator on $L_2(R+)$. Thus for any $f \in L_2(R+)$ the function $\Psi(t,x)=\exp(t\mathcal{K}_{-2})f(x)$ satisfies $\partial_t\Psi=\mathcal{K}_{-2}\Psi$ or $i\partial_t\Psi=-\partial_{xx}\Psi+a\Psi/x^2$ and $\Psi(0,x)=f(x)$. Also the unitary operators $\exp(K)=\exp(t\mathcal{K}_{-2})\exp(\mathcal{K})\exp(-t\mathcal{K}_{-2})$, $\mathcal{K}\in sl(2,R)$ map solutions of the equation $\partial_t\Psi=\mathcal{K}_{-2}\Psi$ into other solutions.

We will soon demonstrate that the operators $\mathcal{K}_{\pm 2}, \mathcal{K}^0$ generate a global unitary irreducible representation of the universal covering group $\widetilde{SL}(2,R)$ of $SL(2,R)$, (1.37)–(1.40), by operators $\mathbf{U}(g), g\in\widetilde{SL}(2,R)$, on $L_2(R+)$. Assuming this, we see that the operators $\mathbf{T}(g)=\exp(t\mathcal{K}_{-2})\mathbf{U}(g)\exp(-t\mathcal{K}_{-2})$ define a group of unitary symmetries for equation (3.1), with associated infinitesimal operators $K=\exp(t\mathcal{K}_{-2})\mathcal{K}\exp(-t\mathcal{K}_{-2})$. This discussion shows the relationship between our Lie algebra of $\mathcal{K}$ operators and the Schrödinger equation for the radial free particle.

Next consider the operator $\mathcal{L}_3\in sl(2,R)$. If $f\in L_2(R+)$, then $\Psi(t,x)=\exp(t\mathcal{L}_3)f(x)$ satisfies $\partial_t\Psi=\mathcal{L}_3\Psi$ or $i\partial_t\Psi=-\partial_{xx}\Psi+a\Psi/x^2+x^2\Psi/4$, the Schrödinger equation for the radial harmonic oscillator. The unitary operators $\mathbf{V}(g)=\exp(t\mathcal{L}_3)\mathbf{U}(g)\exp(-t\mathcal{L}_3)$ are symmetries of this equation and the associated infinitesimal operators $\exp(t\mathcal{L}_3)\mathcal{K}\exp(-t\mathcal{L}_3)$ can be writ-

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ten as first-order linear differential operators in x and t. Similarly, if $f \in L_2(R+)$, then $\Psi(t,x) = \exp(t\mathcal{L}_2)f(x)$ satisfies $\partial_t\Psi = \mathcal{L}_2\Psi$ or $i\partial_t\Psi = -\partial_{xx}\Psi + a\Psi/x^2 - x^2\Psi/4$, the Schrödinger equation for the repulsive radial oscillator. The operators $\mathbf{W}(g) = \exp(t\mathcal{L}_2)\mathbf{U}(g)\exp(-t\mathcal{L}_2)$ determine the symmetry group of this equation and the associated infinitesimal operators $\exp(t\mathcal{L}_2)\mathcal{K}\exp(-t\mathcal{L}_2)$ can be written as first order in x and t.

It follows from these remarks that the Schrödinger equations (5)–(7) in Table 5 have isomorphic symmetry algebras. For each of these equations the algebra of symmetry operators at time $t=0$ is $sl(2,R)$ with basis (3.6). Although we first derived this symmetry algebra through a study of the Schrödinger equation (5), we could also have obtained it from (6) or (7) in Table 5. Moreover, we have shown how to map a solution of any of these equations to a solution of another equation.

From (3.4) we see that the operators $\mathcal{K}_{-2}$, $\mathcal{L}_3$, $\mathcal{L}_2$, corresponding to the radial free particle, attractive, and repulsive harmonic oscillator Hamiltonians, lie on the same $SL(2,R)$ orbits as the three orbit representatives $\mathcal{K}_2, \mathcal{L}_3$, and $\mathcal{K}^0$, respectively. Our three Hamiltonians correspond to the three orbits of $sl(2,R)$. The remarks concerning expressions (1.31)–(1.33) and the invariance of spectra for operators on an orbit carry over without change to this case, except that the inner product is now

$$\langle h_1, h_2\rangle = \int_0^\infty h_1(x)\bar{h}_2(x)\,dx, \qquad h_j \in L_2(R+). \tag{3.9}$$

Note that if $\{f_\lambda\}$ is the basis of generalized eigenvectors for some $\mathcal{K} \in sl(2,R)$, then $\{\Psi_\lambda(t,x) = \exp(t\mathcal{K}_{-2})f_\lambda(x)\}$ is the basis of eigenvectors for $K = \exp(t\mathcal{K}_{-2})\mathcal{K}\exp(-t\mathcal{K}_{-2})$ and the Ψ_λ satisfy the Schrödinger equation for the radial free particle. Similar remarks hold for the other Hamiltonians.

We first present the well-known results for the spectrum of $\mathcal{L}_3$. The eigenfunction equation is

$$i\mathcal{L}_3 f = \lambda f, \qquad (-\partial_{xx} + a/x^2 + x^2/4)f = \lambda f,$$

and the normalized eigenfunctions are

$$\begin{gathered} f_n^{(3)}(x) = \left(\frac{n!\,2^{-\mu/2}}{\Gamma(n+1+\mu/2)}\right)^{1/2} \exp\left(-\frac{x^2}{4}\right) x^{(\mu+1)/2} L_n^{(\mu/2)}\left(\frac{x^2}{2}\right), \\ \lambda = \lambda_n = -2n - \frac{\mu}{2} - 1, \qquad a = \frac{(\mu^2-1)}{4}, \qquad \mu \geqslant 2,\ n = 0,1,2,\ldots, \end{gathered} \tag{3.10}$$

where $L_n^{(\alpha)}(z)$ is a Laguerre polynomial (see (B.9i)). The $\{f_n^{(3)}\}$ form an ON basis for $L_2(R+)$ [123, p. 108].

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Using known recurrence relations for the Laguerre polynomials, (4.9), we can check that the operators $\mathcal{L}_j$ acting on the $f^{(3)}$ basis define an irreducible representation of $sl(2,R)$ belonging to the discrete series. The Casimir operator is $\frac{1}{2}(\mathcal{L}_1{}^2+\mathcal{L}_2{}^2-\mathcal{L}_3{}^2)=-3/16+a/4$. As is well known [10, 115], this Lie algebra representation extends to a global unitary irreducible representation of $\widetilde{SL}(2,R)$. The matrix elements of the operators $\mathbf{U}(g)$ in an $f^{(3)}$ basis can be found in [115] or [87].

We now compute the operators $\mathbf{U}(g)$ directly. Clearly,

$$\exp(a\mathcal{K}^0)h(x)=\exp(a/2)h(e^a x)$$
$$\exp(\alpha\mathcal{K}_2)h(x)=\exp(i\alpha x^2/4)h(x), \qquad h\in L_2(R+).$$

Furthermore,

$$\exp(\beta\mathcal{L}_3)h(x)=\frac{\exp[\mp i\pi(\mu+2)/4]}{2|\sin\beta|}\,\mathrm{l.i.m.}\int_0^\infty (xy)^{1/2}$$
$$\cdot\exp\left(\pm\frac{i}{4}(x^2+y^2)|\cot\beta|\right)J_{\mu/2}\left(\frac{xy}{2|\sin\beta|}\right)h(y)\,dy, \tag{3.11}$$
$$0<|\beta|<\pi,$$

where we take the upper sign for $\beta>0$ and the lower for $\beta<0$. The additional relation $\exp(\pi\mathcal{L}_3)=\exp[-i\pi(1+\mu/2)]$ allows us to determine $\exp(\beta\mathcal{L}_3)$ for any β. To prove these results we apply the integral operator (3.11) to an $f^{(3)}$basis element, using the Hille–Hardy formula, (4.27), and the fact that $\exp(\beta\mathcal{L}_3)f_n^{(3)}=\exp[-i\beta(2n+\mu/2+1)]f_n^{(3)}$ to check its validity. Since (3.11) is valid on an ON basis and $\exp(\beta\mathcal{L}_3)$ is unitary, the expression must be true for all $h\in L_2(R+)$.

The group multiplication formula

$$\exp(\gamma\mathcal{K}_{-2})=\exp(-\sin(\theta)\cos(\theta)\mathcal{K}_2)\exp(\ln\cos(\theta)\mathcal{K}^0)\exp(\theta\mathcal{L}_3)$$

with $\gamma=\tan\theta$ and expressions (3.10), (3.11) easily yield

$$\exp(\gamma\mathcal{K}_{-2})h(x)=\frac{\exp[\mp i\pi(\mu+2)/4]}{2|\gamma|}\,\mathrm{l.i.m.}\int_0^\infty (xy)^{1/2}$$
$$\cdot\exp\left(i\frac{(x^2+y^2)}{4\gamma}\right)J_{\mu/2}\left(\frac{xy}{2|\gamma|}\right)h(y)\,dy, \tag{3.12}$$

where we take the upper sign for $\gamma>0$ and the lower for $\gamma<0$. A similar

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group-theoretic calculation gives

$$\exp(\varphi\mathcal{L}_2)h(x)=\frac{\exp[\mp i\pi(\mu+2)/4]}{2|\sinh\varphi|}\text{l.i.m.}\int_0^\infty (xy)^{1/2}$$
$$\cdot\exp\left(\frac{i}{4}(x^2+y^2)\coth\varphi\right)J_{\mu/2}\left(\frac{xy}{2|\sinh\varphi|}\right)h(y)\,dy. \quad (3.13)$$

From (3.12) we find that the basis functions $f_n^{(3)}(x)$ map to the ON basis functions $\Psi_n^{(3)}(t,x)=\exp(t\mathcal{K}_{-2})f_n^{(3)}(x)$:

$$\Psi_n^{(3)}(t,x)=(-2)^n\exp\left[\pm i\pi\frac{(\mu+2)}{4}\right]\left(\frac{x^2}{1+t^2}\right)^{(\mu+1)/4}(t-i)^{-\mu/4-3/4-n}$$
$$\times(t+i)^{\mu/4+1/4+n}\exp\left[\frac{x^2(it-1)}{4(1+t^2)}\right]L_n^{(\mu/2)}\left(\frac{1}{2}\frac{x^2}{1+t^2}\right),\quad t\neq 0, \quad (3.14)$$

which are R-separable solutions of (3.1). (Indeed we can derive (3.14) from our knowledge that the $\Psi_n^{(3)}$ are R-separable solutions of the form 3 in Table 9.)

The $SL(2,R)$ orbit containing the operator $\mathcal{L}_2$ (repulsive radial oscillator) also contains $\mathcal{K}^0$, so we merely study the spectral theory for $\mathcal{K}^0$. The results are well known [128]. The eigenfunction equation is

$$i\mathcal{K}^0 f=\lambda f,\qquad \mathcal{K}^0=x\partial_x+\tfrac{1}{2}.$$

The spectrum is continuous and covers the real axis with multiplicity one. The generalized eigenfunctions are

$$f_\lambda^{(2)}(x)=(2\pi)^{-1/2}x^{i\lambda-\frac{1}{2}},\quad -\infty<\lambda<\infty,\qquad \langle f_\lambda^{(2)},f_\zeta^{(2)}\rangle=\delta(\lambda-\zeta). \quad (3.15)$$

Using (3.12) and separation of variables, we find $\Psi_\lambda^{(2)}(t,x)=\exp(t\mathcal{K}_{-2})f_\lambda^{(2)}(x)$ where

$$\Psi_\lambda^{(2)}(t,x)=(2\pi)^{-1/2}\Gamma\left(\frac{i\lambda}{2}-\frac{\mu}{4}+\frac{1}{2}\right)\exp\left[-\frac{\pi}{4}\left(\frac{i\mu}{2}+i-\lambda\right)\right]2^{i\lambda-\mu/2-1}t^{i\lambda/2-\frac{1}{4}}$$
$$\times\left(\frac{x^2}{t}\right)^{(\mu+1)/4}L_{i\lambda/2-\mu/2-1/2}^{(\mu/2)}\left(\frac{ix^2}{4t}\right),\qquad t>0,$$
$$\Psi_\lambda^{(2)}(-t,x)=\overline{\Psi}_{-\lambda}^{(2)}(t,x). \quad (3.16)$$

It follows from our procedure that the basis functions satisfy

$$\langle\Psi_\lambda^{(2)}(t,\cdot),\quad \Psi_\zeta^{(2)}(t,\cdot)\rangle=\delta(\lambda-\zeta)$$

and can be used to expand any $h\in L_2(R+)$.

Finally, the orbit containing $\mathcal{K}_{-2}$, corresponding to the radial free particle, also contains $\mathcal{K}_2$. The spectral theory for $\mathcal{K}_2$ is elementary, since this operator is already diagonalized in our realization. The generalized eigenfunctions are (symbolically)

$$f_\lambda^{(1)}(x)=\delta(x-\lambda), \qquad i\mathcal{K}_2 f_\lambda^{(1)}=(\lambda^2/4)f_\lambda^{(1)}, \qquad \lambda\geqslant 0. \tag{3.17}$$

The spectrum is continuous and covers the positive real axis with multiplicity one. We have $\Psi_\lambda^{(1)}(t,x)=\exp(t\mathcal{K}_{-2})f_\lambda^{(1)}(x)$ or

$$\Psi_\lambda^{(1)}(t,x)=\frac{\exp[\mp i\pi(\mu+2)/4]}{2|t|}(x\lambda)^{1/2}\exp\left(\frac{i(x^2+\lambda^2)}{4t}\right)J_{\mu/2}\left(\frac{x\lambda}{2|t|}\right) \tag{3.18}$$

with $\langle\Psi_\lambda^{(1)},\Psi_\zeta^{(1)}\rangle=\delta(\lambda-\zeta)$. Expansions in the basis $\{\Psi_\lambda^{(1)}\}$ are equivalent to the inversion theorem for the Hankel transform [141, p. 199]. The $\Psi_\lambda^{(1)}$ are basis functions for the operator K_2.

Each of our bases has a continuous generating function of the form (1.55) where now

$$k(t,x,y)=\frac{\exp[\pm i\pi(\mu+2)/4]}{2|t|}(xy)^{1/2}\exp\left[\frac{i(x^2+y^2)}{4t}\right]J_{\mu/2}\left(\frac{xy}{2|t|}\right) \tag{3.19}$$

(see [59]).

The overlap functions $\langle f_\lambda^{(i)},f_\zeta^{(j)}\rangle$ have the same significance as described in Section 2.1. Because of the simplicity of the basis $f_\lambda^{(1)}$, the only nontrivial overlap is

$$\langle f_n^{(3)},f_\lambda^{(2)}\rangle=\frac{1}{2}\left[\frac{\Gamma(n+1+\mu/2)}{\pi n!}2^{\mu/2-2i\lambda-1}\right]^{1/2}\frac{\Gamma(-i\lambda/2+\mu/4+1/2)}{\Gamma(1+\mu/2)}$$
$$\times{}_2F_1\left(\begin{matrix}-n, i\lambda/2+\mu/4+1/2\\ 1+\mu/2\end{matrix}\middle|\,2\right). \tag{3.20}$$

In particular, we note that the overlap functions are dependent on the representatives $f_\lambda^{(i)},f_\zeta^{(j)}$ chosen on each orbit. The most general way to define an overlap function is as a mixed-basis matrix element $\langle f_\lambda^{(i)},\mathbf{U}(g)f_\zeta^{(j)}\rangle, g\in\widetilde{SL}(2,R)$. Some of these elements have been computed in [24].

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2.4 The Complex Equation $(\partial_\tau-\partial_{xx}+a/x^2)\Phi(\tau,x)=0$

Here we study the complexification of equation (3.1). The variables t,x are now allowed to take complex values and a is a given nonzero complex constant. Introducing the new variable $\tau=it$, we can write this equation in the form

$$(\partial_\tau-\partial_{xx}+a/x^2)\Phi(\tau,x)=0. \tag{4.1}$$

The complex symmetry algebra for equation (4.1) is easily seen to be three dimensional with basis

$$\begin{aligned} &J^+=\tau^2\partial_\tau+\tau x\,\partial_x+\tau/2+x^2/4,\\ &J^-=-\partial_\tau,\qquad J^0=\tau\,\partial_\tau+\tfrac{1}{2}x\,\partial_x+\tfrac{1}{4} \end{aligned} \tag{4.2}$$

and commutation relations

$$[J^0,J^\pm]=\pm J^\pm,\qquad [J^+,J^-]=2J^0. \tag{4.3}$$

This Lie algebra is isomorphic to $sl(2,\mathcal{C})$, as we shall soon show. Thus the operators (4.2) can be exponentiated to yield a local representation of $SL(2,\mathcal{C})$ by operators $\mathbf{T}(A), A\in SL(2,\mathcal{C})$, acting on the solution space of (4.1). Furthermore, it is obvious that $SL(2,\mathcal{C})$ acts on the symmetry algebra $sl(2,\mathcal{C})$ via the adjoint representation and decomposes the algebra into $SL(2,\mathcal{C})$ orbits. A straightforward computation shows that there are exactly two orbits in $sl(2,\mathcal{C})$ with orbit representatives J^- and J^0. (The orbits in $sl(2,R)$, (3.4), with representatives $K_{-2}-K_2$ and K^0 both belong to the orbit of $sl(2,\mathcal{C})$ with representative J^0, while the orbit with representative K_2 belongs to the complex J^- (same as J^+) orbit.) Similarly, it can be shown that (4.1) R-separates in exactly two complex analytic coordinate systems. (As usual we consider two systems as equivalent if one can be mapped onto the other by one of the operators $\mathbf{T}(A), A\in SL(2,\mathcal{C})$. The separable systems are listed in Table 10.

Table 10 R-Separable Coordinates for the Complex Equation $(\partial_\tau-\partial_{xx}+a/x^2)\Phi=0$

Operator J	Coordinates $\{u,v\}$	Multiplier R	Separated solutions
1 J^-	$x=u,\tau=v$	$R=1$	Bessel times exponential function
2 J^0	$x=u\sqrt{v}\,,\tau=v$	$R=1$	Laguerre times exponential function

To derive identities relating the various separated solutions of (4.1), we can apply Weisner's method and expand arbitrary analytic solutions in terms of the Laguerre functions (orbit 2). These results are worked out in detail in Chapter 5 of [82], so here we will be very brief.

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As suggested by (3.16) and 2 (Table 10), the Laguerre function solutions of (4.1) correspond to the coordinates $\{s,z\}$ where

$$s=\tau, \qquad z=-x^2/4\tau. \tag{4.4}$$

Furthermore, it is convenient to transform (4.1), which we write $Q\Psi=0$, to the equivalent equation $Q'\Psi'=0$ where $\Psi'=R^{-1}\Psi=s^{1/4}z^{\beta}\Psi$ and $Q'=R^{-1}QR$. The symmetry algebra of the primed equation consists of the operators $J'=R^{-1}JR$ where J belongs to the symmetry algebra of (4.1). Explicitly, we have

$$\begin{gathered} J'^{+}=s^2\partial_s+sz\,\partial_z-sz-ls, \qquad J'^{-}=-\partial_s+(z/s)\partial_z-l/s, \qquad J'^{0}=s\,\partial_s, \\ \beta=l+\tfrac{1}{4}, \qquad a=4(\tfrac{1}{16}-l^2), \qquad l\in\mathcal{C}, \end{gathered} \tag{4.5}$$

and the differential equation $Q'\Psi'=0$ reads

$$(z\,\partial_{zz}-(2l+z)\,\partial_z+s\,\partial_s+l)\Psi'(z,s)=0. \tag{4.6}$$

(In the remainder of this section we will use only the operators (4.5) and equation (4.6), so we henceforth drop the primes.)

Now consider the solutions Ψ of (4.6) that are eigenfunctions of J^0:

$$J^0\Psi=m\Psi \Rightarrow \Psi=f_m(z)s^m.$$

Substituting this solution into (4.6), we see that the ordinary differential equation obtained by factoring out s^m is of the form (B.7). Thus the $f_m(z)$ are confluent hypergeometric functions. In particular, the functions

$$\Psi_m(z,s)=L_{m+l}^{(-2l-1)}(z)s^m \tag{4.7}$$

satisfy these equations.

Note that for $l\in\mathcal{C}$, $2l\neq 0,1,2,\ldots$, and $m=-l+n$, $n=0,1,2,\ldots$, the solutions Ψ_m are well defined and reduce to polynomials in the variable z, the Laguerre polynomials $L_n^{(-2l-1)}(z)$, (B.9i). To understand the significance of these polynomial solutions it is helpful to consider the system of equations

$$J^-\Phi=0, \qquad J^0\Phi=-l\Phi,$$

which has the solution $\Phi(z,s)=s^{-l}$, unique to within a multiplicative constant. It is easy to verify that Φ also satisfies the differential equation (4.6). Now we use our knowledge of the symmetry algebra of (4.6) to construct other solutions. If $\Phi(z,s)$ is any analytic function of (z,s), it is a

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simple consequence of local Lie theory that

$$\exp(\alpha J^+)\Phi(z,s)=(1-\alpha s)^l\exp\left[-\frac{\alpha zs}{(1-\alpha s)}\right]\Phi\left(\frac{z}{1-\alpha s},\frac{s}{1-\alpha s}\right)$$
$$=\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}(J^+)^n\Phi(z,s),\qquad |\alpha s|<1. \tag{4.8}$$

Furthermore, if Φ is a solution of (4.6), then so are $(J^+)^n\Phi$ and $\exp(\alpha J^+)\Phi$, provided they are well defined. Substituting our solution $\Phi = s^{-l}$ in this expression, we find

$$(1-\alpha s)^{2l}s^{-l}\exp[-\alpha zs/(1-\alpha s)]=\sum_{n=0}^{\infty}\alpha^n\Phi_n(z,s),$$
$$\Phi_n=(1/n!)(J^+)^n\Phi,\qquad \Phi_0=\Phi,\qquad |\alpha s|<1. \tag{4.8$'$}$$

Using the definition of Φ_n, the commutation relations (4.3), and a straightforward induction argument, we can derive the recurrence formulas

$$J^0\Phi_n=(n-l)\Phi_n,\quad J^+\Phi_n=(n+1)\Phi_{n+1},\quad J^-\Phi_n=(2l-n+1)\Phi_{n-1}, \tag{4.9}$$
$$\Phi_{-1}\equiv 0,\ n=0,1,2,\ldots.$$

Furthermore, comparison of these results with the recurrence formulas (B.8) and $\Phi_0 = s^{-l}$ yields

$$\Phi_n(z,s)=\Psi_{n-l}(z,s)=L_n^{(-2l-1)}(z)s^{n-l},\qquad n=0,1,2,\ldots. \tag{4.10}$$

Thus (4.8′) becomes a generating function for Laguerre polynomials:

$$(1-\alpha)^{2l}\exp[-\alpha z/(1-\alpha)]=\sum_{n=0}^{\infty}\alpha^nL_n^{(-2l-1)}(z),\qquad |\alpha|<1. \tag{4.11}$$

To derive more identities for Laguerre polynomials we need to determine the operators $\mathbf{T}(A)$ that define the action of the local symmetry group $SL(2,\mathcal{C})$ on the solution space of (4.6). Recall that $SL(2,\mathcal{C})$ is the complex Lie group of complex 2×2 matrices A with determinant $+1$:

$$A=\begin{pmatrix} a & b\\ c & d\end{pmatrix},\qquad a,b,c,d\in\mathcal{C},\qquad ad-bc=1. \tag{4.12}$$

The Lie algebra $sl(2,\mathcal{C})$ of this group consists of all complex 2×2 matrices A with trace zero:

$$A=\begin{pmatrix} \alpha & \beta\\ \gamma & -\alpha\end{pmatrix},\qquad \alpha,\beta,\gamma\in\mathcal{C}.$$

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As a basis for $sl(2,\mathbb{C})$ we choose the matrices

$$\mathcal{J}^+=\begin{pmatrix}0&-1\\0&0\end{pmatrix},\quad \mathcal{J}^-=\begin{pmatrix}0&0\\-1&0\end{pmatrix},\quad \mathcal{J}^0=\begin{bmatrix}\frac{1}{2}&0\\0&-\frac{1}{2}\end{bmatrix},\tag{4.13}$$

with commutation relations

$$[\mathcal{J}^0,\mathcal{J}^\pm]=\pm\mathcal{J}^\pm,\qquad [\mathcal{J}^+,\mathcal{J}^-]=2\mathcal{J}^0.$$

Since these relations coincide with (4.3) we see that the symmetry algebra of (4.1) is indeed isomorphic to $sl(2,\mathbb{C})$.

A straightforward computation (see [82, Section 1.4]) shows that if $A\in SL(2,\mathbb{C})$ is given by (4.12) with $d\neq 0$, then

$$\begin{aligned}&A=\exp(\beta\mathcal{J}^+)\exp(\gamma\mathcal{J}^-)\exp(\tau\mathcal{J}^0),\\&e^\tau=d^{-2},\beta=-b/d,\gamma=-cd.\end{aligned}\tag{4.14}$$

This expression enables us to parametrize a neighborhood of the identity in $SL(2,\mathbb{C})$. Next we exponentiate the operators (4.5) to determine the corresponding local multiplier representation of $SL(2,\mathbb{C})$ by operators $\mathbf{T}(A)$ acting on analytic functions $\Phi(z,s)$. According to (4.14) we have

$$\mathbf{T}(A)=\exp((-b/d)J^+)\exp(-cdJ^-)\exp(-2\ln dJ^0)\tag{4.15}$$

for A in a suitably small neighborhood of the identity element. We have already computed $\exp(\alpha J^+)$ in (4.8). Similar computations yield

$$\begin{aligned}&\exp(\gamma J^-)\Phi(z,s)=(1-\gamma/s)^l\Phi\left(z(1-\gamma/s)^{-1},s-\gamma\right),\\&\exp(\tau J^0)\Phi(z,s)=\Phi(z,e^\tau s).\end{aligned}$$

Composing these operators, we find

$$\begin{aligned}\mathbf{T}(A)\Phi(z,s)=&(d+bs)^l\left(a+\frac{c}{s}\right)^l\exp\left[\frac{bzs}{(d+bs)}\right]\\&\times\Phi\left(\frac{zs}{(as+c)(d+bs)},\frac{as+c}{d+bs}\right),\quad \left|\frac{c}{as}\right|<1,\left|\frac{bs}{d}\right|<1,\end{aligned}\tag{4.16}$$

defined for all analytic functions Φ such that the right-hand side makes sense. Note that $\mathbf{T}(A)$ maps an analytic solution of (4.6) to another solution.

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We now apply the operators $\mathbf{T}(A)$ to a basis function Φ_m and expand the resulting function in terms of the $\{\Phi_n\}$ basis:

$$\mathbf{T}(A)\Phi_m(z,s)=\sum_{n=0}^{\infty} T_{nm}(A)\Phi_n(z,s). \tag{4.17}$$

For A close to the identity we can compute the matrix elements $T_{nm}(A)$ directly from the Lie algebra relations (4.9). The computation is simplified through the construction of another model of our Lie algebra representation (4.9). Following Section 5.2 of [82], we choose basis functions

$$f_n(w)=(n!)^{-1}\Gamma(n-2l)w^n, \qquad n=0,1,2,\ldots,$$

and operators

$$J^+=w^2\frac{d}{dw}-2lw, \qquad J^-=-\frac{d}{dw}, \qquad J^0=w\frac{d}{dw}-l. \tag{4.18}$$

It is easy to check that these operators and basis functions satisfy (4.3) and (4.9). Furthermore, by applying (4.15), we can show that the corresponding local group action of $SL(2,\mathbb{C})$ is determined by operators $\mathbf{T}(A)$ where

$$\mathbf{T}(A)f(w)=(bw+d)^{2l}f\left(\frac{aw+c}{bw+d}\right), \qquad |bw/d|<1. \tag{4.19}$$

The matrix elements are given by

$$\mathbf{T}(A)f_m(w)=\sum_{n=0}^{\infty} T_{nm}(A)f_n(w)$$

or

$$d^{2l-m}(1+bw/d)^{2l-m}(aw+c)^m\Gamma(m-2l)/m!$$

$$=\sum_{n=0}^{\infty} T_{nm}(A)\Gamma(n-2l)w^n/n!, \qquad |bw/d|<1. \tag{4.20}$$

Expanding the left-hand side of (4.20) in a power series in w and computing the coefficient of w^n, we obtain

$$T_{nm}(A)=\frac{a^n d^{2l-m}c^{m-n}\Gamma(m-2l)}{\Gamma(m-n+1)\Gamma(n-2l)}\,{}_2F_1\left(\begin{matrix}-n,m-2l\\ m-n+1\end{matrix}\,\middle|\,\frac{bc}{ad}\right). \tag{4.21}$$

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Moreover, the local representation property $\mathbf{T}(AA')=\mathbf{T}(A)\mathbf{T}(A')$ valid for A, A' in a suitably small neighborhood of the identity in $SL(2,\mathcal{C})$ implies

$$T_{nm}(AA')=\sum_{k=0}^{\infty} T_{nk}(A)T_{km}(A').$$

Substituting (4.10) and (4.21) into (4.17), we obtain the identities

$$\begin{aligned}
&(1+bs/d)^{2l-m}(1+c/as)^m \exp[bzs/(d+bs)] \\
&\quad \times L_m^{(-2l-1)}[z(1+bc)^{-1}(1+c/as)^{-1}(1+bs/d)^{-1}] \\
&= \sum_{n=0}^{\infty} \frac{n!}{m!}(-sb/d)^{n-m}\Gamma(n-m+1)^{-1}\,{}_2F_1\left(\begin{matrix}-m, n-2l\\ n-m+1\end{matrix}\,\middle|\,\frac{bc}{ad}\right) \\
&\quad \times L_n^{(-2l-1)}(z), \qquad |bs/d|<1, d=(1+bc)/a,
\end{aligned} \tag{4.22}$$

valid for all integers $m\geqslant 0$ and for all $l\in\mathcal{C}$ such that $2l\neq 0,1,2,\ldots$. (In (4.21) and (4.22) we use the fact that ${}_2F_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix}\,\middle|\,z\right)/\Gamma(\gamma)$ is an entire function of α,β,γ to define this expression for γ a negative integer.)

If $a=d=s=1, c=0$, the identity simplifies to

$$(1-b)^{2l-m}\exp[-bz/(1-b)]L_m^{(-2l-1)}\left(z(1-b)^{-1}\right)$$

$$=\sum_{n=0}^{\infty}\binom{n+m}{n}b^n L_{m+n}^{(-2l-1)}(z), \qquad |b|<1.$$

For $m=0$ this last expression becomes (4.11) with $\alpha=b$. When $a=d=s=1, b=0$, then (4.22) simplifies to

$$(1+c)^m L_m^{(-2l-1)}\left(z(1+c)^{-1}\right)=\sum_{n=0}^{m}\binom{m-2l-1}{n}c^n L_{m-n}^{(-2l-1)}(z).$$

Similar identities can be derived for the basis functions (4.7) that are not polynomials in z, that is, $m+l\neq 0,1,2,\ldots$. For these results see Section 5.8 of [82].

To derive more general identities for Laguerre functions we use the full power of Weisner's method. If $\Psi(z,s)$ is an analytic solution of (4.6) with convergent Laurent series expansion

$$\Psi(z,s)=\sum_m f_m(z)s^m,$$

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then the $f_m(z)$ are confluent hypergeometric (Laguerre) functions (linear combinations of $L_{m+l}^{(-2l-1)}(z)$ and $z^{2l+1}L_{m-l-1}^{(2l+1)}(z)$ for $2l$ not an integer). If in addition Ψ is analytic in a neighborhood of $z=0$, it follows that $f_m(z)=c_m L_{m+l}^{(-2l-1)}(z), c_n \in \not{C}$. Thus,

$$\Psi(z,s)=\sum_m c_m L_{m+l}^{(-2l-1)}(z)s^m. \tag{4.23}$$

The expansion (4.22) is an example of such an identity where we can use Lie algebraic techniques to explicitly compute the coefficients c_m. However, this is no longer true for the example $\mathbf{T}(A)\Psi_p, p\in\not{C}$, where

$$A=\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}$$

and Ψ_p is given by (4.7). Then we have

$$s^{-l}(1-s)^{l-p}\exp\left[-\frac{zs}{1-s}\right]L_{p+l}^{(-2l-1)}\left(\frac{zs}{1-s}\right)$$

$$=\sum_{n=0}^{\infty} c_n L_n^{(-2l-1)}(z)s^{-l+n}, \qquad |s|<1.$$

This expansion is not of the form (4.22) because A is bounded away from the identity and p is not necessarily an integer. We can easily evaluate the constants c_n by setting $z=0$. The result is

$$(1-s)^{l-p}\exp\left(-\frac{zs}{1-s}\right)L_{p+l}^{(-2l-1)}\left(\frac{zs}{1-s}\right)$$

$$=\sum_{n=0}^{\infty}(-s)^n\frac{\Gamma(l-p+1)\Gamma(p-l)}{\Gamma(n-2l)\Gamma(l-p-n+1)\Gamma(l+p+1)}L_n^{(-2l-1)}(z), \tag{4.24}$$

$$|s|<1,$$

a generating function for Laguerre polynomials.

For our next example we choose the generating function Ψ in (4.23) to be an eigenfunction of the operator J^-. Since J^- belongs to orbit 1 in Table 10, we see that in the coordinates x,τ (suitably transformed from (4.1) to (4.6)), we can choose Ψ as separable and expressible in terms of a Bessel function. Indeed, a simultaneous solution Ψ of (4.6) and $J^-\Psi=-\Psi$ is

$$\Psi(z,s)=s^{-l}e^s(zs)^{(2l+1)/2}J_{-2l-1}\left(2(zs)^{1/2}\right). \tag{4.25}$$

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Moreover the function

$$\Psi' = \mathbf{T}(A)\Psi(z,s) = s^{-l}(d+bs)^{-1}\exp\left[\frac{(a+bz)s+c}{d+bs}\right](zs)^{(2l+1)/2}$$

$$\times J_{-2l-1}\left(\frac{2(zs)^{1/2}}{d+bs}\right), \qquad \left|\frac{bs}{d}\right|<1,$$

where A is given by (4.12), also satisfies (4.6) and the equation $(J^-)'\Psi' = -\Psi'$ where

$$(J^-)' = \mathbf{T}(A)J^-\mathbf{T}(A^{-1}) = -b^2J^+ + d^2J^- - 2bdJ^0.$$

Since $w^{-m}J_m(w)$ is an entire function of w for all $m \in \mathcal{C}, \Psi'(z,s)$ has a Laurent expansion in s of the form

$$\Psi'(z,s) = \sum_{n=0}^{\infty} c_n(A)L_n^{(-2l-1)}(z)s^n, \qquad |bs/d|<1.$$

Setting $z=0$, we find

$$(d+bs)^{2l}\exp\left(\frac{as+c}{d+bs}\right) = \sum_{n=0}^{\infty} c_n(A)\Gamma(n-2l)\frac{s^n}{n!}, \qquad \left|\frac{bs}{d}\right|<1,$$

and comparing this expression with (4.22) in the case where $m=0$, we obtain

$$c_n(A) = \exp\left(\frac{ac}{1+bc}\right)\frac{n!}{\Gamma(n-2l)}a^{-2l}\left(-\frac{ab}{1+bc}\right)^n L_n^{(-2l-1)}\left(\frac{a}{b(1+bc)}\right),$$

$$n=0,1,2,\ldots, \quad 2l \neq 0,1,2,\ldots.$$

For the group parameter $c=0$, the result of our computation is the identity

$$(1+abs)^{-1}\exp\left[\frac{(a+bz)s}{a^{-1}+bs}\right](a^2zs)^{(2l+1)/2}J_{-2l-1}\left(\frac{2(zs)^{1/2}}{a^{-1}+bs}\right)$$

$$= \sum_{n=0}^{\infty}\frac{n!}{\Gamma(n-2l)}(-abs)^n L_n^{(-2l-1)}\left(\frac{a}{b}\right)L_n^{(-2l-1)}(z), \quad |abs|<1. \tag{4.26}$$

If $a=1, b=0$, (4.26) reduces to

$$e^s(zs)^{(2l+1)/2}J_{-2l-1}\left(2(zs)^{1/2}\right) = \sum_{n=0}^{\infty} L_n^{(-2l-1)}(z)s^n/\Gamma(n-2l),$$

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while if $a=iy^{1/2}, b=iy^{-1/2}$ it reduces to the Hille–Hardy formula

$$(1-s)^{-1}(-yzs)^{(2l+1)/2}\exp\left[-\frac{s(y+z)}{1-s}\right]J_{-2l-1}\left(\frac{2i(yzs)^{1/2}}{1-s}\right)$$

$$=\sum_{n=0}^{\infty}\frac{n!}{\Gamma(n-2l)}L_n^{(-2l-1)}(y)L_n^{(-2l-1)}(z)s^n, \qquad |s|<1, \quad (4.27)$$

see [37, p. 189].

2.5 Separation of Variables for the Schrödinger Equation $(i\partial_t+\partial_{xx}+\partial_{yy})\Psi=0$

Now we apply the methods of Section 2.1 to time-dependent Schrödinger equations in two space variables:

$$i\partial_t\Psi=-\partial_{xx}\Psi-\partial_{yy}\Psi+V(x,y)\Psi \qquad (5.1)$$

where V is the potential function. Boyer [18] has classified all equations (5.1) that admit a nontrivial symmetry algebra of first-order differential operators. He has shown that (a) the maximal dimension for a symmetry algebra is nine, (b) this maximum occurs only for the four potentials V listed in Table 11, and (c) the algebras of maximal dimension are isomorphic. (There are actually four classes of such potentials corresponding to orbits in the symmetry algebra. We have simply listed a representative from each class in Table 11.) Niederer [102] has shown that the four equations with maximal symmetry are in fact equivalent. Here we will examine this equivalence explicitly and relate it to separation of variables.

As in Section 2.1 we begin with a study of the free-particle Schrödinger equation, which we write in the form

$$Q\Psi=0, \qquad Q=i\partial_t+\partial_{x_1x_1}+\partial_{x_2x_2}, \qquad (x_1,x_2)=(x,y). \quad (5.2)$$

The complex symmetry algebra $\mathcal{G}_3^c$ of this equation is nine dimensional

Table 11 Potentials $V(x,y)$ with Maximal Symmetry

V	Name of System
(1) 0	Free particle
(2) $k(x^2+y^2), k>0$	Harmonic oscillator
(3) $-k(x^2+y^2), k>0$	Repulsive oscillator
(4) $ax, a\neq 0$	Free fall (linear potential)

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with basis

$$K_2=-t^2\partial_t-t(x_1\partial_{x_1}+x_2\partial_{x_2})-t+i(x_1^2+x_2^2)/4, \qquad K_{-2}=\partial_t,$$
$$P_j=\partial_{x_j}, \qquad B_j=-t\partial_{x_j}+ix_j/2, \qquad j=1,2, \tag{5.3}$$
$$M=x_1\partial_{x_2}-x_2\partial_{x_1}, \qquad E=i, \qquad D=x_1\partial_{x_1}+x_2\partial_{x_2}+2t\partial_t+1$$

and commutation relations

$$[D,K_{\pm 2}]=\pm 2K_{\pm 2}, \qquad [D,B_j]=B_j, \quad [D,P_j]=-P_j,$$
$$[D,M]=0, \qquad [M,K_{\pm 2}]=0, \quad [P_j,M]=(-1)^{j+1}P_l,$$
$$[B_j,M]=(-1)^{j+1}B_l, \quad [K_2,K_{-2}]=D, \quad [K_2,B_j]=0,$$
$$[K_{-2},B_j]=-P_j, \qquad [K_{-2},P_j]=0, \quad [P_j,K_2]=B_j,$$
$$[P_j,B_j]=\tfrac{1}{2}E, \qquad [P_j,B_l]=0, \quad j,l=1,2, \quad j\neq l, \tag{5.4}$$

with E in the center of $\mathcal{G}_3^c$. In the following we will study only the *Schrödinger algebra* $\mathcal{G}_3$, the real Lie algebra with basis (5.3).

A second useful basis for $\mathcal{G}_3$ is given by the operators B_j, P_j, E, which generate the five-dimensional Weyl algebra $\mathcal{W}_2$, the operator M, and the three operators L_1, L_2, L_3 where

$$L_1=D, \qquad L_2=K_2+K_{-2}, \qquad L_3=K_{-2}-K_2. \tag{5.5}$$

Here,

$$[L_1,L_2]=-2L_3, \qquad [L_3,L_1]=2L_2, \qquad [L_2,L_3]=2L_1, \tag{5.6}$$

so that the L_ν form a basis for the Lie algebra $sl(2,R)$; compare with (1.11). It follows that $\mathcal{G}_3$ is a semidirect product of $sl(2,R)\oplus o(2)$ and $\mathcal{W}_2$. Here $o(2)$ is the one-dimensional Lie algebra spanned by M.

Using standard results from Lie theory, we can exponentiate the operators (5.3) to obtain a local Lie group G_3 (the *Schrödinger group*) of operators acting on the space $\mathcal{F}$ of locally analytic functions of the real variables t, x_j and mapping solutions of (5.2) into solutions. The required computations can be carried out in simple analogy with expressions (1.15)–(1.19).

The action of the Weyl group W_2 is given by operators

$$\mathbf{T}(\mathbf{w},\mathbf{z},\rho)=\exp(w_1B_1)\exp(z_1P_1)\exp(w_2B_2)\exp(z_2P_2)\exp(\rho E),$$
$$\mathbf{w}=(w_1,w_2), \quad \mathbf{z}=(z_1,z_2),$$

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such that

$$\mathbf{T}(\mathbf{w},\mathbf{z},\rho)\mathbf{T}(\mathbf{w}',\mathbf{z}',\rho') = \mathbf{T}\left(\mathbf{w}+\mathbf{w}',\mathbf{z}+\mathbf{z}',\rho+\rho'+\frac{1}{2}\mathbf{w}'\cdot\mathbf{z}\right), \tag{5.7}$$

where

$$\mathbf{T}(\mathbf{w},\mathbf{z},\rho)\Phi(t,\mathbf{x}) = \exp[i(2\mathbf{x}\cdot\mathbf{w} - t\mathbf{w}\cdot\mathbf{w}+4\rho)/4]\Phi(t,\mathbf{x}-t\mathbf{w}+\mathbf{z}), \qquad \Phi\in\mathfrak{F}.$$

Here $\mathbf{x}\cdot\mathbf{w} = x_1w_1 + x_2w_2$. The action of $SO(2)$ is given by $\mathbf{T}(\theta) = \exp(\theta M)$,

$$\mathbf{T}(\theta)\mathbf{T}(\theta') = \mathbf{T}(\theta+\theta')$$

where

$$\mathbf{T}(\theta)\Phi(t,\mathbf{x}) = \Phi(t,\mathbf{x}\Theta), \quad \Theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{5.8}$$

Finally, the action of $SL(2,R)$ is given by operators $\mathbf{T}(A)$,

$$\mathbf{T}(A)\Phi(t,\mathbf{x}) = \exp\left[\frac{i\beta\mathbf{x}\cdot\mathbf{x}}{4(\delta+t\beta)}\right](\delta+t\beta)^{-1}\Phi\left[\frac{\gamma+t\alpha}{\delta+t\beta},(\delta+t\beta)^{-1}\mathbf{x}\right],$$
$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,R), \tag{5.9}$$

where

$$\mathbf{T}(A)\mathbf{T}(B) = \mathbf{T}(AB), \qquad A,B\in SL(2,R).$$

The one-parameter subgroups of $SL(2,R)$ generated by $K_{\pm 2}, L_1, L_2, L_3$, respectively, are given by expressions (1.17). The adjoint actions of $SO(2)$ and $SL(2,R)$ on W_2 are

$$\mathbf{T}^{-1}(A)\mathbf{T}(\mathbf{w},\mathbf{z},\rho)\mathbf{T}(A) = \mathbf{T}(\mathbf{w}',\mathbf{z}',\rho'),$$
$$\rho' = \rho + (\mathbf{w}'\cdot\mathbf{z}' - \mathbf{w}\cdot\mathbf{z})/4, \mathbf{w}' = \delta\mathbf{w}+\beta\mathbf{z}, \mathbf{z}' = \alpha\mathbf{z}+\gamma\mathbf{w}$$
$$\mathbf{T}^{-1}(\theta)\mathbf{T}(\mathbf{w},\mathbf{z},\rho)\mathbf{T}(\theta) = \mathbf{T}(\mathbf{w}\Theta,\mathbf{z}\Theta,\rho). \tag{5.10}$$

These relations define G_3 as a semidirect product of $SL(2,R)\oplus SO(2)$ and W_2:

$$g = (A,\theta,\mathbf{v})\in G_3, \quad A\in SL(2,R), \quad \theta\in SO(2), \quad \mathbf{v} = (\mathbf{w},\mathbf{z},\rho)\in W_2;$$
$$\mathbf{T}(g) = \mathbf{T}(A)\mathbf{T}(\theta)\mathbf{T}(\mathbf{v}). \tag{5.11}$$

The group G_3 acts on the Lie algebra $\mathcal{G}_3$ of differential operators via the adjoint representation

$$K \to K^g = \mathbf{T}(g) K \mathbf{T}^{-1}(g)$$

and this action splits $\mathcal{G}_3$ into G_3 orbits. We will classify the orbit structure of the factor algebra $\tilde{\mathcal{G}} \cong \mathcal{G}_3/\{E\}$ where $\{E\}$ is the center of $\mathcal{G}_3$. Let $K \in \mathcal{G}_3$ and let a_2, a_0, a_{-2}, respectively, be the coefficients corresponding to K_2, D, K_{-2} in the expansion of $K \neq 0$ in terms of the basis (5.3). Setting $\alpha = a_2 a_{-2} + a_0^2$, we find that α is invariant under the adjoint representation.

The following list is a complete set of orbit representatives in the sense that any $K \neq 0$ lies on the same G_3 orbit as exactly one of the operators in this list:

$$\begin{aligned}
&\text{Case 1} \quad (\alpha<0) \quad K_{-2}-K_2+\beta^2 M, |\beta| \neq 1, K_{-2}-K_2+M+\gamma B_1,\\
&\text{Case 2} \quad (\alpha>0) \quad D+\beta M,\\
&\text{Case 3} \quad (\alpha=0) \quad K_2+M, K_2+P_1, K_2, M, P_1+B_2, P_1.
\end{aligned} \tag{5.12}$$

We next consider the problem of determining higher-order differential operators S that are symmetries of (5.2). The special structure of (5.2) enables us to simplify this problem somewhat. Since we will only apply S to solutions Ψ of $Q\Psi = 0$, without loss of generality we can require that S contain no derivatives in t. In other words, wherever ∂_t appears in S, we can replace it by $i(\partial_{x_1x_1} + \partial_{x_2x_2})$. Another way to view this is to note that if S is a symmetry operator, then so is $S' = S + XQ$ where X is an arbitrary differential operator. Moreover, $S'\Psi = S\Psi$ for any solution Ψ of (5.2). There is a unique choice of X such that S' contains no derivatives with respect to t.

With this in mind we see that only the operators P_j, B_j, E, generating the Weyl algebra, and M are first order or less in the x_j. The elements $K_2 = -i(B_1^2 + B_2^2)$, $K_{-2} = i(P_1^2 + P_2^2)$, and $D = -i(\{B_1, P_1\} + \{B_2, P_2\})$ are second order. (These equalities are valid modulo the replacement of ∂_t by $i(\partial_{x_1x_1} + \partial_{x_2x_2})$.) More generally, we can compute all symmetries S_2 that are second order or less in x_1 and x_2:

$$S = \sum_{i,j=1}^{2} a_{ij}(\mathbf{x},t)\,\partial_{x_ix_j} + \sum_{j=1}^{2} b_j(\mathbf{x},t)\,\partial_{x_j} + c(\mathbf{x},t).$$

A tedious computation shows that such S form a 20-dimensional vector space. A basis for this space is provided by the zeroth-order operator E, the five first-order operators P_j, B_j, M, and the three second-order operators $iK_{\pm 2}, iD$ listed earlier, plus the eleven second-order operators

$$B_1^2 - B_2^2,\quad B_1P_1 - B_2P_2,\quad P_1^2 - P_2^2,\quad \{B_1, M\},\quad \{B_2, M\},\quad \{P_1, M\},$$
$$\{P_2, M\},\quad B_1B_2,\quad P_1P_2,\quad B_1P_2 + B_2P_1,\quad M^2.$$

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It follows that all second-order symmetries are symmetric quadratic forms in B_j, P_j, E, and M.

The problem of R-separation of variables for equation (5.2) was solved in [20]. In the following we will concern ourselves only with those systems which, in addition to admitting R-separable solutions, have the property that the separated factors satisfy three ordinary differential equations, one in each of the separation variables. Since (5.2) is an equation in three variables, there are now two separation constants associated with each separable system.

The relationship between orbits of first-order symmetries (5.12) and separable systems is now rather tenuous. It is true that corresponding to any first-order symmetry K we can find a new system of coordinates $\{u,v,w\}$, not unique, such that the variable u can be separated out of equation (5.2). (See the analogous discussion for the Helmholtz equation in Section 1.2.) However, the resulting equation in v,w may not permit separation of variables. Thus, diagonalization of a symmetry operator K may correspond to a partial, but not total, separation of variables.

The results of [20] are as follows. Corresponding to every R-separation of variables for (5.2) we can find a pair of differential operators K,S such that:

1. K and S are symmetries of (5.2) and $[K,S]=0$.
2. $K\in\tilde{\mathcal{G}}_3$; that is, K is first order in x_1,x_2, and t.
3. S is second order in x_1,x_2 and contains no term in ∂_t.

The R-separation of variables is characterized by the simultaneous equations

$$Q\Psi=0,\qquad K\Psi=i\lambda\Psi,\qquad S\Psi=\mu\Psi. \tag{5.14}$$

In particular, the eigenvalues λ,μ are the usual separation constants for the R-separable solutions Ψ.

It follows that K lies in the symmetry algebra $\tilde{\mathcal{G}}_3$ while S can be expressed as a symmetric quadratic form in B_j,P_j,E, and M. Thus the possible coordinate systems in which (5.2) R-separates can always be characterized by eigenfunction equations for operators at most second order in the enveloping algebra of $\mathcal{G}_3$. The possible commuting operators K,S,R-separable coordinates $\{u,v,w\}$, and separated solutions are listed in Table 12.

The notation for the coordinate systems that we introduce in Table 12 requires some comment. Coordinate systems 13–17 are not of much interest to us because they result from the fact that if P_1 is diagonalized, the free-particle Schrödinger equation (5.2) essentially collapses to the free-particle equation (1.2). However, the remaining coordinate systems are associated with the Hamiltonians for the free-particle, linear potential, harmonic oscillator, and repulsive oscillator in exactly the same manner as

Table 12 Operators and R-Separable Coordinates for the Equation $(i\partial_t + \partial_{xx} + \partial_{yy})\Psi = 0$.

Operators K,S	Coordinates $\{u,v,w\}$	Multiplier $e^{i\mathfrak{R}}$	Separated solutions
1a Fc^1 K_2, B^2	$x=uw$ $y=vw$	$\mathfrak{R}=(u^2+v^2)w/4$	Exponential Exponential
1b Fc^2 K_{-2}, P_1^2	$x=u$ $y=v$	0	Exponential Exponential
2a Fr^1 K_2, M^2	$x=uw\cos v$ $y=uw\sin v$	$u^2w/4$	Bessel Exponential
2b Fr^2 K_{-2}, M^2	$x=u\cos v$ $y=u\sin v$	0	Bessel Exponential
3a Fp^1 $K_2, \{B_2, M\}$	$x=w(u^2-v^2)/2$ $y=uvw$	$(u^2+v^2)^2w/16$	Parabolic cylinder Parabolic cylinder
3b Fp^2 $K_{-2}, \{P_2, M\}$	$x=(u^2-v^2)/2$ $y=uv$	0	Parabolic cylinder Parabolic cylinder
4a Fe^1 $K_2, M^2-B_2^2$	$x=w\cosh u\cos v$ $y=w\sinh u\sin v$	$(\sinh^2 u+\cos^2 v)w/4$	Modified Mathieu Mathieu
4b Fe^2 $K_{-2}, M^2-P_2^2$	$x=\cosh u\cos v$ $y=\sinh u\sin v$	0	Modified Mathieu Mathieu
5a Lc^1 $K_2-2aP_1-2bP_2$, $B_2^2+2bEP_2$	$x=uw+a/w$ $y=vw+b/w$	$(u^2+v^2)w/4$ $-(au+bv)/2w$	Airy Airy
5b Lc^2 $K_{-2}+2aB_1+2bB_2$, $P_1^2-2aEB_1$	$x=u+aw^2$ $y=v+bw^2$	$(au+bv)w$	Airy Airy
6a Lp^1 K_2-aP_1, $\{B_2,M\}-aP_2^2$	$x=(u^2-v^2)w/2$ $+a/w$ $y=uvw$	$(u^2+v^2)^2w/16$ $-(u^2-v^2)a/4w$	Anharmonic oscillator Anharmonic oscillator
6b Lp^2 $K_{-2}-aB_1$, $\{P_2,M\}+aB_2^2$	$x=(u^2-v^2)/2+aw^2/2$ $y=uv$	$(u^2-v^2)aw/4$	Anharmonic oscillator Anharmonic oscillator
7 Oc $K_{-2}-K_2$, $P_1^2+B_1^2$	$x=u(1+w^2)^{1/2}$ $y=v(1+w^2)^{1/2}$	$(u^2+v^2)w/4$	Hermite Hermite
8 Or $K_{-2}-K_2$, M^2	$x=u(1+w^2)^{1/2}\cos v$ $y=u(1+w^2)^{1/2}\sin v$	$u^2w/4$	Laguerre Exponential
9 Oe $K_{-2}-K_2$, $M^2-P_2^2-B_2^2$	$x=(1+w^2)^{1/2}\cosh u\cos v$ $y=(1+w^2)^{1/2}\sinh u\sin v$	$(\sinh^2 u+\cos^2 v)w/4$	Ince Ince

Table 12 (*Continued*)

Operators K, S	Coordinates $\{u,v,w\}$	Multiplier $e^{i\mathcal{R}}$	Separated solutions
10a Rc^1 $D,\{B_1,P_1\}$	$x=u\lvert w\rvert^{1/2}$ $y=v\lvert w\rvert^{1/2}$	0	Parabolic cylinder Parabolic cylinder
10b Rc^2 $K_{-2}+K_2,$ $P_1^2-B_1^2$	$x=u\lvert w^2-1\rvert^{1/2}$ $y=v\lvert w^2-1\rvert^{1/2}$	$\varepsilon(u^2+v^2)w/4$	Parabolic cylinder Parabolic cylinder
11a Rr^1 D,M^2	$x=u\lvert w\rvert^{1/2}\cos v$ $y=u\lvert w\rvert^{1/2}\sin v$	0	Whittaker Exponential
11b Rr^2 $K_{-2}+K_2,$ M^2	$x=\lvert w^2-1\rvert^{1/2}u\cos v$ $y=\lvert w^2-1\rvert^{1/2}u\sin v$	$\varepsilon u^2\, w/4$	Whittaker Exponential
12a Re^1 $D,$ $M^2-\frac{1}{2}\{B_2,P_2\}$	$x=\lvert w\rvert^{1/2}\cosh u\cos v$ $y=\lvert w\rvert^{1/2}\sinh u\sin v$	0	Ince Ince
12b Re^2 $K_{-2}+K_2,$ $M^2-P_2^2+B_2^2$	$x=\lvert w^2-1\rvert^{1/2}\cosh u\cos v$ $y=\lvert w^2-1\rvert^{1/2}\sinh u\sin v$	$\varepsilon(\sinh^2 u+\cos^2 v)/4$	Ince Ince
13 $L1$ $P_1,B_2^2-2bEP_2$	$x=u$ $y=vw+b/w$	$wv^2/4-bv/2w$	Exponential Airy
14 $L2$ $P_1,P_2^2-2aEB_2$	$x=u$ $y=v+aw^2$	avw	Exponential Airy
15 $O1$ $P_1,P_2^2+B_2^2$	$x=u$ $y=v(1+w^2)^{1/2}$	$wu^2/4$	Exponential Hermite
16 $R1$ $P_1,\{B_2,P_2\}$	$x=u$ $y=v\lvert w\rvert^{1/2}$	0	Exponential Parabolic cylinder
17 $R2$ $P_1,P_2^2-B_2^2$	$x=u$ $y=v\lvert w^2-1\rvert^{1/2}$	$\varepsilon v^2 w/4$	Exponential Parabolic cylinder

described in Section 2.1. We denote each system in the form Ab^j. The capital letter corresponds to the type of Hamiltonian; that is, $F\leftrightarrow$free particle, $L\leftrightarrow$linear potential, $O\leftrightarrow$harmonic oscillator, $R\leftrightarrow$repulsive oscillator. The small letter indicates the type of coordinate used in each of these Hamiltonians; that is, $c\leftrightarrow$Cartesian, $r\leftrightarrow$radial (polar) coordinates, $p\leftrightarrow$ parabolic, and $e\leftrightarrow$elliptic coordinates. The superscript j is used to distinguish two systems on the same G_3 orbit.

In each case $w=t$ and the separated solution in the variable w is an exponential function. In the last column of Table 12 we list first the form of the separated solution in u followed by the separated solution in v. The symbol $\varepsilon=\pm 1$ denotes the sign of $1-w^2$ and the anharmonic oscillator

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functions are solutions of a differential equation of the form

$$f''(u)+(\lambda u^2+\alpha u^4-\beta)f(u)=0, \qquad \alpha,\beta\in R. \tag{5.15}$$

From the viewpoint of Galilean and dilatation symmetry alone there are 26 inequivalent coordinate systems. (As indicated in the remarks following Table 6, this list of 26 Galilean–dilatation inequivalent coordinate systems is not precisely a list of orbits because certain pairs of Galilean–dilatation orbits yield separable coordinates that differ only in the sign of a parameter.) However, we can also regard two coordinate systems as equivalent if the first can be transformed to the second under the action of some $g\in G_3$. In terms of operators, the system described by K,S is equivalent to the system described by K',S' if, under the adjoint action of G_3 on the enveloping algebra of $\mathcal{G}_3$, the two-dimensional space spanned by K,S can be mapped onto the two-dimensional space spanned by K',S'. Under this more general equivalence relation not all of the 26 systems are inequivalent. Indeed, the systems denoted Ab^1 and Ab^2 lie on the same two-dimensional orbits, so that there are only 17 equivalence classes of orbits. (For convenience in applications, the representatives of orbits 5, 6, 13, 14 contain parameters a,b. Some of these parameters can be normalized to ± 1 with the dilatation symmetry.)

We can describe these equivalences in terms of the operator $J=\exp[\pi(K_2-K_{-2})/4]$:

$$J\Phi(t,\mathbf{x})=\frac{\sqrt{2}}{(1+t)}\exp\Big[i(1+t)^{-1}\frac{\mathbf{x}\cdot\mathbf{x}}{4}\Big]\Phi\Big(\frac{t-1}{t+1},\sqrt{2}\,(t+1)^{-1}\mathbf{x}\Big), \qquad \Phi\in\mathcal{F}. \tag{5.16}$$

Note that $J^2=\exp[\pi(K_2-K_{-2})/2]$, and

$$\begin{aligned} &J^2\Phi(t,\mathbf{x})=t^{-1}\exp[i\mathbf{x}\cdot\mathbf{x}/4t]\Phi(-t^{-1},t^{-1}\mathbf{x}),\\ &J^4\Phi(t,\mathbf{x})=-\Phi(t,-\mathbf{x}), \qquad J^8\Phi(t,\mathbf{x})=\Phi(t,\mathbf{x}). \end{aligned} \tag{5.17}$$

It is easy to show that $J(K_{-2}+K_2)J^{-1}=D$, and, checking the adjoint action of J on second-order operators, we can verify that the three coordinate systems Rc^2, Rr^2, Re^2 are equivalent under J to the three systems Rc^1, Rr^1, Re^1, respectively.

Denoting the adjoint action of J^2 on $K\in\mathcal{G}_3$ by $K'=J^2KJ^{-2}$, we find $P'_j=-B_j, B'_j=P_j, K'_{-2}=-K_2, K'_2=-K_{-2}, D'=-D, M'=M, E'=E$, so that the six pairs of the form Fa^1, Fa^2 or La^1, La^2 are equivalent under J^2.

We next demonstrate that the operators (5.3) can be interpreted as a Lie algebra of skew-symmetric operators on the Hilbert space $L_2(R_2)$ of complex-valued Lebesgue square-integrable functions on the plane. This is accomplished by considering t as a fixed parameter and replacing ∂_t by

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$i(\partial_{x_1x_1}+\partial_{x_2x_2})$ in expressions (5.3). It is then straightforward to show that the resulting operators, multiplied by i and restricted to the domain of infinitely differentiable functions on R_2 with compact support, have unique self-adjoint extensions. In fact, these operators are real linear combinations of the operators

$$\begin{aligned}&\mathcal{K}_2=i(x_1^2+x_2^2)/4, \qquad \mathcal{K}_{-2}=i(\partial_{x_1x_1}+\partial_{x_2x_2}), \qquad \mathcal{P}_j=\partial_{x_j},\\ &\mathcal{B}_j=ix_j/2, \qquad \mathfrak{M}=x_1\partial_{x_2}-x_2\partial_{x_1}, \qquad \mathcal{E}=i,\\ &\mathcal{D}=x_1\partial_{x_1}+x_2\partial_{x_2}+1, \qquad j=1,2.\end{aligned} \tag{5.18}$$

Note that when the parameter $t=0$, the operators (5.3) reduce to (5.18). Thus the script operators (5.18) satisfy the same commutation relations (5.4) as do the italic operators (5.3). More specifically, we have the general identity

$$\exp(t\mathcal{K}_{-2})\mathcal{K}\exp(-t\mathcal{K}_{-2})=K \tag{5.19}$$

relating corresponding script ($\mathcal{K}$) and italic (K) operators. Here exp $(t\mathcal{K}_{-2})$ is a unitary operator on $L_2(R_2)$ which corresponds to time translation for the free-particle system. It is shown in [67, p. 493] that

$$\exp(a\mathcal{K}_{-2})f(\mathbf{x})=\text{l.i.m.}(4\pi ia)^{-1}\int\int_{-\infty}^{\infty}\exp\left[-\frac{(\mathbf{x}-\mathbf{y})^2}{4ia}\right]f(\mathbf{y})\,dy_1\,dy_2, \tag{5.20}$$
$$f\in L_2(R_2).$$

If $f\in L_2(R_2)$, then we can show that $\Psi(t,\mathbf{x})=\exp(t\mathcal{K}_{-2})f(\mathbf{x})$ satisfies $\partial_t\Psi=\mathcal{K}_{-2}\Psi$ or $i\partial_t\Psi=-\Delta_2\Psi$ (for almost every t) whenever f is in the domain of $\mathcal{K}_{-2}$, and $\Psi(0,\mathbf{x})=f(\mathbf{x})$. Also, the unitary operators $\exp(\alpha K)=\exp(t\mathcal{K}_{-2})\exp(\alpha\mathcal{K})\exp(-t\mathcal{K}_{-2})$ map Ψ into $\Phi=\exp(\alpha K)\Psi$, which also satisfies $i\partial_t\Phi=-\Delta_2\Phi$ for each linear combination $\mathcal{K}$ of the operators (5.18). Thus the operators $\exp(\alpha K)$ are symmetries of (5.2).

We will see later that the operators (5.18) generate a global unitary irreducible representation of G_3 on $L_2(R_2)$. Assuming this here, we let $\mathbf{U}(g), g\in G_3$, be the corresponding unitary operators and set $\mathbf{T}(g)=\exp(t\mathcal{K}_{-2})\mathbf{U}(g)\exp(-t\mathcal{K}_{-2})$. It then follows that the $\mathbf{T}(g)$ are unitary symmetries of (5.2) with associated infinitesimal operators $K=\exp(t\mathcal{K}_{-2})\,\mathcal{K}\exp(-t\mathcal{K}_{-2})$.

Next consider the operator $\mathcal{L}_3=\mathcal{K}_{-2}-\mathcal{K}_2=i\left[\Delta_2-\frac{1}{4}(x_1^2+x_2^2)\right]$. If $f\in L_2(R_2)$, then $\Psi(t,\mathbf{x})=\exp(t\mathcal{L}_3)f(\mathbf{x})$ satisfies $\partial_t\Psi=\mathcal{L}_3\Psi$ or

$$i\partial_t\Psi=-\Delta_2\Psi+\frac{1}{4}(x_1^2+x_2^2)\Psi \tag{5.21}$$

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and $\Psi(0,\mathbf{x})=f(\mathbf{x})$. Similarly, the unitary operators $\mathbf{V}(g)=\exp(t\mathcal{L}_3)\times \mathbf{U}(g)\exp(-t\mathcal{L}_3)$ are symmetries of (5.21), the Schrödinger equation for the harmonic oscillator, and the associated infinitesimal operators $\exp(t\mathcal{L}_3)\times \mathcal{K}\exp(-t\mathcal{L}_3)$ can be expressed as first-order differential operators in t and $\mathbf{x}$. Analogous statements hold for the operator $\mathcal{L}_2=\mathcal{K}_{-2}+\mathcal{K}_2= i\left(\Delta_2+\frac{1}{4}(x_1^2+x_2^2)\right)$ with associated equation $\partial_t\Psi=\mathcal{L}_2\Psi$ or

$$i\,\partial_t\Psi=-\Delta_2\Psi-\frac{1}{4}(x_1^2+x_2^2)\Psi \tag{5.22}$$

(Schrödinger equation for the repulsive oscillator) and the operator $\mathcal{K}_{-2}-\mathcal{B}_1=i(\Delta_2-x_1/2)$ with associated equation $\partial_t\Psi=(\mathcal{K}_{-2}-\mathcal{B}_1)\Psi$,

$$i\,\partial_t\Psi=-\Delta_2\Psi+\frac{1}{2}x_1\Psi \tag{5.23}$$

(linear potential).

These remarks show explicitly the equivalence of equations (5.2), (5.21)–(5.23). Though we have chosen to start with equation (5.2), an analysis of any of the other equations would have led to the same (script) symmetry algebra (5.18).

From Table 12 we see that, except for coordinates 13–17, which are essentially the same as those discussed in Section 2.1, every R-separable coordinate system corresponds to a G_3 orbit that contains exactly one of the Hamiltonian operators $i\mathcal{K}_{-2}, i\mathcal{L}_3, i\mathcal{L}_2$, or $i(\mathcal{K}_{-2}-\mathcal{B}_1)$. Thus each coordinate system is naturally associated with one of these four Hamiltonians. Moreover, the remarks accompanying expressions (1.29), (1.30) also hold here: an R-separable coordinate system for the free-particle equation corresponds to a truly separable coordinate system for one of the other three Schrödinger equations, namely, that equation whose Hamiltonian is diagonalized by the system.

Consider a pair of commuting self-adjoint operators $i\mathcal{K}$, $\mathcal{S}$ where $\mathcal{K}\in\mathcal{G}_3$ and $\mathcal{S}$ is a symmetric quadratic operator in the enveloping algebra of $\mathcal{G}_3$. These operators have a common spectral resolution; that is, there is a complete set of (generalized) eigenfunctions $f_{\lambda,\mu}(\mathbf{x})$ in $L_2(R_2)$ with

$$i\mathcal{K}f_{\lambda,\mu}=\lambda f_{\lambda,\mu},\qquad \mathcal{S}f_{\lambda,\mu}=\mu f_{\lambda,\mu},\qquad \langle f_{\lambda,\mu},f_{\lambda',\mu'}\rangle=\delta_{\lambda\lambda'}\delta_{\mu\mu'}, \tag{5.24}$$

where

$$\langle h_1,h_2\rangle=\int\!\!\int_{-\infty}^{\infty}h_1(\mathbf{x})\bar{h}_2(\mathbf{x})\,dx_1,dx_2,\qquad h_j\in L_2(R_2) \tag{5.25}$$

(see [77, p. 76]). Now suppose $i\mathcal{K}',\mathcal{S}'$ are another pair of commuting self-adjoint operators on the same G_3 orbit as $i\mathcal{K},\mathcal{S}$. Then by renormaliza-

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tion of these operators if necessary, it follows that there is a $g\in G_3$ such that

$$\mathcal{K}'=\mathbf{U}(g)\mathcal{K}\mathbf{U}(g^{-1}),\qquad \mathcal{S}'=\mathbf{U}(g)\mathcal{S}\mathbf{U}(g^{-1}).$$

Thus the spectral resolution of the primed pair is identical to that for the unprimed pair. Indeed for $f'_{\lambda,\mu}=\mathbf{U}(g)f_{\lambda,\mu}$ we have

$$i\mathcal{K}'f'_{\lambda,\mu}=\lambda f'_{\lambda,\mu},\qquad \mathcal{S}'f'_{\lambda,\mu}=\mu f'_{\lambda,\mu},\quad \langle f'_{\lambda,\mu},f'_{\lambda',\mu'}\rangle=\delta_{\lambda\lambda'}\delta_{\mu\mu'},\tag{5.26}$$

and the $f'_{\lambda,\mu}$ form a complete ON set in $L_2(R_2)$.

In the following we will frequently need the spectral resolution of a pair $i\mathcal{K},\mathcal{S}$ where $i\mathcal{K}$ is one of the four Hamiltonians listed earlier. However, in many cases we will be able to use the unitary symmetry operators $\mathbf{U}(g)$ to construct an equivalent pair $i\mathcal{K}',\mathcal{S}'$ whose spectral resolution is much simpler to compute. This information will then provide the spectral resolution of the original pair.

As a special case of these remarks, consider the operator $\mathcal{K}_{-2}=i\Delta_2$. If $\{f_{\lambda,\mu}\}$ is the basis (5.24) of generalized eigenfunctions for the pair $\mathcal{K},\mathcal{S}$, then $\{f'_{\lambda,\mu}(t,\mathbf{x})=\exp(t\mathcal{K}_{-2})f_{\lambda,\mu}(\mathbf{x})\}$ is the corresponding basis of generalized eigenvectors for the italic operators

$$K=\exp(t\mathcal{K}_{-2})\mathcal{K}\exp(-t\mathcal{K}_{-2}),\qquad S=\exp(t\mathcal{K}_{-2})\mathcal{S}\exp(-t\mathcal{K}_{-2})$$

and the $f'_{\lambda,\mu}(t,\mathbf{x})$ are also solutions of the free-particle Schrödinger equation (5.2). Similar remarks hold for the other Hamiltonians. This clarifies the relationship between the two $(\mathbf{x})$ and three $(\mathbf{x},t)$ variable models of G_3.

We now explicitly compute the spectral resolutions of the pairs of commuting operators listed in Table 12. We begin with the Oc orbit and the two-variable model; that is, we determine the spectral resolution of the pair $\mathcal{L}_3=\mathcal{K}_{-2}-\mathcal{K}_2,\ \mathcal{P}_1^2+\mathcal{B}_1^2$. Equations (5.24) are

$$\left[-\Delta_2+(x_1^2+x_2^2)/4\right]f=\lambda f,\qquad \left(\partial_{x_1x_1}-x_1^2/4\right)f=\mu f.$$

Note that these equations are separable in the variables x_1,x_2. Comparing with (1.34) we find the well-known ON basis of eigenfunctions

$$\begin{aligned} f_{\lambda,\mu}&=oc_{n,m}(\mathbf{x})=(2^{m+n}\pi n!m!)^{-1/2}\exp(-\mathbf{x}\cdot\mathbf{x}/4)H_n(x_1/\sqrt{2})\\ &\quad\times H_n(x_2/\sqrt{2}),\qquad \mu=-n-\tfrac{1}{2},\quad \lambda+\mu=m+\tfrac{1}{2},\quad n,m=0,1,2,\ldots,\\ &\langle oc_{n,m},oc_{n',m'}\rangle=\delta_{nn'}\delta_{mm'},\end{aligned}\tag{5.27}$$

where $H_n(x)$ is a Hermite polynomial.

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At this point we can show directly that the operators (5.18) exponentiate to yield a global unitary irreducible representation of G_3. Indeed, from the recurrence relations (2.28), (2.29), (2.33) for the Hermite polynomials we can see that the operators $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ acting on the *oc* basis define a unitary representation of $sl(2,R)$ that is a direct sum of representations from the discrete series, and the $\mathcal{W}_2$ operators define a unitary irreducible representation of $\mathcal{W}_2$. As follows from the work of Bargmann [10, 115], this Lie algebra representation extends to a global representation of G_3, irreducible since its restriction to W_2 is already irreducible.

We now compute the unitary operators $\mathbf{U}(g)$ on $L_2(R_2)$. The operators

$$\mathbf{U}(\mathbf{w},\mathbf{z},\rho)=\exp(w_1\mathcal{B}_1)\exp(z_1\mathcal{P}_1)\exp(w_2\mathcal{B}_2)\exp(z_2\mathcal{P}_2)\exp(\rho\mathcal{E})$$

defining the irreducible representation of W_2 are

$$\mathbf{U}(\mathbf{w},\mathbf{z},\rho)h(\mathbf{x})=\exp(i\rho+i\mathbf{w}\cdot\mathbf{x}/2)h(\mathbf{x}+\mathbf{z}),\qquad h\in L_2(R_2).\quad(5.28)$$

The operator $\mathbf{U}(\theta)=\exp(\theta\mathcal{M})$ is

$$\mathbf{U}(\theta)h(\mathbf{x})=h(\mathbf{x}\Theta)\quad(5.29)$$

where Θ is given by (5.8). The operators $\mathbf{U}(A), A\in SL(2,R)$, are more difficult to compute. We have the integral operator $\exp(a\mathcal{K}_{-2})$ from (5.20). Also,

$$\exp(b\mathcal{K}_2)h(\mathbf{x})=\exp(ib\mathbf{x}\cdot\mathbf{x}/4)h(\mathbf{x}),\qquad \exp(c\mathcal{D})h(\mathbf{x})=e^ch(e^c\mathbf{x}).\quad(5.30)$$

Using group multiplication in $SL(2,R)$, we find

$$\exp(\varphi\mathcal{L}_2)=\exp(\tanh(\varphi)\mathcal{K}_2)\exp(\sinh(\varphi)\cosh(\varphi)\mathcal{K}_{-2})\exp(-\ln(\cosh\varphi)\mathcal{D})$$

so that

$$\exp(\varphi\mathcal{L}_2)h(\mathbf{x})=\frac{\exp(i\coth(\varphi)\mathbf{x}\cdot\mathbf{x}/4)}{4\pi i\sinh\varphi}$$

$$\cdot\int\int_{-\infty}^{\infty}\exp\left[\frac{i}{4}\left(-\frac{2}{\sinh\varphi}\mathbf{x}\cdot\mathbf{y}+\coth(\varphi)\mathbf{y}\cdot\mathbf{y}\right)\right]$$

$$\cdot h(\mathbf{y})\,dy_1\,dy_2,\qquad \varphi\neq 0\quad(5.31)$$

(In this and the following two integrals, use of the short form l.i.m. is to be

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understood.) Similar computations yield

$$\exp(\theta \mathcal{L}_3)h(\mathbf{x}) = \frac{\exp(i\cot(\theta)\mathbf{x}\cdot\mathbf{x}/4)}{4\pi i \sin\theta} \cdot \int\int_{-\infty}^{\infty} \exp\left[\frac{i}{4}\left(-\frac{2}{\sin\theta}\mathbf{x}\cdot\mathbf{y} + \cot(\theta)\mathbf{y}\cdot\mathbf{y}\right)\right] \cdot h(\mathbf{y})\, dy_1\, dy_2, \qquad \theta \neq n\pi, \tag{5.32}$$

$$\exp[\rho(\mathcal{K}_{-2} + a\mathcal{B}_1)]h(\mathbf{x}) = \frac{\exp[i(a\rho x_1/2 - a^2\rho^3/12)]}{4\pi i\rho} \cdot \int\int_{-\infty}^{\infty} \exp\left\{\frac{i}{4\rho}\left[(x_1 - a\rho^2 - y_1)^2 + (x_2 - y_2)^2\right]\right\} \cdot h(\mathbf{y})\, dy_1\, dy_2, \qquad \rho \neq 0. \tag{5.33}$$

2.6 Bases and Overlaps for the Schrödinger Equation

From (5.20) and (5.27) it follows that the basis functions $oc_{n,m}(\mathbf{x})$ map to the ON basis functions $Oc_{n,m}(t,\mathbf{x}) = \exp(t\mathcal{K}_{-2})oc_{n,m}(\mathbf{x})$, solutions of the Schrödinger equation, where

$$Oc_{n,m}(t,\mathbf{x}) = (2^{m+n+1}\pi n!\, m!)^{1/2} \exp\left[\frac{i\pi(m+n-1)}{2} - \frac{(u^2+v^2)(1-iw)}{4}\right] \times \left(\frac{w+i}{w-i}\right)^{(m+n)/2} (w-i)^{-1} H_m(u/\sqrt{2}\,) H_n(v/\sqrt{2}\,) \tag{6.1}$$

with

$$x_1 = u(1+w^2)^{1/2}, \qquad x_2 = v(1+w^2)^{1/2}, \qquad t = w.$$

The functions (6.1) correspond to the separable coordinate system Oc in Table 12.

Next we compute the spectral resolution for the system Or:

$$i(\mathcal{K}_{-2} - \mathcal{K}_2)f = \lambda f, \qquad \mathfrak{M}^2 f = \mu f.$$

The basis of eigenfunctions is

$$\begin{aligned} or^{+}_{n,m}(\mathbf{x}) &= [\,m!/2^m\pi(n+m)!\,]^{1/2} \exp(-r^2/4) r^m L_n^{(m)}(r^2/2) \cos m\theta, \\ or^{-}_{n,m}(\mathbf{x}) &= \tan(n\theta)\, or^{+}_{n,m}(\mathbf{x}), \end{aligned} \tag{6.2}$$

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where n,m are integers with $m \geqslant 1, n \geqslant 0$ and $x_1 = r\cos\theta, x_2 = r\sin\theta$. The eigenvalues λ,μ are related to m,n via $\lambda = 2n+m+1, \mu = -m^2$. For $m=0$ there is one additional eigenvector

$$or^{+}_{\lambda,0}(\mathbf{x}) = (2/\pi n!)^{1/2}\exp(-r^2/4)L_n^{(0)}(r^2/2). \tag{6.3}$$

Here, the $L_n^{(\alpha)}(r)$ are Laguerre polynomials. The orthogonality relations are

$$\langle or^{\varepsilon}_{n,m}, or^{\varepsilon'}_{n',m'}\rangle = \delta_{\varepsilon\varepsilon'}\delta_{nn'}\delta_{mm'}, \qquad \varepsilon,\varepsilon' = \pm .$$

The three-variable basis functions $Or_{n,m}(t,\mathbf{x}) = \exp(t\mathcal{K}_{-2})or_{n,m}(\mathbf{x})$ are

$$Or^{+}_{n,m}(t,\mathbf{x}) = K\left(\frac{m!}{\pi^3 2^m (n+m)!}\right)^{1/2}\frac{(-1)^{m+n}}{2^{2m}}\frac{(w+i)^{m/2+n}}{(w-i)^{m/2+n+1}}$$

$$\cdot\exp\left[\frac{u^2(iw-1)}{4}\right]L_n^{(m)}\left(\frac{u^2}{2}\right)\cos mv,$$

$$Or^{-}_{n,m}(t,\mathbf{x}) = \tan(mv)Or^{+}_{n,m}(t,\mathbf{x}), \qquad m \geqslant 1. \tag{6.4}$$

For $m=0$ we have $K=\sqrt{2}$; otherwise $K=1$. Also,

$$x_1 = (1+w^2)^{1/2}u\cos v, \qquad x_2 = (1+w^2)^{1/2}u\sin v, \qquad t = w.$$

The equations for the system Oe,

$$i(\mathcal{K}_{-2} - \mathcal{K}_2)f = \lambda f, \qquad (\mathfrak{M}^2 - \mathcal{P}_2^2 - \mathcal{B}_2^2)f = \mu f,$$

separate in elliptic coordinates $x_1 = \cosh\zeta\cos\eta, x_2 = \sinh\zeta\sin\eta$. We obtain the ON basis

$$\begin{aligned} oe^{+}_{p,m}(\mathbf{x}) &= \pi^{-1}hc_p^m(i\zeta,\tfrac{1}{2})hc_p^m(\eta,\tfrac{1}{2}),\\ oe^{-}_{p,m}(\mathbf{x}) &= \pi^{-1}hs_p^m(i\zeta,\tfrac{1}{2})hs_p^m(\eta,\tfrac{1}{2}), \end{aligned} \tag{6.5}$$

where

$$hc_p^m(\eta,\tfrac{1}{2}) = \exp(-\tfrac{1}{4}\cos 2\eta)C_p^m(\eta,\tfrac{1}{2})$$

$$hs_p^m(\eta,\tfrac{1}{2}) = \exp(-\tfrac{1}{4}\cos 2\eta)S_p^m(\eta,\tfrac{1}{2}),$$

and m,p are integers with $o \leqslant m \leqslant p, (-1)^{m-p} = 1$. The eigenvalues λ,μ are related to p,m via $\lambda = p+1, \mu = \lambda/2 + a_p^m(\frac{1}{2})$ or $\mu = \lambda/2 + b_p^m(\frac{1}{2})$ and the

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orthogonality relations are

$$\langle oe^{\varepsilon}_{p,m}, oe^{\varepsilon'}_{p',m'}\rangle = \delta_{\varepsilon\varepsilon'}\delta_{pp'}\delta_{mm'}, \qquad \varepsilon, \varepsilon' = \pm.$$

The functions $C_p^m(\eta,\zeta), S_p^m(\eta,\zeta)$ are Ince polynomials [7], that is, polynomial solutions of period 2π for the Whittaker–Hill equation

$$\frac{d^2v}{d\eta^2} + \zeta \sin 2\eta \frac{dv}{d\eta} + (a - p\zeta \cos 2\eta)v = 0. \qquad (6.6)$$

This equation has been investigated in detail by Arscott [8], and it is his notation for the solutions and eigenvalues that we use. The $C_p^m(\eta,\zeta)$ are polynomials of order p in $\cos\eta$ and correspond to the eigenvalues $a = a_p^m(\zeta)$, while the $S_p^m(\eta,\zeta)$ are polynomials of order p in $\sin\eta$ and correspond to the eigenvalues $a = b_p^m(\zeta)$.

The three-variable basis functions $Oe_{p,m}(t,\mathbf{x}) = \exp(t\mathcal{K}_{-2})oe_{p,m}(\mathbf{x})$ are

$$Oe^+_{p,m}(t,\mathbf{x}) = (\lambda_p^{m+}/\pi)\exp\left[iw(\sinh^2 u + \cos^2 v)/4\right]$$
$$\cdot (w-i)^{(p/2)+1}(w+i)^{-p/2} hc_p^m\left(iu, \tfrac{1}{2}\right) hc_p^m\left(v, \tfrac{1}{2}\right) \qquad (6.7)$$

where

$$x_1 = (1+w^2)^{1/2}\cosh u \cos v, \qquad x_2 = (1+w^2)^{1/2}\sinh u \sin v, \qquad t = w.$$

The expression for $Oe^-_{p,m}(t,\mathbf{x})$ is as above except the phase factor $\lambda_p^{m+}, |\lambda_p^{m+}| = 1$, is replaced by λ_p^{m-} and the functions $hc_p^m(\eta,\zeta)$ are replaced by $hs_p^m(\eta,\zeta)$. The constants $\lambda_p^{m\pm}$ are calculable in principle from a knowledge of the explicit form of the Ince polynomials. Note that the expression $Oe_{p,m} = \exp(t\mathcal{K}_{-2})oe_{p,m}$ is a nontrivial relation satisfied by products of Ince polynomials. We are able to evaluate this integral (in a manner analogous to the evaluation of (3.38), Section 1.3) because we know in advance that the integral is an R-separable solution of the Schrödinger equation in the variables u, v, w.

For the remaining cases in Table 12 there are always two coordinate systems associated with each orbit. For simplicity we shall always treat the coordinate system with superscript 1. The corresponding results for system 2 follow immediately upon application of the unitary operators J or J^2, (5.16), (5.17).

The Fc system is defined by equations

$$i\mathcal{K}_2 f = -\tfrac{1}{4}\gamma^2 f, \qquad \mathcal{B}_1 f = \tfrac{1}{2}i\gamma\cos(\alpha) f,$$

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and has a basis of generalized eigenfunctions

$$fc_{\gamma,\alpha}(\mathbf{x})=r^{-1/2}\delta(r-\gamma)\delta(\theta-\alpha), \qquad 0\leqslant\alpha<2\pi, \quad 0\leqslant\gamma,$$
$$\langle fc_{\gamma,\alpha}, fc_{\gamma',\alpha'}\rangle=\delta(\gamma-\gamma')\delta(\alpha-\alpha'), \qquad x_1=r\cos\theta, \qquad x_2=r\sin\theta. \tag{6.8}$$

The three-variable basis functions $Fc_{\gamma,\alpha}(t,\mathbf{x})=\exp(t\mathcal{K}_{-2})fc_{\gamma,\alpha}(\mathbf{x})$ are

$$Fc_{\gamma,\alpha}(t,\mathbf{x})=\frac{\gamma^{1/2}}{4\pi it}\exp\left\{\frac{i\left[(x_1-\gamma\cos\alpha)^2+(x_2-\gamma\sin\alpha)^2\right]}{4t}\right\}. \tag{6.9}$$

The Fr system is defined by

$$i\mathcal{K}_2 f=-\tfrac{1}{4}\gamma^2 f, \qquad i\mathfrak{M}f=-mf$$

with basis

$$fr_{\gamma,m}(\mathbf{x})=(2\pi r)^{-1/2}\delta(r-\gamma)e^{im\theta}, \qquad \langle fr_{\gamma,m}, fr_{\gamma',m'}\rangle=\delta(\gamma-\gamma')\delta_{mm'}. \tag{6.10}$$

Here $0\leqslant\gamma$, $m=0,\pm1,\ldots$ and r,θ are polar coordinates. The three-variable basis functions are

$$Fr_{\gamma,m}(t,\mathbf{x})=\left(\frac{\gamma}{2\pi}\right)^{1/2}\frac{i^{m-1}}{2t}\exp\left[\frac{i(r^2+\gamma^2)}{4t}\right]\exp(im\theta)J_m\left(\frac{-r\gamma}{2t}\right) \tag{6.11}$$

where $J_m(z)$ is a Bessel function.

The F_p system is determined by equations

$$i\mathcal{K}_2 f=-\tfrac{1}{4}\gamma^2 f, \qquad \{\mathcal{B}_2, \mathfrak{M}\}f=-\mu\gamma f.$$

with eigenbasis

$$fp^{+}_{\gamma,\mu}(\mathbf{x})=(2\pi r)^{-1/2}(1+\cos\theta)^{-i\mu/2-\frac{1}{4}}(1-\cos\theta)^{i\mu/2-\frac{1}{4}}\delta(r-\gamma), \quad -\pi\leqslant\theta<0,$$
$$=0, \qquad 0\leqslant\theta\leqslant\pi,$$
$$fp^{-}_{\gamma,\mu}(\mathbf{x})=fp^{-}_{\gamma,\mu}(r,\theta)=fp^{+}_{\gamma,\mu}(r,-\theta). \tag{6.12}$$

Here r,θ are polar coordinates, $0\leqslant\gamma$, $-\infty<\mu<\infty$, and the spectrum is continuous with multiplicity two. The orthogonality relations are

$$\langle fp^{\pm}_{\gamma,\mu}, fp^{\pm}_{\gamma',\mu'}\rangle=\delta(\gamma-\gamma')\delta(\mu-\mu'), \qquad \langle fp^{\pm}_{\gamma,\mu}, fp^{\mp}_{\gamma',\mu'}\rangle=0.$$

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The three-variable basis functions are

$$Fp^{+}_{\gamma,\mu}(t,\mathbf{x})=\frac{i\gamma^{1/2}\exp(i\gamma^2/4t)}{2^3\pi t\cos(i\mu\pi)}\exp\left(\frac{i(\xi^2+\eta^2)^2}{16t}\right)$$

$$\cdot[D_{-i\mu/2-\frac{1}{2}}(\sigma\xi t^{-1/2})D_{i\mu/2-\frac{1}{2}}(\sigma\eta t^{-1/2})+D_{-i\mu/2-\frac{1}{2}}(-\sigma\xi t^{-1/2})$$

$$\cdot D_{i\mu/2-\frac{1}{2}}(-\sigma\eta t^{-1/2})],\qquad t>0,$$

$$Fp^{+}_{\gamma,\mu}(-t,\mathbf{x})=\overline{Fp}^{\,+}_{\gamma,-\mu}(t,\mathbf{x}),$$

$$Fp^{-}_{\gamma,\mu}(t,x_1,x_2)=Fp^{+}_{\gamma,\mu}(t,x_1,-x_2), \tag{6.13}$$

where $\sigma=\gamma^{1/2}\exp(i\pi/4)$ and ξ,η are parabolic coordinates

$$2x_1=\xi^2-\eta^2,\qquad x_2=\xi\eta.$$

The *Fe* system is defined by equations

$$i\mathcal{K}_2 f=-\tfrac{1}{4}\gamma^2 f,\qquad (\mathfrak{M}^2+4\mathfrak{B}_1^2-4\mathfrak{B}_2^2)f=-\mu f$$

(equivalent to 4a in Table 12 since $\mathcal{K}_2=-i(\mathfrak{B}_1^2+\mathfrak{B}_2^2)$). The basis functions are

$$fe_{\gamma,n}(\mathbf{x})=(r\pi)^{-1/2}\delta(r-\gamma)\begin{cases} ce_n(\theta,\gamma^2/2), & n=0,1,2,\ldots,\\ se_{-n}(\theta,\gamma^2/2), & n=-1,-2,\ldots,\end{cases}$$

$$\gamma\geqslant 0,\qquad \langle fe_{\gamma,n},fe_{\gamma',n'}\rangle=\delta(\gamma-\gamma')\delta_{nn'}, \tag{6.14}$$

where $ce_n(\theta,q),se_n(\theta,q)$ are the periodic Mathieu functions (B.26) and r,θ are polar coordinates. The eigenvalues $\mu=\mu_n$ are discrete and all of multiplicity one. The basis functions $Fe_{\gamma,n}(t,\mathbf{x})=\exp(t\mathcal{K}_{-2})fe_{\gamma,n}(\mathbf{x})$ are

$$Fe_{\gamma,n}(t,\mathbf{x})=\frac{A_{\gamma,n}}{4\pi i\tau}\left(\frac{\gamma}{\pi}\right)^{1/2}\exp\left[i\tau(\cos^2\sigma+\sinh^2\rho+\gamma^2)\right]$$

$$\cdot\begin{cases} ce_n(\sigma,\gamma^2/2)Ce_n(\rho,\gamma^2/2), & n=0,1,2,\ldots,\\ se_{-n}(\sigma,\gamma^2/2)Se_{-n}(\rho,\gamma^2/2), & n=-1,-2,\ldots,\end{cases} \tag{6.15}$$

where $A_{\gamma,n}$ is a normalization constant, $Se_n(\rho,q)$ and $Ce_n(\rho,q)$ are modified Mathieu functions (3.40), Section 1.3, and

$$x_1=-2\tau\cosh\rho\cos\sigma,\qquad x_2=-2\tau\sinh\rho\sin\sigma,\qquad t=\tau.$$

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The Lc system (transformed so that $b=0$) can be defined by equations

$$i(\mathcal{K}_2+a\mathcal{P}_1)f=\lambda f, \qquad \mathcal{B}_2^2 f=-\tfrac{1}{4}\rho^2 f, \qquad a\neq 0,$$

with basis functions

$$\begin{aligned} lc_{\lambda,\rho}(\mathbf{x}) &= \frac{\delta(x_2-\rho)}{(2\pi|a|)^{1/2}}\exp\left[-ia^{-1}\left(\lambda x_1+\frac{\rho^2 x_1}{4}+\frac{x_1^3}{12}\right)\right], \\ \langle lc_{\lambda,\rho}, lc_{\lambda',\rho'}\rangle &= \delta(\lambda-\lambda')\delta(\rho-\rho'), \qquad -\infty<\lambda,\rho<\infty. \end{aligned} \tag{6.16}$$

The three-variable basis functions are

$$\begin{aligned} Lc_{\lambda,\rho}(t,\mathbf{x}) &= \frac{(9a)^{-1/3}}{8iw(2\pi|a|)^{1/2}}\exp\left[i\left((u^2+v^2)\frac{w}{4}-\frac{au}{w}-\frac{\rho v}{2}-\frac{a}{3w^3}-\lambda/w\right)\right] \\ &\quad\cdot \mathrm{Ai}\left[(36a)^{-1/3}\left(\frac{u}{a}+\frac{\lambda}{a}+\frac{\rho^2}{4a}\right)\right] \end{aligned} \tag{6.17}$$

where $\mathrm{Ai}(z)$ is an Airy function ((1.52), Section 2.1). Here

$$x_1=uw+a/w, \qquad x_2=vw, \qquad t=w.$$

The Lp system is defined by

$$i(\mathcal{K}_2+a\mathcal{P}_1)f=\lambda f, \qquad (\{\mathcal{B}_2,\mathcal{M}\}+a\mathcal{P}_2^2)f=\mu f$$

with basis functions

$$\begin{aligned} lp_{\lambda,n}(\mathbf{x}) &= (2\pi|a|)^{-1/2}h_n(x_2)\exp\left[-i(\lambda x_1+x_1x_2^2/4+x_1^3/12)/a\right], \\ \langle lp_{\lambda,n}, lp_{\lambda',n'}\rangle &= \delta(\lambda-\lambda')\delta_{nn'}, \qquad -\infty<\lambda<\infty, \qquad n=0,1,2,\ldots. \end{aligned} \tag{6.18}$$

Here the anharmonic oscillator function $h_n(x)$ is a solution of

$$\frac{d^2h(x)}{dx^2}-\left(\frac{\mu}{a}+\frac{\lambda x^2}{a^2}+\frac{x^4}{4a^2}\right)h(x)=0, \qquad \lambda,a \text{ fixed}, \tag{6.19}$$

such that

$$\int_{-\infty}^{\infty}|h_n(x)|^2\,dx=1. \tag{6.20}$$

The eigenvalues $\mu=\mu_n(\lambda)$ of (6.19) subject to condition (6.20) are discrete [100, p. 250] with multiplicity one, and we assume them ordered so that $\mu_0<\mu_1<\mu_2<\ldots$. Here $h_n(x)$ is either even or odd for each value of n.

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Denote a general solution of (6.19) by $h_{\mu,\lambda,a}(x)$. Then using separation of variables it is straightforward to show that the basis functions $Lp_{\lambda,n}(t,\mathbf{x})=\exp(t\mathcal{K}_{-2})lp_{\lambda,n}(\mathbf{x})$ are

$$Lp_{\lambda,n}(t,\mathbf{x})=C_{\lambda,n}w^{-1}\exp\left\{i\left[\frac{(u^2+v^2)^2w}{16}-\frac{a(u^2-v^2)}{4w}-\frac{a^2}{12w^2}-\frac{\lambda}{w}\right]\right\}$$
$$\cdot h_{2\mu_n,\lambda,a/2}(u)h_{2\mu_n,\lambda,a/2}(iv) \tag{6.21}$$

where the two h functions have the same parity as $h_n(x)$ and $C_{\lambda,n}$ is a normalization constant. (Note that since (6.19) is invariant under the replacement $x\to -x$, for each μ,λ,a this equation has a single even solution in x and a single odd solution to within multiplication by a normalization constant.) Also,

$$x_1=(u^2-v^2)w/2+a/w, \qquad x_2=uvw, \qquad t=w.$$

The Rc system is defined by the equations

$$i\mathfrak{D}f=\rho f, \qquad \{\mathcal{B}_1,\mathcal{P}_1\}f=\mu f$$

with basis eigenfunctions

$$rc_{\lambda\mu}^{\varepsilon\varepsilon'}(\mathbf{x})=(2\pi)^{-1}(x_1)_{\varepsilon}^{-i\lambda-\frac{1}{2}}(x_2)_{\varepsilon'}^{-i\mu-\frac{1}{2}}, \tag{6.22}$$
$$-\infty<\lambda,\mu<\infty,\ \varepsilon,\varepsilon'=\pm,\ \lambda=\rho-\mu;$$

see (1.46). The orthogonality relations are

$$\langle rc_{\lambda\mu}^{\varepsilon\varepsilon'},rc_{\bar\lambda\bar\mu}^{\bar\varepsilon\bar\varepsilon'}\rangle=\delta_{\varepsilon\bar\varepsilon}\delta_{\varepsilon'\bar\varepsilon'}\delta(\lambda-\bar\lambda)\delta(\mu-\bar\mu).$$

The three-variable eigenfunctions are

$$Rc_{\lambda\mu}^{++}(t,\mathbf{x})=(8\pi^2iw)^{-1}\left[\exp(i\pi/4)(2w)^{1/2}\right]^{-i(\lambda+\mu)+1}$$
$$\cdot\Gamma\left(\tfrac{1}{2}-i\lambda\right)\Gamma\left(\tfrac{1}{2}-i\mu\right)\exp\left[\frac{i(u^2+v^2)}{8}\right]$$
$$\cdot D_{i\lambda-\frac{1}{2}}\left(\frac{-u}{(2i)^{1/2}}\right)D_{i\mu-\frac{1}{2}}\left(\frac{-v}{(2i)^{1/2}}\right), \qquad t>0, \tag{6.23}$$

where $x_1=|w|^{1/2}u$, $x_2=|w|^{1/2}v$, $t=w$. The remaining three-variable basis

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functions are given by

$$\begin{aligned} Rc^{++}_{\lambda\mu}(u,v) &= \exp[\pi(i+\lambda+\mu)]Rc^{--}_{\lambda\mu}(-u,-v) \\ &= \exp[\pi(i/2+\lambda)]Rc^{-+}_{\lambda\mu}(-u,v) \\ &= \exp[\pi(i/2+\mu)]Rc^{+-}_{\lambda\mu}(u,-v). \end{aligned} \tag{6.24}$$

The Rr system is defined by the equations

$$\mathcal{D}f = i\rho f, \qquad \mathcal{M}f = imf.$$

The eigenfunctions are

$$\begin{aligned} rr_{\rho,m}(\mathbf{x}) &= (2\pi)^{-1}r^{i\rho-1}e^{im\theta}, \qquad -\infty<\rho<\infty, \quad m=0,\pm1,\ldots, \\ x_1 &= r\cos\theta, \qquad x_2 = r\sin\theta, \end{aligned} \tag{6.25}$$

satisfying the orthogonality relations

$$\langle rr_{\rho,m}, rr_{\rho',m'}\rangle = \delta_{mm'}\delta(\rho-\rho').$$

The three-variable basis functions are

$$\begin{aligned} Rr_{\rho,m}(t,\mathbf{x}) &= 2^{-m+i\rho-2}\exp\left[i\pi\frac{(3m-1+i\rho)}{4}\right]\pi^{-1}w^{i\rho/2-\frac{1}{2}} \\ &\cdot\Gamma\left(\frac{m+i\rho+1}{2}\right)\frac{u^m}{\Gamma(m+1)}\,{}_1F_1\left(\begin{matrix}(m+1-i\rho)/2\\ m+1\end{matrix}\,\middle|\,\frac{iu^2}{4}\right)\exp(imv) \end{aligned} \tag{6.26}$$

where

$$x_1 = \sqrt{w}\,u\cos v, \qquad x_2 = \sqrt{w}\,u\sin v, \qquad t = w > 0.$$

The Re system is defined by equations

$$\mathcal{D}f = i\lambda f, \qquad \left(\mathcal{M}^2 + \tfrac{1}{2}\{\mathcal{B}_2, \mathcal{P}_2\}\right)f = \mu f.$$

The ON basis of eigenfunctions is

$$re^{+}_{\lambda m}(\mathbf{x}) = (2\pi)^{-1/2}r^{i\lambda-1}Gc_m(\theta,\tfrac{1}{4},-\lambda), \tag{6.27a}$$

$$re^{-}_{\lambda m}(\mathbf{x}) = (2\pi)^{-1/2}r^{i\lambda-1}Gs_m(\theta,\tfrac{1}{4},-\lambda), \tag{6.27b}$$

$$m = 0,1,2,\ldots, -\infty<\lambda<\infty,$$

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where $x_1 = r\cos\theta$, $x_2 = r\sin\theta$. Here we have introduced the notation

$$Gc_m\left(\theta, \tfrac{1}{4}, -\lambda\right) = \exp\left[i\cos(2\theta)/16\right] gc_m\left(\theta, \tfrac{1}{4}, -\lambda\right),$$

$$Gs_m\left(\theta, \tfrac{1}{4}, -\lambda\right) = \exp\left[i\cos(2\theta)/16\right] gs_m\left(\theta, \tfrac{1}{4}, -\lambda\right).$$

The functions $gc_m(\theta,\alpha,\beta)$, $gs_m(\theta,\alpha,\beta)$ are even and odd nonpolynomial solutions of the Whittaker–Hill equation

$$\frac{d^2g}{d\theta^2} + \left(\mu + \frac{\alpha^2}{8} + \alpha\beta\cos 2\theta - \frac{\alpha^2}{8}\cos 4\theta\right) g = 0 \tag{6.28}$$

with period 2π. The subscript m (the number of zeros in the interval $[0,2\pi]$) labels the discrete eigenvalues $\mu = \mu_m$ of the operator $\mathfrak{M}^2 + \frac{1}{2}\{\mathfrak{B}_2, \mathfrak{P}_2\}$. This notation is due to Urwin and Arscott [127]. Each of the solutions Gc_m, Gs_m can be written as an infinite trigonometric series in $\cos n\theta$, $\sin n\theta$, respectively, which converges for the discrete eigenvalues μ_m. The orthogonality relations are

$$\langle re^{\pm}_{\lambda m}, re^{\pm}_{\lambda' m'}\rangle = \delta(\lambda - \lambda')\delta_{mm'}, \qquad \langle re^{\pm}_{\lambda m}, re^{\mp}_{\lambda' m'}\rangle = 0$$

and the three-variable basis functions are

$$\begin{aligned} Re^{+}_{\lambda m}(t,\mathbf{x}) &= K^{\lambda+}_m w^{(i\lambda-1)/2} Gc_m\left(iu, \tfrac{1}{4}, -\lambda\right) Gc_m\left(v, \tfrac{1}{4}, -\lambda\right), \\ Re^{-}_{\lambda m}(t,\mathbf{x}) &= K^{\lambda-}_m w^{(i\lambda-1)/2} Gs_m\left(iu, \tfrac{1}{4}, -\lambda\right) Gs_m\left(v, \tfrac{1}{4}, -\lambda\right), \end{aligned} \tag{6.29}$$

where

$$x_1 = \sqrt{w}\,\cosh u \cos v, \qquad x_2 = \sqrt{w}\,\sinh u \sin v, \qquad t = w > 0.$$

The constants $K^{\lambda\pm}_m$ are in principle calculable by choosing special values of the parameters u, v, w. In fact, in the process of calculating the functions $Re^{\pm}$ by separation of variables, we obtain relations

$$\begin{aligned} K^{\lambda+}_m Gc_m\left(iu, \tfrac{1}{4}, -\lambda\right) Gc_m\left(v, \tfrac{1}{4}, -\lambda\right) &= \exp\left[i(\sinh^2 u + \cos^2 v)/4\right] \int_{-\pi}^{\pi} d\theta \\ &\times Gc_m\left(\theta, \tfrac{1}{4}, -\lambda\right) \exp\left[-i(\cosh u \cos v \cos\theta + \sinh u \sin v \sin\theta)^2/8\right] \\ &\times D_{i\lambda-1}\left(-\left[\cosh u \cos v \cos\theta + \sinh u \sin v \sin\theta\right]/(2i)^{1/2}\right) \end{aligned}$$

with a similar expression for the functions $Gs_n(\theta, \frac{1}{4}, -\lambda)$. The constants $K^{\lambda\pm}_m$ can be calculated for particular values of the arguments; for example,

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if $Gc_m(\theta,\frac{1}{4},-\lambda)=\sum_{k=0}^{\infty}A_k^m\cos 2k\theta$, then

$$K_m^{\lambda+}=2\pi D_{i\lambda-1}(0)A_0^m\left[Gc_m\left(\pi/2,\tfrac{1}{4},-\lambda\right)Gc_m\left(0,\tfrac{1}{4},-\lambda\right)\right]^{-1}.$$

This completes our determination of bases for the solution space of the Schrödinger equation (5.2).

Exactly as in Section 2.1 we can show that these results lead to a number of Hilbert space expansion theorems. Indeed, if $\{f_{\lambda\mu}\}$ is an ON basis for $L_2(R_2)$, then $\{\mathbf{U}(g)f_{\lambda\mu}\}$ for any $g\in G_3$ is also an ON basis. In particular, each of the three-variable models constructed above provides a basis for $L_2(R_2)$ (as well as a basis of solutions for (5.2)). Furthermore, we can derive discrete and continuous generating functions for each basis.

Now we compute some overlap functions $\langle Aa_{\lambda\mu},Bb_{\lambda'\mu'}\rangle$ that allow us to expand eigenfunctions $Aa_{\lambda\mu}$ in terms of eigenfunctions $Bb_{\lambda'\mu'}$. The utility of these formulas is that they are invariant under the action of G, so the same expressions allow us to expand $\mathbf{U}(g)Aa_{\lambda\mu}$ in terms of $\mathbf{U}(g)Bb_{\lambda'\mu'}$, where the results may be much less obvious. In the following we use the two-variable bases to compute some overlaps of interest. Because of G_3 invariance, identical results hold for the three-variable bases.

Here we omit overlaps involving the discrete basis *oe*. This basis is of special interest but the overlap computation involves use of the Bargmann–Segal Hilbert space of analytic functions, which we will not discuss here. Detailed results and an interesting connection between Ince polynomials and the representation theory of $SU(2)$ are presented in [21]. For most of the other bases we give an overlap with either of the discrete bases *oc* or *or*. The principle behind these computations should now be obvious, so the interested reader can derive for himself any of the other overlaps.

$$\langle fc_{\gamma,\alpha},or_{nm}^{\pm}\rangle=\gamma^{1/2}or_{nm}^{\pm}(\gamma\cos\alpha,\gamma\sin\alpha); \tag{6.30}$$

$$\langle fr_{\gamma,p},or_{nm}^{\pm}\rangle=\begin{cases}0\\ \text{if}\quad p\neq\pm m,\\ \left[\dfrac{\gamma m!}{2^{m+1}(n+m)!}\right]^{1/2}\exp\left(\dfrac{-\gamma^2}{4}\right)\gamma^m L_n^{(m)}\left(\dfrac{\gamma^2}{2}\right)\\ \text{if}+\text{and}\ p=\pm m\neq 0,\\ \dfrac{ip}{m}\left[\dfrac{\gamma m!}{2^{m+1}(n+m)!}\right]^{1/2}\exp\left(\dfrac{-\gamma^2}{4}\right)\gamma^m L_n^{(m)}\left(\dfrac{\gamma^2}{2}\right)\\ \text{if}-\text{and}\ p=\pm m\neq 0\\ \left(\dfrac{4\gamma}{n!}\right)^{1/2}\exp\left(\dfrac{-\gamma^2}{4}\right)L_n^{(o)}\left(\dfrac{\gamma^2}{2}\right)\\ \text{if}\quad p=m=0;\end{cases} \tag{6.31}$$

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$$\langle fp^{+}_{\gamma\mu}, or^{\pm}_{nm}\rangle = \left[\frac{\gamma m!}{2^m \pi (n+m)!}\right]^{1/2} \exp\left(\frac{-\gamma^2}{4}\right)\gamma^m L_n^{(m)}\left(\frac{\gamma^2}{2}\right)$$
$$\times \exp\left[-\pi i \frac{(1\mp 1)}{4}\right](a_m \pm a_{-m});$$

$$\langle f\bar{p}_{\gamma\mu}, or^{\pm}_{nm}\rangle = \langle fp^{+}_{\gamma\mu}, or^{\pm}_{n,-m}\rangle;$$

$$a_m = \exp\left[\pi\frac{(i/2-\mu)}{2}\right]\Gamma\left(m+\tfrac{1}{2}\right)\left[\frac{(-1)^m\Gamma\left(\tfrac{1}{2}+i\mu\right)}{\Gamma\left(m+i\mu+\tfrac{1}{2}\right)}\right.$$
$$\times {}_2F_1\left[\left.\begin{matrix}\tfrac{1}{2}+i\mu, \tfrac{1}{2}+m\\ m+i\mu+1\end{matrix}\right| -1\right] - \frac{i\Gamma\left(\tfrac{1}{2}-i\mu\right)}{\Gamma(m-i\mu+1)}$$
$$\left.\times {}_2F_1\left[\left.\begin{matrix}\tfrac{1}{2}-i\mu, \tfrac{1}{2}+m\\ m-i\mu+1\end{matrix}\right| -1\right]\right]; \tag{6.32}$$

$$\langle fe_{\gamma,p}, or^{+}_{nm}\rangle = \theta(p)(1+(-1)^{m-p})A^p_m\left[\frac{\gamma m!}{2^{m+2}\pi^2(n+m)!}\right]^{1/2}$$
$$\times \exp\left(\frac{-\gamma^2}{4}\right)\gamma^m L_n^{(m)}\left(\frac{\gamma^2}{2}\right); \tag{6.33}$$

where $\theta(x)=1$ for $x\geqslant 0$, and $\theta(x)=0$ otherwise. A similar expression for $\langle fe_{\gamma,p}, or_{nm}\rangle$ can be obtained by replacing $\theta(p)$ by $\theta(-p)$ and A^p_m by B^p_m in expression (6.33). Here A^p_m, B^p_m are the coefficients in the trigonometric expansions of the even and odd Mathieu functions, respectively. Also,

$$\langle lc_{\lambda,\rho}, oc_{n,m}\rangle = \exp(-\rho^2/4)(2^{m-1}\pi m!)^{-1/2}H_m(\rho/\sqrt{2}\,)C_n \tag{6.34}$$

where

$$2^{2/3}\exp\left[-i\left(^{1/6}+\lambda+\rho^2/4+\sqrt{2y}\,\right)\right]\mathrm{Ai}\left[2^{2/3}\left(1/4-i\lambda-i\rho^2/4-i(2y)^{1/2}\right)\right]$$
$$=\sum_{n=0}^{\infty}\left[\left((2i)^{1/2}y\right)^n/n!\right]C_n,$$

and we have normalized so that $a=-1$,

$$\langle lc_{\lambda,\rho}, lp_{\mu,n}\rangle = (2\pi|a|)^{-1}\bar{h}_n(\rho)\delta[(\lambda-\mu)/a], \tag{6.35}$$
$$\langle rc^{++}_{\lambda\mu}, oc_{n,m}\rangle = \pi^{-2}(2^{m+n+3}n!m!)^{-1/2}\mathfrak{L}^{\lambda}_m\mathfrak{L}^{\mu}_n \tag{6.36}$$

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where

$$\mathfrak{L}_m^\lambda = \begin{cases} 2^{m+i\lambda-\frac{1}{2}}\Gamma(i\lambda/2+1/4)\Gamma((m+1)/2)\,{}_2F_1\left(\begin{matrix}-m/2, i\lambda/2+1/4\\ 1/2\end{matrix}\middle|\, 2\right), \\ m \text{ even}, \\ 2^{m+i\lambda+1}\Gamma(i\lambda/2+1/4)\Gamma(m/2)\,{}_2F_1\left(\begin{matrix}(1-m)/2, i\lambda/2+3/4\\ 3/2\end{matrix}\middle|\, 2\right), \\ m \text{ odd}. \end{cases}$$

The remaining overlaps for rc^{+-}, rc^{-+}, and rc^{--} can be calculated by using relations (6.24).

$$\langle rr^+_{\lambda m}, or_{nm'}\rangle = \delta_{mm'}(2/n!)^{m/2-i\lambda}\left[(m+n)!/m!\right]^{1/2}$$
$$\times\Gamma((m+1-i\lambda)/2)\,{}_2F_1\left(\begin{matrix}-n, (m+1-i\lambda)/2\\ m+1\end{matrix}\middle|\, 2\right), \tag{6.37}$$

$$\langle rr^-_{\lambda m}, or_{nm'}\rangle = -i(-1)^{\operatorname{sign} m}\langle rr^+_{\lambda m}, or_{nm'}\rangle, \tag{6.38}$$

$$\langle rr^+_{\lambda 0}, or_{n,m'}\rangle = \delta_{om'}\left(2^{-\frac{1}{2}-i\lambda}/(n!)^{1/2}\right)\Gamma((1-i\lambda)/2)$$
$$\times{}_2F_1\left(\begin{matrix}-n, (1-i\lambda)/2\\ 1\end{matrix}\middle|\, 2\right). \tag{6.39}$$

$$\langle oc_{n_1,n_2}, or^{\pm}_{nm}\rangle = K\delta_{n_1+n_2,2n+m}\, i^{n_1}\left(2^{2n+m+1}n!\,n_2!(n+m)!/n_1!\right)^{1/2}$$
$$\times\binom{1}{-i}\Bigg[i^m\{\Gamma((n_1+n_2-m)/2)\Gamma((n_2-n_1+m)/2)\}^{-1}$$
$$\times{}_2F_1\left[\begin{matrix}-n_1, 1-(n_1+n_2+m)/2\\ (n_2-n_1-m)/2\end{matrix}\middle|\, -1\right] \pm i^{-m}\{\Gamma((n_1+n_2-m)/2)$$
$$\times\Gamma((n_2-n_1+m)/2)\}^{-1}\,{}_2F_1\left[\begin{matrix}-n_1, 1-(n_1+n_2-m)/2\\ (n_2-n_1+m)/2\end{matrix}\middle|\, -1\right]\Bigg]. \tag{6.40}$$

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For the basis *re* we have

$$\langle re^{+}_{\lambda m}, or^{+}_{nm'}\rangle = \tfrac{1}{2}\left(1+(-1)^{m-m'}\right)\overline{A}^{m}_{m'}(2\pi)^{1/2}\langle rr^{+}_{\lambda m'}, or_{n,m'}\rangle, \qquad (6.41)$$

$$\langle re^{-}_{\lambda m}, or^{-}_{nm'}\rangle = \tfrac{1}{2}\left(1+(-1)^{m-m'}\right)\overline{B}^{m}_{m'}(2\pi)^{1/2}\langle rr^{-}_{\lambda m'}, or_{n,m'}\rangle, \qquad (6.42)$$

where $\overline{A}^{m}_{m'}$ and $\overline{B}^{m}_{m'}$ are the coefficients for the expansion of the functions $gc_m(\theta,\frac{1}{4},-\lambda)$ and $gs_m(\theta,\frac{1}{4},-\lambda)$, respectively, in trigonometric series analogous to (B.26) [127].

2.7 The Real and Complex Heat Equations $(\partial_t - \partial_{xx} - \partial_{yy})\Phi = 0$

The heat equation in three-dimensional space-time (suitably normalized) is

$$Q\Phi = 0, \qquad Q = \partial_t - \partial_{x_1x_1} - \partial_{x_2x_2} \qquad (7.1)$$

where t, x_1, x_2 are the real time and space variables, respectively. Since this equation can be obtained from the Schrödinger equation by replacing t in (5.2) with $-it$, the symmetry algebras of these two equations are closely related. The symmetry algebra of (7.1) is nine dimensional with basis

$$\begin{aligned} &H_2 = t^2\partial_t + tx_1\partial_{x_1} + tx_2\partial_{x_2} + t + \left(x_1^2 + x_2^2\right)/4, \qquad H_{-2} = \partial_t,\\ &P_j = \partial_{x_j}, \qquad B_j = t\,\partial_{x_j} + x_j/2, \qquad M = x_1\partial_{x_2} - x_2\partial_{x_1}, \qquad (7.2)\\ &H^0 = x_1\partial_{x_1} + x_2\partial_{x_2} + 2t\partial_t + 1, \qquad H_0 = 1, \qquad j = 1,2, \end{aligned}$$

and commutation relations

$$\begin{aligned} &[H^0, H_{\pm 2}] = \pm 2H_{\pm 2}, \qquad [H^0, B_j] = B_j, \qquad [H^0, P_j] = -P_j,\\ &[P_j, H_2] = B_j, \qquad [P_j, B_j] = \tfrac{1}{2}H_0, \qquad [P_j, B_l] = 0,\\ &[H_{-2}, H_2] = H^0, \qquad [H_{\pm 2}, M] = [H_2, B_j] = [H_{-2}, P_j] = [H^0, M] = 0,\\ &\left[B_j, M\right] = (-1)^{j+1}B_l, \qquad \left[H_{-2}, B_j\right] = P_j, \qquad \left[P_j, M\right] = (-1)^{j+1}P_l, \qquad (7.3)\\ &\qquad\qquad j,l = 1,2,\ j \neq l, \end{aligned}$$

where H_0 is in the center of the algebra. We denote by $\mathcal{G}_3'$ the real Lie algebra with basis (7.2). The operators B_j, P_j, H_0 span the five-dimensional Weyl subalgebra $\mathcal{W}_2$ of $\mathcal{G}_3'$ and local Lie theory yields the associated local

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group action

$$\mathbf{T}(\mathbf{w},\mathbf{z},\rho)\Psi(t,\mathbf{x})=\exp\left[\frac{1}{2}\mathbf{x}\cdot\mathbf{w}+\frac{t}{4}\mathbf{w}\cdot\mathbf{w}+\rho\right]\Psi(t,\mathbf{x}+t\mathbf{w}+\mathbf{z}). \quad (7.4)$$

Here, $\mathbf{w}=(w_1,w_2)$, $\mathbf{z}=(z_1,z_2)$, $\mathbf{x}\cdot\mathbf{w}=x_1w_1+x_2w_2$, and $w_j,z_j,\rho\in R$. The operators act on the space $\mathfrak{F}$ of functions $\Psi(t,\mathbf{x})$, analytic in some given domain $\mathfrak{D}$ in three-space. Furthermore, these operators map solutions of the heat equation into solutions.

Similarly, the operators $H_{\pm 2}$, H_0 span the three-dimensional subalgebra $sl(2,R)$ and determine operators

$$\mathbf{T}(A)\Psi(t,\mathbf{x})=\exp\left[-\frac{\beta}{4}(\delta+t\beta)^{-1}\mathbf{x}\cdot\mathbf{x}\right](\delta+t\beta)^{-1}\Psi\left(\frac{\gamma+t\alpha}{\delta+t\beta},(\delta+t\beta)^{-1}\mathbf{x}\right),$$

$$A=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in SL(2,R), \qquad \Psi\in\mathfrak{F}, \quad (7.5)$$

which define a local representation of $SL(2,R)$. The operator M determines a local representation of $SO(2)$:

$$\mathbf{T}(\theta)\Psi(t,\mathbf{x})=\Psi(t,\mathbf{x}\Theta), \qquad \Theta=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}. \quad (7.6)$$

The local Lie group G_3' of symmetry operators $\mathbf{T}$ can be represented as a semidirect product of W_2 and $SL(2,R)\times SO(2)$ by means of expressions analogous to (5.10) and (5.11).

For future use we point out explicitly the special case of (7.5) where

$$A_0=2^{-1/2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}, \qquad A_0^8=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}:$$

$$\mathbf{T}(A_0)\Psi(t,\mathbf{x})=\exp\left[-\tfrac{1}{4}(1+t)^{-1}\mathbf{x}\cdot\mathbf{x}\right]\frac{\sqrt{2}}{1+t}\Psi\left(\frac{t-1}{t+1},\frac{\sqrt{2}}{t+1}\mathbf{x}\right),$$

$$\mathbf{T}(A_0^2)\Psi(t,\mathbf{x})=\exp\left[-\frac{1}{4t}\mathbf{x}\cdot\mathbf{x}\right]t^{-1}\Psi(-t^{-1},t^{-1}\mathbf{x}),$$

$$\mathbf{T}(A_0^4)\Psi(t,\mathbf{x})=-\Psi(t,-\mathbf{x}). \quad (7.7)$$

Here, $\mathbf{T}(A_0^2)$ is the *Appell transform* [4, 13].

The problem of R-separation of variables for the heat equation (7.1) is analogous to that for the free-particle Schrödinger equation (5.2) and the results are similar [56]. Here the R-separable solutions of (7.1) take the

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form

$$\Phi(t,\mathbf{x}) = \exp[\mathfrak{R}(u,v,w)]U(u)V(v)W(w), \qquad \mathfrak{R} \text{ real}, \tag{7.8}$$

where either $\mathfrak{R} \equiv 0$ or $\mathfrak{R} \neq 0$ cannot be written as a sum $\mathfrak{R} = A(u) + B(v) + C(w)$. We require that $\{u,v,w\}$ be a real analytic coordinate system such that substitution of (7.8) into (7.1) reduces the partial differential equation to three ordinary differential equations, one for each of the factors U, V, W. Two coordinate systems are considered equivalent if they can be obtained from one another under the adjoint action of G_3'.

The results announced in [56] are as follows. Corresponding to every R-separation of variables for (7.1) we can find a pair of differential operators H, S such that:

1. H and S are symmetries of (7.1) and $[H,S]=0$.
2. $H \in \mathcal{G}_3'$; that is, H is first order in x_1, x_2, and t.
3. S is second order in x_1, x_2, and contains no term in ∂_t.

The R-separation of variables is characterized by the simultaneous equations

$$Q\Phi = 0, \qquad H\Phi = i\lambda\Phi, \qquad S\Phi = \mu\Phi. \tag{7.9}$$

The eigenvalues λ, μ are the usual separation constants for R-separable solutions Φ.

It follows from these remarks that S can always be expressed as a symmetric quadratic form in B_j, P_j, E, and M. The possible coordinates and their characterizations are listed in Table 13.

For each system in Table 13 we have $w = t$ and the separated solution in the variable w is exponential. In the last column of the table we list first the form of the separated solution in u followed by the separated solution in v. The anharmonic oscillator functions are solutions of a differential equation of the form

$$f''(u) + (\lambda u^2 + \alpha u^4 - \beta) f(u) = 0. \tag{7.10}$$

From the viewpoint of Galilean and dilatation symmetry alone there are 26 distinct coordinate systems. However, from the viewpoint of G_3' symmetry there are only 17 systems. It is easy to show that two systems whose labels differ only in the superscripts lie on the same G_3' orbit. Indeed, systems of the form Fa^1, Fa^2 or La^1, La^2 are related by $\mathbf{T}(A_0^2)$, (7.7), and systems of the form Ra^1, Ra^2 are related by $\mathbf{T}(A^0)$. These are the only G_3' equivalences.

The eigenfunctions of the commuting pair H^0, M^2 are of special interest for this equation. From Table 13, the corresponding eigenfunctions separate in the variables $u = [(x^2+y^2)/t]^{1/2} = rt^{-1/2}$, $v = \theta$, $w = t$, where $x =$

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Table 13 Operators and R-Separable Coordinates for the Equation $(\partial_t - \partial_{xx} - \partial_{yy})\Phi = 0$

Operators H,S	Coordinates $\{u,v,w\}$	Multiplier $e^{\mathfrak{R}}$	Separated Solutions
1a Fc^1 H_2, B_1^2	$x = uw$ $y = vw$	$\mathfrak{R} = -(u^2+v^2)w/4$	Exponential Exponential
1b Fc^2 H_{-2}, P_1^2	$x = u$ $y = v$	0	Exponential Exponential
2a Fr^1 H_2, M^2	$x = uw\cos v$ $y = uw\sin v$	$-u^2w/4$	Bessel Exponential
2b Fr^2 H_{-2}, M^2	$x = u\cos v$ $y = u\sin v$	0	Bessel Exponential
3a Fp^1 $H_2, \{B_2, M\}$	$x = (u^2-v^2)w/2$ $y = uvw$	$-(u^2+v^2)^2w/16$	Parabolic cylinder Parabolic cylinder
3b Fp^2 $H_{-2}, \{P_2, M\}$	$x = (u^2-v^2)/2$ $y = uv$	0	Parabolic cylinder Parabolic cylinder
4a Fe^1 $H_2, M^2 - B_2^2$	$x = w\cosh u\cos v$ $y = w\sinh u\sin v$	$-(\sinh^2 u + \cos^2 v)w/4$	Modified Mathieu Mathieu
4b Fe^2 $H_{-2}, M^2 - P_2^2$	$x = \cosh u\cos v$ $y = \sinh u\sin v$	0	Modified Mathieu Mathieu
5a Lc^1 $H_2 - 2aP_1 - 2bP_2$, $B_2^2 - 2bP_2H_0$	$x = uw + a/w$ $y = vw + b/w$	$-(u^2+v^2)w/4$ $+(au+bv)/2w$	Airy Airy
5b Lc^2 $H_{-2} + 2aB_1 + 2bB_2$ $P_1^2 + 2aB_1H_0$	$x = u + aw^2$ $y = v + bw^2$	$-(au+bv)w$	Airy Airy
6a Lp^1 $H_2 - aP_1$, $\{B_2, M\} - aP_2^2$	$x = (u^2-v^2)w/2 + a/w$ $y = uvw$	$-(u^2+v^2)^2w/16$ $+a(u^2-v^2)/4w$	Anharmonic oscillator Anharmonic oscillator
6b Lp^2 $H_{-2} - 2aB_1$, $\{P_2, M\} + 2aB_2^2$	$x = (u^2-v^2)/2 + aw^2$ $y = uv$	$-a(u^2-v^2)w/2$	Anharmonic oscillator Anharmonic oscillator
7 Oc $H_{-2} + H_2$ $P_1^2 + B_1^2$	$x = u(1+w^2)^{1/2}$ $y = v(1+w^2)^{1/2}$	$-(u^2+v^2)w/4$	Parabolic cylinder Parabolic cylinder
8 Or $H_{-2} + H_2, M^2$	$x = (1+w^2)^{1/2}u\cos v$ $y = (1+w^2)^{1/2}u\sin v$	$-u^2w/4$	Whittaker Exponential
9 Oe $H_{-2} + H_2$, $M^2 - P_2^2 - B_2^2$	$x = (1+w^2)^{1/2}\cosh u\cos v$ $y = (1+w^2)^{1/2}\sinh u\sin v$	$-(\sinh^2 u + \cos^2 v)$ $\cdot w/4$	Ince Ince

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Table 13 (*Continued*)

	Operators H, S	Coordinates $\{u, v, w\}$	Multiplier $e^{\mathcal{R}}$	Separated Solutions
10a	Rc^1 $H^0, \{B_1, P_1\}$	$x = \lvert w\rvert^{1/2} u$ $y = \lvert w\rvert^{1/2} v$	0	Hermite Hermite
10b	Rc^2 $H_{-2} - H_2,$ $P_1^2 - B_1^2$	$x = u\lvert 1 - w^2\rvert^{1/2}$ $y = v\lvert 1 - w^2\rvert^{1/2}$	$-\varepsilon(u^2 + v^2)w/4$ $\varepsilon = \operatorname{sign}(1 - w^2)$	Hermite Hermite
11a	Rr^1 H^0, M^2	$x = \lvert w\rvert^{1/2} u \cos v$ $y = \lvert w\rvert^{1/2} u \sin v$	0	Laguerre Exponential
11b	Rr^2 $H_{-2} - H_2, M^2$	$x = \lvert 1 - w^2\rvert^{1/2} u \cos v$ $y = \lvert 1 - w^2\rvert^{1/2} u \sin v$	$-\varepsilon u^2 w/4$	Laguerre Exponential
12a	Re^1 $H^0,$ $M^2 - \frac{1}{2}\{B_2, P_2\}$	$x = \lvert w\rvert^{1/2} \cosh u \cos v$ $y = \lvert w\rvert^{1/2} \sinh u \sin v$	0	Finite Ince Finite Ince
12b	Re^2 $H_{-2} - H_2,$ $M^2 - P_2^2 + B_2^2$	$x = \lvert 1 - w^2\rvert^{1/2} \cosh u \cos v$ $y = \lvert 1 - w^2\rvert^{1/2} \sinh u \sin v$	$-\varepsilon(\sinh^2 u$ $+ \cos^2 v)w/4$	Finite Ince Finite Ince
13	$L1$ $P_1, B_2^2 + 2bP_2H_0$	$x = u$ $y = vw + b/w$	$-v^2 w/4 + bv/2w$	Exponential Airy
14	$L2$ $P_1, P_2^2 + 2aB_2H_0$	$x = u$ $y = v + aw^2$	$-avw$	Exponential Airy
15	$O1$ $P_1, P_2^2 + B_2^2$	$x = u$ $y = v(1 + w^2)^{1/2}$	$-v^2 w/4$	Exponential Parabolic cylinder
16	$R1$ $P_1, \{B_2, P_2\}$	$x = u$ $y = v\lvert w\rvert^{1/2}$	0	Exponential Hermite
17	$R2$ $P_1, P_2^2 - B_2^2$	$x = u$ $y = v\lvert 1 - w^2\rvert^{1/2}$	$-\varepsilon v^2 w/4$	Exponential Hermite

$r\cos\theta$, $y = r\sin\theta$. Moreover, the solutions $\Phi_{m,n}(t, \mathbf{x})$ of the heat equation (bounded at $\mathbf{x} = \mathbf{0}$) that satisfy

$$H^0\Phi_{m,n} = (m + 2n + 1)\Phi_{m,n}, \qquad M\Phi_{m,n} = im\Phi_{m,n}, \qquad (7.11)$$
$$n = 0, 1, 2, \ldots, m = n, n-1, \ldots, -n$$

are expressible in terms of Laguerre polynomials

$$\Phi_{m,n}(t, \mathbf{x}) = t^n (re^{i\theta})^m L_n^{(m)}(-r^2/4t). \qquad (7.12)$$

Studies of expansions of solutions of the heat equation in terms of these polynomials can be found in [25] and [29].

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It is well known that if $f(x)$ is a bounded continuous function defined in the plane R_2, then there is a unique solution $\Phi(t,\mathbf{x})$ of the heat equation, bounded and continuous in $(t,\mathbf{x})$ for all $\mathbf{x}\in R_2$, $t\geq 0$, and continuously differentiable in t, twice continuously differentiable in x_1,x_2 for all $\mathbf{x}\in R, t>0$, such that $\Phi(0,\mathbf{x})=f(\mathbf{x})$ [107]. This solution is

$$\Phi(t,\mathbf{x})=(4\pi t)^{-1}\iint_{-\infty}^{\infty}\exp[-(\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y})/4t]f(\mathbf{y})\,dy_1\,dy_2$$
$$=I^t(f),\qquad t>0. \tag{7.13}$$

In analogy with our work in Section 2.2 we can construct another model of the symmetry algebra (7.2). First we restrict the operators (7.2) to the solution space of the heat equation. This allows us to replace ∂_t by Δ_2 in the expressions for these operators and to consider $t\geq 0$ as a fixed parameter. With this interpretation the operators (7.2) are the symmetry operators at a fixed time t. At time $t=0$ they become

$$\begin{aligned}&\mathcal{K}_2=(x_1^2+x_2^2)/4,\qquad \mathcal{K}_{-2}=\Delta_2,\qquad \mathcal{P}_j=\partial_{x_j},\qquad \mathcal{B}_j=x_j/2,\\ &\mathcal{M}=x_1\partial_{x_2}-x_2\partial_{x_1},\qquad \mathcal{K}^0=x_1\partial_{x_1}+x_2\partial_{x_2}+1,\\ &\mathcal{K}_0=1,\qquad j=1,2,\end{aligned}\tag{7.14}$$

and when acting on, say, the space $\mathcal{F}_0$ of infinitely differentiable functions $f(\mathbf{x})$ on R_2 with compact support, these operators satisfy the usual commutation relations (7.3).

Exactly as in Section 2.2 we can interpret (7.13) in the form

$$\Phi(t,\mathbf{x})=I^t(f)=\exp(t\Delta_2)f(\mathbf{x})=\exp(t\mathcal{K}_{-2})f(\mathbf{x}),\qquad f\in\mathcal{F}_0, t>0, \tag{7.15}$$

and show that the connection between the italic operators H, (7.2), and the corresponding script operators $\mathcal{K}$, (7.14), is

$$H\exp(t\mathcal{K}_{-2})=\exp(t\mathcal{K}_{-2})\mathcal{K} \tag{7.16}$$

where $H\in\mathcal{G}_3'$ and $\mathcal{K}$ is obtained from H by setting $t=0$. In addition, we can obtain results of the form

$$\exp(aH)\exp(t\mathcal{K}_{-2})=\exp(t\mathcal{K}_{-2})\exp(a\mathcal{K}) \tag{7.17}$$

and show that the equations

$$\begin{aligned}\partial_t\Phi=\big(&\Delta_2+a_1\mathbf{x}\cdot\mathbf{x}+a_2\partial_{x_1}+a_3\partial_{x_2}+a_4(x_1\partial_{x_2}-x_2\partial_{x_1})\\ &+a_5x_1+a_6x_2+a_7(x_1\partial_{x_1}+x_2\partial_{x_2})+a_8\big)\Phi,\qquad a_j\in R,\end{aligned}\tag{7.18}$$

have isomorphic symmetry algebras and are equivalent to (7.1). The

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techniques for solving the Cauchy problem, discussed in Section 2.2, work for all of the equations (7.18).

At this point it is useful to discuss a method for constructing explicit solutions of the heat equation, which applies equally well to many other equations studied in this book. Every R-separable coordinate system for (7.1) is associated with a pair of commuting operators, one of which is first order. By diagonalizing this first-order operator, we can separate out the corresponding coordinate and reduce the heat equation to an equation with one less variable. For example, diagonalizing the symmetry operator ∂_t, we can separate out the t variable and obtain solutions

$$\Phi(t,\mathbf{x}) = \exp(-k^2 t)F(\mathbf{x})$$

where F is any solution of the Helmholtz equation

$$\Delta_2 F + k^2 F = 0. \tag{7.19}$$

This rather obvious remark becomes less trivial when we realize that each of the symmetry operators $\mathbf{T}(g)$, (7.4)–(7.6), maps Φ into another solution $\mathbf{T}(g)\Phi$. For example, if $g = A_0$, (7.7), we have the result that

$$\begin{aligned}\mathbf{T}(A_0)\Phi(t,\mathbf{x}) = &\exp\left[\left(-k^2(t-1) - \tfrac{1}{2}\mathbf{x}\cdot\mathbf{x}\right)/(t+1)\right]\\ &\times 2^{1/2}(t+1)^{-1}F\left(2^{1/2}\mathbf{x}/(t+1)\right)\end{aligned} \tag{7.20}$$

is a solution of the heat equation for any solution F of the Helmholtz equation (7.19). By choosing appropriate group elements g and solutions F, we can construct solutions of the heat equation satisfying a wide variety of initial and boundary conditions. Some examples are given by Bateman [13, p. 340].

Now we proceed to a study of the complex heat equation. This is equation (7.1) where now t, x_1, x_2 are complex variables. It is obvious that the symmetry algebra $\mathcal{G}_3^c$ of this equation is (complex) nine dimensional with basis (7.2). The basis operators can be exponentiated to yield the local Lie group G_3^c of symmetry operators acting on the space $\mathfrak{F}$ of functions $\Psi(t,\mathbf{x})$ analytic in some domain $\mathfrak{D}$ in complex (t, x_1, x_2) space. The group action is given by (7.4)–(7.6) where now the parameters $\mathbf{w}, \mathbf{z}, \rho$ are allowed to take arbitrary complex values and the matrices $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ range over the group $SL(2, \mathcal{C})$. Of course these operators map solutions of the complex heat equation into solutions.

The problem of R-separation of variables for this equation can be formulated in a manner analogous to that of the complex heat equation $(\partial_t - \partial_{xx})\Phi = 0$ in Section 2.2. We expect that all such R-separable systems will correspond to a pair of commuting symmetry operators in the envelop-

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ing algebra of $\mathcal{G}_3^c$. It is clear that all the real R-separable systems listed in Tables 12 and 13 can be analytically continued to yield R-separable systems for the complex heat equation. However, each system Aa in Table 12 is complex equivalent to the system Aa in Table 13. Furthermore, the systems Oc, Or, Oe are complex equivalent to the systems Rc, Rr, Re, respectively.

There exist other R-separable systems complex inequivalent to these. For example, if we diagonalize the operator ∂_t, we can reduce (7.1) to the complex Helmholtz equation and, from Table 3, find separable solutions that are products of Bessel functions, clearly inequivalent to any of the entries in Tables 12 and 13.

The separation of variables problem for (7.1) has been completely solved by E. G. Kalnins (private communication), who finds 38 nontrivial separable systems, each system characterized by a pair of commuting symmetry operators. Rather than discuss these results here, we will simply make use of the separable systems at hand and use them to apply Weisner's method.

As suggested by (7.12), for Laguerre polynomial solutions of (7.1) it is appropriate to introduce new coordinates

$$z = -\left(x_1^2 + x_2^2\right)/4t, \qquad s = i(x_1 + ix_2)/2, \qquad \tau = t. \tag{7.21}$$

In terms of these coordinates the basis functions (7.12) become (suitably normalized)

$$\begin{aligned} &\Phi_{m,n}(t,s,z) = \tau^n s^m L_n^{(m)}(z), \qquad H^0\Phi_{m,n} = (m+2n+1)\Phi_{m,n}, \\ &M\Phi_{m,n} = im\Phi_{m,n}, \qquad n = 0,1,2,\ldots. \end{aligned} \tag{7.22}$$

These expressions make sense for any $m \in \mathcal{C}$ such that m is not a negative integer. Since the Laguerre polynomial $L_n^{(m)}(z)$ can be expressed as a confluent hypergeometric function (see (B.9i)), we can choose another set of eigenfunctions

$$\Psi_{m,n}(\tau,s,z) = \tau^n s^m {}_1F_1\left(\begin{matrix} -n \\ m+1 \end{matrix}\middle| z\right), \qquad \Phi_{m,n} = \binom{m+n}{n}\Psi_{m,n} \tag{7.23}$$

and a linearly independent set of eigenfunctions

$$\Psi'_{m,n}(\tau,s,z) = \tau^n s^m z^{-m} {}_1F_1\left(\begin{matrix} -n-m \\ -m+1 \end{matrix}\middle| z\right). \tag{7.24}$$

That is, $\Psi_{m,n}$ and $\Psi'_{m,n}$ form a basis for the space of solutions of the eigenvalue equations (7.22) for fixed n, m.

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In terms of the τ,s,z coordinates, the operators (7.2) become

$$\begin{aligned}&H_2=\tau^2\partial_\tau+\tau s\,\partial_s+\tau z\,\partial_z+\tau(1-z),\quad H_{-2}=\tau^{-1}(\tau\,\partial_\tau-z\,\partial_z),\\&H_{-1}^-=\partial_s+zs^{-1}\partial_z,\quad H_{-1}^+=s\tau^{-1}\partial_z,\quad H_1^-=\tau\,\partial_s+\tau zs^{-1}\partial_z-\tau zs^{-1}\\&H_1^+=s\,\partial_z-s,\qquad \tilde H^0=s\,\partial_s,\qquad H^0=s\,\partial_s+2\tau\,\partial_\tau+1,\qquad H_0=1,\end{aligned}\tag{7.25}$$

where

$$\begin{aligned}&H_{-1}^-=-iP_1-P_2,\quad H_{-1}^+=-iP_1+P_2,\quad H_1^-=-iB_1-B_2,\\&H_1^+=-iB_1+B_2,\quad \tilde H^0=iM.\end{aligned}\tag{7.26}$$

Note that

$$\left[H^0,H_j^\alpha\right]=jH_j^\alpha,\qquad \left[\tilde H_j^0,H_j^\alpha\right]=\alpha H_j^\alpha.\tag{7.27}$$

It is clear from the explicit expressions (7.25) that each of the Lie algebra operators maps a polynomial in z to another such polynomial. It follows from this and the commutation relations (7.27) that $H_j^\alpha\Psi_{m,n}$ must be a constant times $\Psi_{m+(\alpha)1,n+[j-(\alpha)1]/2}$. Differentiating the power series (7.23) term by term, we can verify

$$\begin{aligned}&H_2\Psi_{m,n}=(m-n+1)\Psi_{m,n+1},\quad H_{-2}\Psi_{m,n}=n\Psi_{m,n-1},\\&H_{-1}^-\Psi_{m,n}=m\Psi_{m-1,n},\qquad H_{-1}^+\Psi_{m,n}=-n(m+1)^{-1}\Psi_{m+1,n-1},\\&H_1^-\Psi_{m,n}=m\Psi_{m-1,n+1},\qquad H_1^+\Psi_{m,n}=-(n+m+1)(m+1)^{-1}\Psi_{m+1,n},\\&\tilde H^0\Psi_{m,n}=m\Psi_{m,n},\qquad H^0\Psi_{m,n}=(m+2n+1)\Psi_{m,n}.\end{aligned}\tag{7.28}$$

Note that the first six of these relations agree exactly with the six differential recurrence formulas (B.8) for the functions ${}_1F_1$. Thus we have an interpretation of the recurrence formulas in terms of the action of the symmetry algebra of the complex heat equation.

Note also that the operators $H_{\pm 2},H^0$, which form a basis for an $sl(2,\mathbb{C})$ subalgebra of $\mathcal{G}_3^c$, yield the same recurrence formulas for Laguerre functions as did the operators $J^\pm,J^0$ in Section 2.4 (see (4.9)). This is due to the fact that equation (4.1) can be obtained from (7.1) by introducing polar coordinates and separating out the angular variable. Thus all the results of Section 2.4 can be obtained as special cases of results concerning solutions of (7.1).

Moreover, most of Chapter 4 in the author's book [82] is concerned with identities for Laguerre functions that can be obtained from a study of the

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subalgebra of $\mathcal{G}_3^c$ with basis $\{H_{-1}^-, H_1^+, \tilde{H}^0, H_0\}$ and commutation relations $(H_{\pm 1}^{\pm}=H^{\pm}, \tilde{H}^0=H^0)$,

$$[H^0,H^{\pm}]=\pm H^{\pm}, \qquad [H^+,H^-]=H_0, \qquad [H_0,H]=0. \quad (7.29)$$

(See also [78].) Thus it is clear that the special function theory associated with the heat equation is rich in useful results. Here, we will present only a few examples illustrating the interplay between the symmetry of this equation and identities obeyed by the separated solutions.

It is easy to see that the fundamental generating function (4.11) for Laguerre polynomials arises when the solution $\exp(\alpha H_2)\Phi_{-2l-1,0}$ of the complex heat equation is evaluated in two different ways. Similarly, if we apply the operator $\exp(\alpha H_1^-)$ to the basis function $\Phi_{m,0}(\tau,s,z)=s^m$ and make use of the recurrence formula $H_1^-\Phi_{l,n}=(n+1)\Phi_{l-1,n+1}$ and the Lie theory relation

$$\exp(\alpha H_1^-)\Psi(\tau,s,z)=\exp(-\tau z\alpha/s)\Psi(\tau,s+\alpha\tau,z[1+\alpha\tau/s]),$$

we obtain the generating function

$$e^{-\alpha z}(1+\alpha)^m=\sum_{n=0}^{\infty}\alpha^n L_n^{(m-n)}(z), \qquad m\in\mathcal{C}, \quad |\alpha|<1. \quad (7.30)$$

(Here we have set $\tau=s$ and factored s^m out of both sides of this expression.)

We can obtain the action of the local symmetry group G_3^c in terms of the coordinates t,s,z by combining expressions (7.21) with (7.4)–(7.7). In particular, the Appell transform has the simple appearance

$$\mathbf{T}(A_0^2)\Phi(\tau,s,z)=\tau^{-1}e^z\Phi(-\tau^{-1},s\tau^{-1},-z). \quad (7.31)$$

Applying this operator to the basis function $\Psi_{m,n}$, (7.23), with $m,n\in\mathcal{C}$ such that m is not a negative integer, we obtain

$$\mathbf{T}(A_0^2)\Psi_{m,n}=(-1)^n\tau^{-m-n-1}s^m e^z\,{}_1F_1\left(\begin{matrix}-n\\ m+1\end{matrix}\middle|-z\right).$$

This expression is a simultaneous eigenfunction of H^0 and M again, with eigenvalues $-m-2n-1$ and im, respectively. Furthermore, it is analytic in z at $z=0$. Hence, there exists a constant $c_{m,n}$ such that

$$\mathbf{T}(A_0^2)\Psi_{m,n}=c_{m,n}\Psi_{m,-m-n-1}.$$

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Setting $z=0$ on both sides of this expression, we obtain $c_{m,n}=(-1)^n$ or

$$e^z\,{}_1F_1\left(\begin{matrix}-n\\ m+1\end{matrix}\middle|-z\right)={}_1F_1\left(\begin{matrix}m+n+1\\ m+1\end{matrix}\middle|z\right). \tag{7.32}$$

This is the important transformation formula for the ${}_1F_1$ listed in Appendix B.

The heat equation can be written in the form $(H_{-2}-P_1^2-P_2^2)\Phi=0$ or, what is the same thing, $(H_{-2}+H_{-1}^+H_{-1}^-)\Phi=0$. It follows from (7.25) that in terms of coordinates $\{\tau,s,z\}$ this equation reads

$$(z\,\partial_{zz}+(s\,\partial_s-z+1)\,\partial_z+\tau\,\partial_\tau)\Phi=0. \tag{7.33}$$

A straightforward application of Weisner's method shows that any solution Φ of (7.33) that is analytic in τ, s, z in a suitable region, such that Φ can be expanded in a Laurent series in τ, s about $\tau=0$, $s=0$ and such that $\Phi(\tau,s,0)$ is bounded in this region, must satisfy an identity of the form

$$\Phi(\tau,s,z)=\sum_{m,n} c_{m,n}L_n^{(m)}(z)\tau^n s^m \tag{7.34}$$

where the $c_{m,n}$ are complex constants. Conversely, a uniformly convergent series of the form (7.34) in some region of τ,s,z space defines a solution of the complex heat equation. We conclude that all generating functions of the form (7.34) are obtainable as solutions of the heat equation. One way to find such functions Φ is to characterize them as simultaneous eigenfunctions of a pair of commuting operators in the enveloping algebra of $\mathcal{G}_3^c$. For example, the equations

$$\{B_1,P_1\}\Phi=(4\alpha+2)\Phi, \qquad H^0\Phi=(\lambda+1)\Phi, \qquad \alpha,\lambda\in\mathcal{C}, \tag{7.35}$$

correspond to the coordinates u,v,w where

$$u=\tau^{-1/2}(s+\tau z/s), \qquad v=\tau^{-1/2}(-s+\tau z/s), \qquad w=\tau; \tag{7.36}$$

see 10a in Table 13. In terms of the new coordinates we have

$$H^0=2w\,\partial_w+1, \qquad \{B_1,P_1\}=-8\left(\partial_{uu}-\tfrac{1}{2}u\,\partial_u-\tfrac{1}{4}\right)$$

and solutions $\Phi^{\alpha,\lambda}$ of (7.35) can be written in the form $\Phi^{\alpha,\lambda}=w^{\lambda/2}U(u)V(v)$ where

$$2U''-uU'+\alpha U=0, \qquad 2V''+vV'+(\alpha-\lambda)V=0. \tag{7.37}$$

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Comparing these equations with (2.24) and (2.25), we find the independent solutions $H_\alpha(u/2)$ and $\exp(u^2/4)H_{-\alpha-1}(iu/2)$ for U and $H_{\lambda-\alpha}(iv/2)$, $\exp(-v^2/4)H_{\alpha-\lambda-1}(v/2)$ for V. To be definite we choose the solutions

$$\Phi^{\alpha,\lambda}(u,v,w)=w^{\lambda/2}H_\alpha(u/2)H_{\lambda-\alpha}(iv/2). \tag{7.38}$$

By changing to u,v,w coordinates in the expressions (7.25) for the $\mathcal{G}_3^c$ symmetry operators and applying these operators to the functions (7.38), the reader can obtain a family of simple recurrence formulas obeyed by products of Hermite functions. Let us apply Weisner's method to the generating function (7.38) in the case where λ and α are positive integers with $\alpha\leqslant\lambda$. In this case the Hermite functions appearing in (7.38) are Hermite polynomials. From (7.34) and (7.36) we obtain

$$\begin{aligned}\tau^{\lambda/2}H_\alpha\left[2^{-1}(\tau^{-1/2}s+\tau^{1/2}z/s)\right]&H_{\lambda-\alpha}\left[i2^{-1}(-\tau^{-1/2}s+\tau^{1/2}z/s)\right]\\&=\sum_{k=0}^{\lambda}s^{\lambda-2k}\tau^k c_k L_k^{(\lambda-2k)}(z).\end{aligned} \tag{7.39}$$

(We have used the facts that $H_\alpha(x)$ is a polynomial of order α and $H_\alpha(-x)=(-1)^\alpha H_\alpha(x)$ to obtain this result.) Setting $x=s\tau^{-1/2}$, we find

$$\begin{aligned}H_\alpha\left[2^{-1}(x+z/x)\right]&H_{\lambda-\alpha}\left[i2^{-1}(-x+z/x)\right]\\&=\sum_{k=0}^{\lambda}x^{\lambda-2k}c_k L_k^{(\lambda-2k)}(z).\end{aligned} \tag{7.40}$$

To obtain a simple generating function for the coefficients c_k, we set $z=0$ and use the fact that $L_n^{(m)}(0)=\binom{m+n}{n}$ where $\binom{m+n}{n}$ is a binomial coefficient (B.1):

$$H_\alpha(x/2)H_{\lambda-\alpha}(-ix/2)=\sum_{k=0}^{\lambda}\binom{\lambda-k}{k}c_k x^{\lambda-2k}. \tag{7.41}$$

By explicitly computing the coefficient of $x^{\lambda-2k}$ on the left-hand side of this equation, we can express c_k as a terminating hypergeometric series ${}_3F_2$.

The polynomial functions (7.38) can be used as an alternative (but less useful) basis for solutions of the heat equation. Thus, one can compute matrix elements of the group operators $\mathbf{T}(g)$ in this basis, expand an arbitrary solution Φ in terms of the basis and so on.

For λ and α complex numbers, we can derive infinite series identities that are similar to (7.40) but slightly more complicated.

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2.8 Concluding Remarks

We close this chapter by pointing out several important research results closely related to our subject but which will not be treated in detail here.

In [140], Winternitz, Smorodinsky, Uhlir, and Fris determined all potentials $V(x,y)$ such that the time-independent Schrödinger equation

$$(-\Delta_2+V(x,y))\Phi=\lambda\Phi \tag{8.1}$$

admits a first- or second-order symmetry operator. They showed that the possible symmetry operators are of the form $L+f(x,y)$ where $L\in\mathcal{E}(2)$ ((1.6), (1.7) in Section 1.1) for first-order symmetries, and of the form $S+f(x,y)$ where S is a second-order symmetric operator in the enveloping algebra of $\mathcal{E}(2)$ for second-order symmetries. The equations that admit first-order symmetries separate in corresponding coordinate systems (2.31) or (2.32), Section 1.2. Equations that admit no first-order symmetries but do admit second-order symmetries separate in one of the four coordinate systems listed in Table 1. The latter equations are class II. Note that the Lie algebra $\mathcal{E}(2)$ appears in this study even though it is *not* the symmetry algebra of equation (8.1) except in the trivial case in which $V(x,y)$ is constant. The reason that $\mathcal{E}(2)$ appears here is that a first- or second-order symmetry operator for (8.1) has the property that its derivative terms L or S must necessarily commute with the Laplace operator Δ_2; hence L and S must belong to the enveloping algebra of $\mathcal{E}(2)$, as follows from the results of Section 1.1. Usually however, the complete symmetry operator will not belong to the enveloping algebra of $\mathcal{E}(2)$ because the functional part $f(x,y)$ of the operator will not be zero or even a constant. Here, $f(x,y)$ will depend on the potential.

Just as in Section 1.1, separation of variables corresponding to first-order symmetries turns out to be rather trivial. The interesting cases are the class II equations that admit second-order symmetries but no nontrivial first-order symmetries. Such equations admit separation of variables in one or more of the four coordinate systems listed in Table 1. Each separable coordinate system is determined by the pure differential part of the symmetry operator, that is, the part which belongs to the enveloping algebra of $\mathcal{E}(2)$. Thus, the occurrence of the operator Δ_2 in (8.1) limits the number of separable coordinate systems to four at most. Whether or not a given equation (8.1) separates in one of these coordinate systems depends on the explicit form of the potential V. One finds that (8.1) separates in one of the coordinate systems if and only if this equation admits the second-order symmetry $S+f(x,y)$ where the pure differential operator S corresponds to the coordinate system.

Although the interesting cases of (8.1) as treated in [140] are all class II, it frequently happens that such cases arise from class I equations by a

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partial separation of variables. For example, (8.1) arises from the time-dependent Schrödinger equation (5.1) if we split off the time variable by the assumption $\Psi(x,y,t)=e^{-i\lambda t}\Phi(x,y)$. The class I harmonic oscillator, repulsive oscillator, and linear potential equations which we have studied in the time-dependent case become class II equations in [140] when the t variable is separated. This is because separation of the t variable drastically reduces the symmetry of the Schrödinger equation.

An especially interesting class II equation that appears in [140] corresponds to the potential $v(x,y)=-\alpha/x^2-\beta/y^2$ where α,β are real constants such that $\alpha^2+\beta^2>0$. In [18] and [103], Boyer and Niederer have both listed this potential in their classifications of all potentials $V(x,y)$ such that the time-dependent Schrödinger equation admits nontrivial first-order symmetry operators. In [19], Boyer studied the time-dependent Schrödinger equation

$$\left(i\partial_t+\partial_{xx}+\partial_{yy}-\alpha/x^2-\beta/y^2\right)\Psi=0 \tag{8.2}$$

from the point of view presented in this book. He showed that this equation is still class II. However, it is highly tractable since it can be obtained from the free-particle Schrödinger equation (class I), $(i\partial_t+\Delta_4)\Psi=0$ by a partial separation of variables. Boyer showed that (8.2) R-separates in 25 coordinate systems for $\alpha=0$, $\beta\neq 0$ and in 15 coordinate systems for $\alpha\neq 0$, $\beta\neq 0$. Moreover, he found that each separable coordinate system corresponded to a pair of commuting second-order symmetry operators of (8.2). The special function identities that he obtained from this study are similar to, but not the same as, those obtained in Section 2.5.

In [5, 6], Armstrong used methods due to the author and the Wigner–Echart theorem to study the quantum-mechanical systems of Section 2.3, all of which admit $SL(2,R)$ as a dynamical symmetry group. He considered infinite families of self-adjoint operators on $L_2(R+)$ that transform irreducibly under the adjoint action of $SL(2,R)$ and used group theory to compute the matrix elements of these operators with respect to a basis of eigenvectors of L_3. See also [99]. An extension of the theory which is similar in viewpoint to the methods of this book is contained in [86] and [87].

Finally, in [23] group theory and separation of variables are used to determine all possible first- and second-order raising operators for Hamiltonians of the form $H=-\Delta_2+V(x,y)$. (A *raising operator* R for H satisfies the commutation relation $[H,R]=\mu R$, $\mu>0$. If the eigenvector Ψ of H satisfies $H\Psi=\lambda\Psi$, then formally $H(R\Psi)=(\lambda+\mu)R\Psi$, so R maps an eigenvector corresponding to eigenvalue λ to an eigenvector corresponding to eigenvalue $\lambda+\mu$.) In [23] it is shown that a necessary condition for H to admit a second-order raising operator is that equation (8.1) separate in one of the four coordinate systems listed in Table 1. A complete list of possible raising operators is given.

Exercises

1. Compute the symmetry algebra of the free-particle Schrödinger equation (1.2).
2. Determine the decomposition of the Schrödinger algebra $\mathcal{G}_2$ into orbits under the adjoint action of G_2.
3. Show that under the adjoint action of $SL(2,R)$, the Lie algebra $sl(2,R)$ decomposes into three orbits.
4. Expression (1.30) shows explicitly the equivalence between the Schrödinger equations for the free particle and the harmonic oscillator. Derive the corresponding expression giving the equivalence between the free-particle and linear potential equations.
5. Derive the bilinear expansions (1.55) for the fundamental solution of the Schrödinger equation

$$k(t,x-y)=(4\pi it)^{-1/2}\exp\left[-(x-y)^2/4it\right]$$

with respect to the bases $\{F_\lambda^{(j)}\}$, $j=2$, 4. (See [35, 136] for detailed discussions of such continuous generating functions.)

6. Use the methods of Section 2.2 to solve the Cauchy problem for

$$\partial_t\Phi=\partial_{xx}\Phi+x\Phi.$$

That is, find a bounded solution $\Phi(t,x)$ of this equation for $t>0$, continuous for $t\geqslant 0$, such that $\Phi(0,x)=f(x)$ where $f(x)$ is bounded and continuous on the real line.

7. The Hermite functions $H_n(z)$, (2.26), are polynomials for $n=0,1,2,\ldots$ and for $n=-1,-2,\ldots$ they are called Hermite functions of the second kind. Show that the second-kind functions can be expressed in terms of the error function and its derivatives [37]. Verify that the functions $\Phi_n(z,s)=H_n(z)s^n$, $n=0,\pm 1,\pm 2,\ldots$, satisfy the recurrence relations

$$H_1\Phi_n=\Phi_{n+1},\qquad H^0\Phi_n=\left(n+\tfrac{1}{2}\right)\Phi_n,\qquad H_{-1}\Phi_n=(n/2)\Phi_{n-1},$$
$$H_2\Phi_n=\Phi_{n+2},\qquad H_{-2}\Phi_n=\tfrac{1}{2}n(n-1)\Phi_{n-2},$$

where the operators H_j, (2.23), form a basis for the symmetry algebra of the complex heat equation. Show that this representation is not irreducible. Use the simple models constructed in Section 2.2 to compute the matrix elements of this representation and obtain the corresponding special function identities. In particular, derive the identity associated with the expression $\exp(\alpha H_1)\Phi_{-1}$.

8. Compute the bilinear expansion for the fundamental solution $k(t,x,y)$, (3.19), of the radial free-particle Schrödinger equation in terms of the Laguerre polynomial basis. Show that the expansion is a special case of the Hille–Hardy formula (4.27). Determine the bilinear expansion of $k(t,x,y)$ in terms of the continuum basis $\{\Psi_\lambda^{(2)}\}$, (3.16).
9. Compute the symmetry algebra of the complex heat equation $\partial_t\Phi-\partial_{xx}\Phi-\partial_{yy}\Phi=0$.