CHAPTER 5

Lie Theory and Hypergeometric Functions

With respect to a suitable basis, the matrix elements of irreducible representations of $\mathscr{G}(1, 0) \simeq sl(2) \oplus (\mathscr{E})$ can be expressed in terms of hypergeometric functions. In Sections 5-1 to 5-7 these matrix elements will be computed and employed to derive identities and differential relations for the hypergeometric functions. Sections 5-8 to 5-15 are devoted to the realization of irreducible representations of $sl(2)$ in terms of *type A* and *type B* differential operators. It will be shown that the most general hypergeometric function can be obtained as a basis vector transforming under the *type A* operators. Similarly, the most general confluent hypergeometric (Laguerre) function will arise as a basis vector transforming under *type B* operators. This connection between *type A* and *B* operators and special functions will prove to be a powerful tool for deriving identities involving hypergeometric and Laguerre functions.

Up to isomorphism the complex Lie algebra $sl(2)$ has two distinct real forms: $su(2)$ and $L(G_3)$. In Sections 5-16 to 5-18 all of the unitary irreducible representations of the real Lie groups $SU(2)$ and G_3 will be obtained by restriction of corresponding local multiplier representations of $SL(2)$. The implications of unitarity for special function theory will be briefly discussed.

Finally, in Section 5-19 the Lie algebraic fact that $\mathscr{T}_3$ is a contraction of $sl(2)$ will be used to derive a formula expressing Bessel functions as limits of Jacobi polynomials.

5-1 The Representation $D^{\mu}(u, m_o)$

According to Theorem 2.3 the irreducible representation $D^{\mu}(u, m_o)$ of $\mathscr{G}(1, 0)$ is defined by complex constants μ, u, m_o such that neither $m_o + u$ or $m_o - u$ is an integer, and $0 \leqslant \operatorname{Re} m_o < 1$. The spectrum of this representation is the set $S = \{m_o + n\colon n \text{ an integer}\}$. There is a basis $\{f_m : m \in S\}$ for the representation space V such that

$$\begin{aligned} J^3 f_m &= m f_m, \quad & J^+ f_m &= (m-u) f_{m+1}, \quad & J^- f_m &= -(m+u) f_{m-1}, \\ E f_m &= \mu f_m, \quad & C_{1,0} f_m &= (J^+J^- + J^3J^3 - J^3) f_m = u(u+1) f_m \end{aligned}$$

for all $m \in S$. These operators satisfy the commutation relations

$$[J^3, J^{\pm}] = \pm J^{\pm}, \qquad [J^+, J^-] = 2J^3, \qquad [J^{\pm}, E] = [J^3, E] = 0. \tag{5.1}$$

We will construct a realization of the algebraic representation $D^{\mu}(u, m_o)$ such that $J^{\pm}$, J^3, E take the form of the differential operators (2.35) acting on a vector space of analytic functions of the complex variable z. Let $\mathscr{V}_1$ be the complex vector space of all finite linear combinations of the functions $h_n(z) = z^n$, $n = 0, \pm 1, \pm 2, \ldots$. In Eqs. (2.35) set $\lambda = m_o$, $c_3 = -u - m_o$, to obtain

$$\begin{aligned} J^3 &= m_o + z\frac{d}{dz}, \qquad & E &= \mu, \\ J^+ &= (m_o - u)z + z^2\frac{d}{dz}, \qquad & J^- &= -\left(\frac{u + m_o}{z} + \frac{d}{dz}\right). \end{aligned} \tag{5.2}$$

The basis vectors f_m, $m \in S$, are defined by $f_m(z) = h_n(z)$ for $m = m_o + n$ and all integers n. Thus,

$$\begin{aligned} J^3 f_m &= \left(m_o + z\frac{d}{dz}\right) z^n = (m_o + n)z^n = m f_m, \\ J^+ f_m &= \left((m_o - u)z + z^2\frac{d}{dz}\right) z^n = (m-u)z^{n+1} = (m-u) f_{m+1}, \\ J^- f_m &= -\left(\frac{u+m_o}{z} + \frac{d}{dz}\right) z^n = -(m+u)z^{n-1} = -(m+u) f_{m-1}, \\ E f_m &= \mu f_m. \end{aligned} \tag{5.3}$$

Clearly, the operators (5.2) define a realization of $D^{\mu}(u, m_o)$ on $\mathscr{V}_1$.

We can now apply the procedure described in Section 2-2 to extend this realization of $D^{\mu}(u, m_o)$ on $\mathscr{V}_1$ to a local multiplier representation of the local Lie group $G(1, 0) = GL(2)$. However, since $\mathcal{G}(1, 0) \simeq sl(2) \oplus (\mathscr{E})$ and $\rho(\mathscr{E}) = \mathrm{E}$ is a multiple of the identity operator for every irreducible representation ρ of $\mathcal{G}(1, 0)$ listed in Theorem 2.3, the nontrivial part of the representation theory of $\mathcal{G}(1, 0)$ is concerned solely with the action of ρ on $sl(2)$. We can set $\mathrm{E} = \mu = 0$ without loss of generality for special function theory. It is clear, moreover, that the representation $D^0(u, m_o)$ of $\mathcal{G}(1, 0)$ induces an irreducible representation $D(u, m_o)$ of $sl(2)$. The action of $D(u, m_o)$ is obtained from Eqs. (5.1)–(5.3) by suppressing the operator E. In the remainder of this section we will study the representation $D(u, m_o)$ of $sl(2)$ and extend it to a local multiplier representation of the local group $SL(2)$. Similarly, rather than study the representations $\uparrow_u^\mu$, $\downarrow_u^\mu$, $D(2u)$ of $\mathcal{G}(0, 1)$ given by Theorem 2.3, we will restrict ourselves to the representations $\uparrow_u$, $\downarrow_u$, $D(2u)$ of $sl(2)$ obtained by setting $\mathrm{E} = \mu = 0$.

It was shown in Section 1-2 that $sl(2)$ is the Lie algebra of the local Lie group $SL(2)$,

$$SL(2) \equiv \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d, \in \mathcal{C}, \det g = 1 \right\},$$

i.e., the group of all 2×2 complex matrices with determinant equal to $+1$. Theorem 1.10 can be applied to extend the realization of $D(u, m_o)$ defined on $\mathscr{V}_1$ to a local multiplier representation of $SL(2)$ defined on $\mathcal{O}_1$, where $\mathcal{O}_1$ is the complex vector space of all functions of z analytic in some neighborhood of the point $z = 1$. Clearly, $\mathcal{O}_1 \supset \mathscr{V}_1$ and $\mathcal{O}_1$ is invariant under the differential operators (5.2). Thus, these operators generate a Lie algebra of generalized Lie derivatives corresponding to a local multiplier representation A of $SL(2)$ acting on $\mathcal{O}_1$.

The action of the 1-parameter subgroup $\{\exp c\mathscr{J}^-, c \in \mathcal{C}\}$ of $SL(2)$ on $\mathcal{O}_1$ is obtained by solving the equations

$$\frac{dz}{dc} = -1, \qquad \frac{d}{dc}\nu(z^o, \exp c\mathscr{J}^-) = -\frac{u + m_o}{z}\nu(z^o, \exp c\mathscr{J}^-)$$

with initial conditions $z(0) = z^o \neq 0$, $\nu(z^o, \mathbf{e}) \equiv 1$, where

$$\exp c\mathscr{J}^- = \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix}.$$

The solution of these equations is

$$z(c) = z^o - c, \qquad \nu(z^o, \exp c\mathscr{J}^-) = (1 - c/z^o)^{u+m_o}.$$

If $f \in \mathcal{O}_1$ is analytic in a neighborhood of z^o we have

$$[\mathbf{A}(\exp c\mathscr{J}^-)f](z^o) = (1 - c/z^o)^{u+m_o} f(z^o - c)$$

and $\mathbf{A}(\exp c\mathcal{J}^-) f \in \mathcal{O}_1$ if $|c/z^o| < 1$ and $z^o - c$ is in the domain of f. Similarly,

$$[\mathbf{A}(\exp b\mathcal{J}^+)f](z^o) = (1 - bz^o)^{u-m_0} f\left(\frac{z^o}{1 - bz^o}\right)$$

where $\mathbf{A}(\exp b\mathcal{J}^+) f \in \mathcal{O}_1$ if $|bz^o| < 1$ and $z^o/(1 - bz^o)$ is in the domain of f; and

$$[\mathbf{A}(\exp \tau\mathcal{J}^3)f](z^o) = e^{m_0\tau} f(e^{\tau} z^o)$$

where $\mathbf{A}(\exp \tau\mathcal{J}^3) f \in \mathcal{O}_1$ if $e^{\tau} z^o$ is in the domain of f. If $g \in SL(2)$ and $d \neq 0$ it is a straightforward computation to show that

$$g = (\exp b'\mathcal{J}^+)(\exp c'\mathcal{J}^-)(\exp \tau'\mathcal{J}^3)$$

where $b' = -b/d$, $c' = -cd$, $e^{\tau'/2} = d^{-1}$, $0 \leqslant \operatorname{Im} \tau' < 4\pi$. Thus, for $|b'|, |c'|, |\tau'|$ sufficiently small, the operator $\mathbf{A}(g)$ is given by

$$\begin{aligned}[\mathbf{A}(g)f](z) &= [\mathbf{A}(\exp b'\mathcal{J}^+)\, \mathbf{A}(\exp c'\mathcal{J}^-)\, \mathbf{A}(\exp \tau'\mathcal{J}^3) f](z) \\ &= (d + bz)^{u-m_0} \left(a + \frac{c}{z}\right)^{u+m_0} f\left(\frac{az + c}{bz + d}\right) \qquad (5.4)\end{aligned}$$

for $f \in \mathcal{O}_1$ and $z \neq 0$ in the domain of f. We have used the fact $ad - bc = 1$. The multiplier ν takes the form

$$\nu(z, g) = (d + bz)^{u-m_0}(a + c/z)^{u+m_0}.$$

As it stands, the final expression in (5.4) is not well defined; it is not even a single-valued function of the group parameters. However, it is obvious that for g in a sufficiently small neighborhood of the identity element, $\nu(z, g)$ has a unique Laurent series expansion in z. To give a precise definition of the operators $\mathbf{A}(g)$ we restrict g to the open set $N \subset SL(2)$,

$$N = \{g \in SL(2) : |c/a| < 1 < |d/b|, -\pi < \arg a, \arg d < \pi\}.$$

As a local Lie group, N is isomorphic to $SL(2)$. Given $f \in \mathcal{O}_1$ let D_f be the domain of f, i.e., the open set in $\mathcal{C}$, containing 1, on which f is defined and analytic. For every $g \in N$, $z \in \mathcal{C}$ let zg be the complex number

$$zg = \frac{az + c}{bz + d}$$

and define $\mathcal{D}_1(g)$, the domain of g, by

$$\mathcal{D}_1(g) = \{f \in \mathcal{O}_1 : 1g \in D_f\}, \qquad 1 \in \mathcal{C}.$$

Then, if $g \in N$, and $f \in \mathscr{D}_1(g)$ we have $\mathbf{A}(g)\, f \in \mathscr{O}_1$ where $\mathbf{A}(g)f$ is defined by the Laurent series expansion of ν in (5.4). In particular,

$$D_{\mathbf{A}(g)f} = \{| c/a | < | z | < | d/b |\} \cap \{z : zg \in D_f\}.$$

To give a precise statement of the representation property of the operators $\mathbf{A}(g)$ we define the set $N(g_o)$, for every $g_o \in N$, by $N(g_o) = \{g \in N: gg_o \in N\}$ and let $N'(g_o)$ be the connected component of $N(g_o)$ containing the identity element $\mathbf{e} \in SL(2)$. Then, if $g_2 \in N, g_1 \in N'(g_2)$ we have $g_1g_2 \in N$; and, for every $f \in \mathscr{O}_1$ such that $f \in \mathscr{D}_1(g_2)$ and $\mathbf{A}(g_2)f \in \mathscr{D}_1(g_1)$, we can conclude that $f \in \mathscr{D}_1(g_1g_2)$ and

$$[\mathbf{A}(g_1g_2)f](z) = [\mathbf{A}(g_1)(\mathbf{A}(g_2)f)](z) \tag{5.5}$$

for all z in some nonzero neighborhood of $z = 1$. (The restriction of g_1 to $N'(g_2)$ is needed to make the representation single-valued.)

Following the procedure of Section 2-2 we can restrict the multiplier representation A from $\mathscr{O}_1$ to $\bar{\mathscr{V}}_1$. ($\bar{\mathscr{V}}_1$ is the space of all finite linear combinations of functions of the form $\mathbf{A}(g)f$ where $g \in N$, $f \in \mathscr{V}_1$.) Clearly $\bar{\mathscr{V}}_1$ is invariant under A. Furthermore, every $h \in \bar{\mathscr{V}}_1$ has a unique Laurent expansion

$$h(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \qquad a_n \in \mathscr{C},$$

which converges absolutely in an open annulus D_h about $z = 0$, including the point $z = 1$ in its interior. Thus the basis functions $f_m(z) = h_n(z) = z^n$, $m = m_o + n$, n an integer, form an analytic basis for $\bar{\mathscr{V}}_1$ (see Section 2-2).

The matrix elements $A_{lk}(g)$ of the operators $\mathbf{A}(g)$ on $\bar{\mathscr{V}}_1$ are defined by

$$[\mathbf{A}(g)h_k](z) = \sum_{l=-\infty}^{\infty} A_{lk}(g)h_l(z), \qquad g \in N, \quad k = 0, \pm 1, \pm 2, \ldots, \tag{5.6}$$

or

$$z^k \left(1 + \frac{c}{az}\right)^{u+m_o+k} \left(1 + \frac{bz}{d}\right)^{u-m_o-k} a^{u+m_o+k} d^{u-m_o-k} = \sum_{l=-\infty}^{\infty} A_{lk}(g) z^l, \tag{5.7}$$

where $| c/a | < | z | < | d/b |$, $-\pi < \arg a$, $\arg \mathrm{d} < \pi$, and $ad - bc = 1$. From (5.5),

$$\mathbf{A}(g_1g_2)h_k = \mathbf{A}(g_1)[\mathbf{A}(g_2)h_k],$$

which leads to the addition theorem

$$A_{lk}(g_1g_2) = \sum_{j=-\infty}^{\infty} A_{lj}(g_1)A_{jk}(g_2), \qquad l, k \text{ integers}, \tag{5.8}$$

valid for all $g_1, g_2 \in N$ such that $g_1 \in N'(g_2)$. Expanding the left-hand side of (5.7) by means of the binomial theorem and computing the coefficient of z^l we obtain an explicit expression for the matrix elements:

$$A_{lk}(g) = \frac{a^{s+l}d^{t-k}c^{k-l}\Gamma(s+k+1)}{\Gamma(s+l+1)(k-l)!}F(-s-l, -t+k; k-l+1; bc/ad) \quad \text{if} \quad k \geqslant l,$$
$$A_{lk}(g) = \frac{a^{s+k}d^{t-l}b^{l-k}\Gamma(t-k+1)}{\Gamma(t-l+1)(l-k)!}F(-s-k, -t+l; l-k+1; bc/ad) \quad \text{if} \quad l \geqslant k. \tag{5.9}$$

Here $g \in N$, Γ is the gamma function, F is the hypergeometric function defined by its power series expansion, (A.4), and $s = u + m_o$, $t = u - m_o$. Note that neither s nor t is an integer. The two expressions (5.9) can be combined into one by noting that

$$\frac{F(-s-l, -t+k; k-l+1; bc/ad)}{\Gamma(k-l+1)}$$

is defined even when $k - l + 1$ is a negative integer. In fact, using (A.5) we obtain

$$A_{lk}(g) = a^{s+l}d^{t-k}c^{k-l}\frac{\Gamma(s+k+1)}{\Gamma(s+l+1)}\frac{F(-s-l, -t+k; k-l+1; bc/ad)}{\Gamma(k-l+1)} \tag{5.10}$$

for all integers l, k. The matrix elements can be written in other equivalent forms by making use of the transformation formulas (A.8), but this will be left to the reader.

It will sometimes be convenient to adopt a different coordinate system for N. Namely, we write

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} e^{\alpha/2} & 0 \\ 0 & e^{-\alpha/2} \end{pmatrix} \begin{pmatrix} \cosh \frac{w}{2} & \sinh \frac{w}{2} \\ \sinh \frac{w}{2} & \cosh \frac{w}{2} \end{pmatrix} \begin{pmatrix} e^{\beta/2} & 0 \\ 0 & e^{-\beta/2} \end{pmatrix}$$
$$= (\exp \alpha \mathscr{J}^3) \left(\exp -w \left(\frac{\mathscr{J}^+ + \mathscr{J}^-}{2}\right)\right) (\exp \beta \mathscr{J}^3).$$

A simple computation shows that the coordinates $a, b, c, d = (1 + bc)/a$ are related to the coordinates α, β, w by

$$a = e^{(\alpha+\beta)/2} \cosh \frac{w}{2}, \qquad b = e^{(\alpha-\beta)/2} \sinh \frac{w}{2},$$
$$c = e^{(-\alpha+\beta)/2} \sinh \frac{w}{2}, \qquad d = e^{(-\alpha-\beta)/2} \cosh \frac{w}{2}, \qquad \cosh w = 2bc + 1. \tag{5.11}$$

To make this change of variable one-to-one on N we would have to put some restrictions on the range of the complex variables α, β, w. Thus, (α, β, w) and $(\alpha + i\pi, \beta - i\pi, -w)$ correspond to the same coordinates a, b, c. Further the coordinate transformation is not defined for $bc = 0$, so the coordinates (α, β, w) cannot be extended over all of N. For our purposes, however, it is enough to note that if we can assign to $g \in N$ coordinates (α, β, w) satisfying (5.11), then for suitably restricted values of the parameters the matrix elements $A_{lk}(g)$ are given by

$$A_{lk}(g) = e^{\alpha(m_o+l)}e^{\beta(m_o+k)} \frac{\Gamma(u + m_o + k + 1)}{\Gamma(u + m_o + l + 1)} \mathfrak{B}_u^{-m_o-l, m_o+k}(\cosh w) \quad (5.12)$$

where $\cosh w = 2bc + 1$ and $\mathfrak{B}_u^{r,m}(z)$ is defined by (A.9), (iii). (Indeed (5.12) is obviously true for α, β real and $w > 0$. Its domain of validity can then be extended by analytic continuation.) Clearly, the branch points of $\mathfrak{B}_u^{r,m}(z)$ are located at $z = -1, +1, \infty$. Thus, if the z-plane is cut along the real axis from $-\infty$ to $+1$ this function can be analytically continued to the whole z-plane outside the cut. Furthermore, we have the relations

$$\mathfrak{B}_u^{r,m}(z) = \mathfrak{B}_u^{m,r}(z), \qquad \mathfrak{B}_u^{r,m}(z) = \mathfrak{B}_{-u-1}^{r,m}(z). \quad (5.13)$$

An important special case is

$$\mathfrak{B}_\nu^{o,-\mu}(z) = \mathfrak{B}_\nu^{\mu}(z)$$

where the generalized spherical harmonics $\mathfrak{B}_\nu^{\mu}$ are defined by (A.9), (ii). The functions $\mathfrak{B}_u^{r,m}(z)$ will be encountered frequently in the remainder of this chapter.

If the expression (5.10) for the matrix elements is inserted into (5.7) one obtains the generating function

$$(1 + z^{-1})^s(1 + bz)^t = \sum_{l=-\infty}^{\infty} z^l \frac{\Gamma(s + 1)}{\Gamma(s + l + 1)} \frac{F(-s - l, -t; -l + 1; b)}{\Gamma(-l + 1)},$$
$$1 < |z| < |b^{-1}|. \quad (5.14)$$

We can derive addition theorems for the hypergeometric functions by substituting (5.10) into (5.8). The result is (after some manipulation)

$$(1 + b/a)^{-s}(1 + ac)^{-t}(a + 1)^k \frac{F\left(s, t; k + 1; \dfrac{(c + b/a)(a + 1)}{(1 + b/a)(1 + ac)}\right)}{\Gamma(k + 1)}$$
$$= \sum_{j=-\infty}^{\infty} a^j \frac{F(s, t + j - k; j + 1; b)}{\Gamma(j + 1)} \frac{F(s - j, t; -j + k + 1; c)}{\Gamma(-j + k + 1)}, \quad (5.15)$$

where s, t are arbitrary complex numbers, not integers, and k is an integer. When k is a negative integer, the function $[\Gamma(k)]^{-1} F(s, t; k; z)$ is defined by (A.5).

As a power series in z, $F(s, t; k; z)$ converges absolutely for $|z| < 1$. However, if a cut is made in the complex plane from $+1$ to $+\infty$ along the real axis, this function can be analytically continued to a function analytic and single-valued in z throughout the cut plane. Thus, Eq. (5.15) can be analytically continued in the variables a, b, c to a larger domain of validity than that originally implied by the addition theorem (5.8). In fact, the left-hand side of (5.15) is defined and analytic for all $a, b, c \in \mathcal{C}$ such that $|b| < |a| < |1/c|$ and $(c + b/a)(a + 1)(1 + b/a)^{-1}(1 + ac)^{-1}$ does not cross the cut. Hence, the right-hand side of (5.15) must also be defined and analytic for these values of the variables, where in addition we must require that b and c do not cross the cut.

Two special cases of this formula are of interest. For $b = 0$,

$$(1 + ac)^{-t}(a + 1)^k \frac{F\left(s, t; k + 1; \dfrac{c(a + 1)}{1 + ac}\right)}{\Gamma(k + 1)} = \sum_{j=0}^{\infty} \frac{a^j}{j!} \frac{F(s - j, t; -j + k + 1; c)}{\Gamma(-j + k + 1)}, \qquad |ac| < 1,$$

while if $c = 0$,

$$(1 + b/a)^{-s}(1 + 1/a)^k \frac{F\left(s, t; k + 1; \dfrac{(a + 1)(b/a)}{(1 + b/a)}\right)}{\Gamma(k + 1)} = \sum_{j=-\infty}^{k} \frac{a^{j-k}}{(k - j)!} \frac{F(s, t + j - k; j + 1; b)}{\Gamma(j + 1)}, \qquad |b/a| < 1.$$

5-2 The Representation $\uparrow_u$

The irreducible representation $\uparrow_u^\mu$ of $\mathcal{G}(1, 0)$ is defined for all $\mu, u \in \mathcal{C}$ such that $2u$ is not a nonnegative integer. The spectrum of this representation is

$$S = \{-u + n : n \text{ a nonnegative integer}\},$$

and the representation space V has a basis $\{f_m, m \in S\}$ such that

$$\mathrm{J}^3 f_m = m f_m, \qquad \mathrm{E} f_m = \mu f_m, \qquad \mathrm{J}^+ f_m = (m - u) f_{m+1},$$
$$\mathrm{J}^- f_m = -(m + u) f_{m-1}, \qquad \mathrm{C}_{1\,0} f_m = (\mathrm{J}^+\mathrm{J}^- + \mathrm{J}^3\mathrm{J}^3 - \mathrm{J}^3) f_m = u(u + 1) f_m.$$

(Here $f_{-u-1} \equiv 0$, so $\mathrm{J}^- f_{-u} = 0$.)

As in the last section we can construct a realization of $\uparrow_u^\mu$ on a space of analytic functions of one complex variable z such that $J^\pm$, J^3, E are the differential operators (2.35). Let $\mathscr{V}_2$ be the complex vector space of all finite linear combinations of the functions $h_n(z) = z^n$, $n = 0, 1, 2, \ldots$, and define the basis functions f_m, $m \in S$, by $f_m(z) = h_n(z)$ where $m = -u + n$. In Eqs. (2.35) set $\lambda = -u$, $c_3 = -2u$ to obtain the differential operators

$$J^3 = -u + z\frac{d}{dz}, \qquad E = \mu, \qquad J^+ = -2uz + z^2\frac{d}{dz}, \qquad J^- = -\frac{d}{dz}.$$

These operators and basis functions satisfy the relations

$$\begin{aligned}
J^3 f_m &= \left(-u + z\frac{d}{dz}\right) z^n = (-u+n)z^n = m f_m,\\
J^+ f_m &= \left(-2uz + z^2\frac{d}{dz}\right) z^n = (-2u+n)z^{n+1} = (m-u) f_{m+1},\\
J^- f_m &= -\frac{d}{dz} z^n = -nz^{n-1} = -(u+m) f_{m-1},\\
E f_m &= \mu f_m
\end{aligned} \tag{5.16}$$

for all $m \in S$ and, thus, determine a realization of $\uparrow_u^\mu$.

In the usual manner this realization can be extended to a multiplier representation of $G(1, 0)$ defined on $\mathscr{A}_2' \supset \mathscr{V}_2$. Here $\mathscr{A}_2'$ is the complex vector space of all functions f analytic in some neighborhood of $z = 0$, i.e., the space of all functions f of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad a_n \in \mathscr{C}, \tag{5.17}$$

where the power series converges in a nonzero neighborhood of $z = 0$. However, as was shown in Section 5-1, without loss of generality we can set $E = \mu = 0$ in (5.16) and consider the representation $\uparrow_u$ of $sl(2)$ induced by the representation $\uparrow_u^0$ of $\mathscr{G}(1, 0)$. Hence, we shall suppress the operator E in the following paragraphs and extend the realization of $\uparrow_u$ given by (5.16) to a local multiplier representation B of $SL(2)$ on $\mathscr{A}_2'$.

The multiplier representation B was computed in Section 1-4. Indeed, from Eq. (1.69)′ we have

$$[\mathbf{B}(g)f](z) = (bz+d)^{2u} f\left(\frac{az+c}{bz+d}\right), \qquad f \in \mathscr{A}_2', \tag{5.18}$$

where $g \in SL(2)$ is given by (1.15). However, to define this multiplier representation precisely we restrict g to the open set $P \subset SL(2)$ where

$$P = \{g \in SL(2) : d \neq 0, \mid \arg d \mid < \pi\}.$$

P is a local Lie group isomorphic to $SL(2)$, i.e., P and $SL(2)$ have the same Lie algebra. For every $f \in \mathcal{A}'_2$ let $D_f \subset \mathcal{C}$ be the domain of f. Furthermore, if $zg = (az + c)/(bz + d)$ for every $g \in P$, $z \in \mathcal{C}$, define $\mathcal{D}_2(g)$, the domain of g, by

$$\mathcal{D}_2(g) = \{f \in \mathcal{A}'_2 : 0g \in D_f\}, \qquad 0 \in \mathcal{C}. \tag{5.19}$$

Then if $g \in P$ and $f \in \mathcal{D}_2(g)$ we have $\mathbf{B}(g)\, f \in \mathcal{A}'_2$ where $\mathbf{B}(g)\, f$ is given by the power series (5.18). In fact,

$$D_{\mathbf{B}(g)f} = \{\mid z \mid < \mid d/b \mid\} \cap \{z : zg \in D_f\}.$$

Given $g_o \in P$ define the set $P(g_o) \subset P$ by

$$P(g_o) = \{g \in P : gg_o \in P\}$$

and let $P'(g_o)$ be the connected component of $P(g_o)$ containing $\mathbf{e}$ (the identity element of $SL(2)$. The fact that B is a local representation of P is expressed as follows: (1) If $g_2 \in P$, $g_1 \in P'(g_2)$, then $g_1 g_2 \in P$; and (2) for every $f \in \mathcal{A}'_2$ such that $f \in \mathcal{D}_2(g_2)$ and $\mathbf{B}(g_2)\, f \in \mathcal{D}_2(g_1)$, we can conclude that $f \in \mathcal{D}_2(g_1 g_2)$ and

$$[\mathbf{B}(g_1 g_2) f](z) = [\mathbf{B}(g_1)(\mathbf{B}(g_2) f)](z)$$

for all z in a nonzero neighborhood of $z = 0$. The restriction of g_1 to $P'(g_2)$ is required to make the representation single-valued ($2u$ may not be an integer). However, if $2u$ happens to be a negative integer this restriction is not necessary.

According to our usual procedure we define the matrix elements $B_{lk}(g)$ of the operators $\mathbf{B}(g)$ by

$$[\mathbf{B}(g) h_k](z) = \sum_{l=0}^{\infty} B_{lk}(g) h_l(z), \qquad g \in P, \quad k = 0, 1, 2, \ldots, \tag{5.20}$$

or

$$d^{2u-k}\left(1 + \frac{bz}{d}\right)^{2u-k} (az + c)^k = \sum_{l=0}^{\infty} B_{lk}(g) z^l, \qquad \mid bz/d \mid < 1, \tag{5.21}$$

where $|\arg d| < \pi$ and $ad - bc = 1$. From the representation property of the operators $\mathbf{B}(g)$ we obtain the addition theorem

$$B_{lk}(g_1 g_2) = \sum_{j=0}^{\infty} B_{lj}(g_1) B_{jk}(g_2), \qquad l, k \geqslant 0,$$

which holds for all $g_1, g_2 \in P$ such that $g_1 \in P'(g_2)$. The matrix elements were computed explicitly in Section 1-4:

$$B_{lk}(g) = \frac{a^l d^{2u-k} c^{k-l} \Gamma(k+1)}{\Gamma(l+1)\Gamma(k-l+1)} F(-l, -2u+k; k-l+1; bc/ad) \quad \text{if } k \geqslant l \geqslant 0, \tag{5.22}$$

$$B_{lk}(g) = \frac{a^k d^{2u-l} b^{l-k} \Gamma(2u-k+1)}{\Gamma(2u-l+1)\Gamma(l-k+1)} F(-k, -2u+l; l-k+1; bc/ad) \quad \text{if } l \geqslant k \geqslant 0, \tag{5.23}$$

However, from (A.5) it follows that both (5.22) and (5.23) retain their validity for all $k, l \geqslant 0$ without restriction. The hypergeometric series in these expressions break off after a finite number of terms. Hence they are hypergeometric polynomials in bc/ad (the Jacobi polynomials).

In terms of the variables (α, β, w), Eq. (5.11), the matrix elements are

$$\begin{aligned} B_{lk}(g) &= e^{\alpha(-u+l)} e^{\beta(-u+k)} \left(\frac{k!}{l!}\right) \mathfrak{B}_u^{u-l,-u+k}(\cosh w) \\ &= e^{\alpha(-u+l)} e^{\beta(-u+k)} \frac{\Gamma(2u-k+1)}{\Gamma(2u-l+1)} \mathfrak{B}_u^{-u+l,u-k}(\cosh w) \end{aligned} \tag{5.24}$$

where the functions $\mathfrak{B}_u^{r,m}(z)$ are defined by (A.9). (See the remarks immediately following Eq. (5.12).)

From (5.21) and (5.23) there follows the generating function

$$(1+z)^{2u-k}(b+z)^k = \sum_{l=0}^{\infty} z^l \frac{\Gamma(2u-k+1)}{\Gamma(2u-l+1)} \frac{F(-k, -2u+l; l-k+1; b)}{\Gamma(l-k+1)}$$

valid for $|z| < 1$. The addition theorem for the matrix elements B_{lk} takes the form

$$\begin{aligned} &(a+b)^l (1+ac)^{2u} (a+1)^{k-l} \\ &\quad \cdot \frac{F(-l, -2u; k-l+1; (ac+b)(a+1)(a+b)^{-1}(1+ac)^{-1})}{\Gamma(k-l+1)} \\ &= \sum_{j=0}^{\infty} a^j \frac{F(-l, -2u-k+j; j-l+1; b)}{\Gamma(j-l+1)} \\ &\quad \cdot \frac{F(-j, -2u; k-j+1; c)}{\Gamma(k-j+1)}, \qquad |ac| < 1, \end{aligned} \tag{5.25}$$

valid for all nonnegative integers k, l, and all complex numbers u such that $2u$ is not a nonnegative integer. The function $[\Gamma(n)]^{-1}F(s, t; n; z)$ is defined by (A.5) when n is a negative integer. If $b = 0$, Eq. (5.25) simplifies to

$$(1 + ac)^{2u}(a + 1)^{k-l}\frac{F(-l, -2u; k - l + 1; c(a + 1)(1 + ac)^{-1})}{\Gamma(k - l + 1)}$$
$$= \sum_{j=0}^{\infty}\frac{a^j}{j!}\frac{F(-j - l, -2u; k - j - l + 1; c)}{\Gamma(k - j - l + 1)}, \qquad |ac| < 1,$$

while if $c = 0$, it becomes

$$(a + b)^l\frac{(a + 1)^{k-l}}{\Gamma(k - l + 1)}F\left(-l, -2u; k - l + 1; \frac{b(a + 1)}{a + b}\right)$$
$$= \sum_{j=0}^{k}\frac{a^j}{(k - j)!}\frac{F(-l, -2u - k + j; j - l + 1; b)}{\Gamma(j - l + 1)}$$

5-3 The Representation $\downarrow_u$

According to Theorem 2.3 the irreducible representation $\downarrow_u^\mu$ of $\mathscr{G}(1, 0)$ is defined for each $\mu, u \in \mathscr{C}$ such that $2u$ is not a nonnegative integer. The spectrum of this representation is the set $S = \{u - n: n$ a nonnegative integer$\}$. The representation space V has a basis $\{f_m, m \in S\}$ such that

$$\mathrm{J}^3 f_m = m f_m, \qquad \mathrm{E} f_m = \mu f_m, \qquad \mathrm{J}^+ f_m = (m - u) f_{m+1},$$
$$\mathrm{J}^- f_m = -(m + u) f_{m-1}, \qquad \mathrm{C}_{1,0} f_m = (\mathrm{J}^+\mathrm{J}^- + \mathrm{J}^3\mathrm{J}^3 - \mathrm{J}^3) f_m = u(u + 1) f_m.$$

(Here $f_{u+1} = 0$.) As in the first two sections of this chapter we could find a realization of $\downarrow_u^\mu$ in terms of the differential operators (2.35). However, it is more convenient to use the operators

$$\mathrm{J}^3 = u - z\frac{d}{dz}, \qquad \mathrm{E} = \mu, \qquad \mathrm{J}^+ = -\frac{d}{dz}, \qquad \mathrm{J}^- = -2uz + z^2\frac{d}{dz} \tag{5.26}$$

which are a modification of (2.35). These operators satisfy the commutation relations (5.1), hence, they generate a Lie algebra isomorphic to $\mathscr{G}(1, 0)$.

We will realize $\downarrow_u^\mu$ on the space $\mathscr{V}_2$ of all finite linear combinations of the functions $h_n(z) = z^n$, $n \geqslant 0$, where $\mathrm{J}^\pm$, J^3, E are given by (5.26).

Thus, if the basis functions f_m, $m \in S$, for $\mathscr{V}_2$ are defined by $f_m(z) = z^n$ where $m = u - n$, $n \geqslant 0$, we have

$$\begin{aligned}
J^3 f_m &= \left(u - z\frac{d}{dz}\right) z^n = (u-n)z^n = m f_m, \\
J^+ f_m &= \left(-\frac{d}{dz}\right) z^n = -nz^{n-1} = (m-u) f_{m+1}, \\
J^- f_m &= \left(-2uz + z^2\frac{d}{dz}\right) z^n = (-2u+n)z^{n+1} = -(m+u) f_{m-1}, \\
E f_m &= \mu f_m,
\end{aligned} \tag{5.27}$$

which yields a realization of $\downarrow_u^\mu$.

As usual we set $E = \mu = 0$ in (5.26), (5.27), and restrict ourselves to consideration of the representation $\downarrow_u$ of $sl(2)$ induced by $\downarrow_u^0$. The realization of $\downarrow_u$ given by (5.27) can be extended to a local multiplier representation of $SL(2)$ on the space $\mathcal{O}l_2'$ of all functions of the complex variable z, analytic in some neighborhood of $z = 0$. Since the procedures involved in deriving this multiplier representation and computing its matrix elements are so similar to the work of Section 5-2, only the results will be presented. Consider the neighborhood of U of the identity element of $SL(2)$ defined by

$$U = \{g \in SL(2) : a \neq 0, \mid \arg a \mid < \pi\}$$

and the set $\mathscr{D}_3(g)$,

$$\mathscr{D}_3(g) = \{f \in \mathcal{O}l_2' : b/a \in D_f\},$$

where $D_f \subset \mathcal{C}$ is the domain of f. The action of the local Lie group U on $\mathcal{O}l_2'$ is determined by the operators $\mathbf{C}(g)$ where

$$[\mathbf{C}(g)f](z) = (cz+a)^{2u} f\left(\frac{dz+b}{cz+a}\right), \tag{5.28}$$

defined for all $f \in \mathcal{O}l_2'$, $g \in U$, and $z \in \mathcal{C}$ such that

$$| z | \leqslant \left|\frac{a}{c}\right| \quad \text{and} \quad \frac{dz+b}{cz+a} \in D_f.$$

Given $g_o \in U$, define the set $U(g_o)$ by $U(g_o) = \{g \in U: gg_o \in U\}$ and let $U'(g_o)$ be the connected component of $U(g_o)$ containing $\mathbf{e}$. The fact that the operators $\mathbf{C}(g)$ form a local representation of U is expressed by the relation

$$[\mathbf{C}(g_1 g_2)f](z) = [\mathbf{C}(g_1)(\mathbf{C}(g_2)f)](z),$$

valid for $g_2 \in U$, $g_1 \in U'(g_2)$, $f \in \mathscr{D}_3(g_2)$, $\mathbf{C}(g_2)f \in \mathscr{D}_3(g_1)$ and z in a suitable nonzero neighborhood of $z = 0$. The matrix elements for this multiplier representation are defined by

$$[\mathbf{C}(g)h_k](z) = \sum_{l=0}^{\infty} C_{lk}(g)h_l(z), \qquad k \geqslant 0, \quad g \in U, \tag{5.29}$$

which leads to

$$(cz + a)^{2u-k}\,(dz + b)^k = \sum_{l=0}^{\infty} C_{lk}(g)z^l, \qquad |\,cz/a\,| < 1. \tag{5.30}$$

A comparison of (5.21) and (5.30) shows that the matrix elements can be obtained from those of the preceding section by making the interchanges $a \leftrightarrow d$, $b \leftrightarrow c$. Thus,

$$\begin{aligned} C_{lk}(g) &= \frac{d^l a^{2u-k} b^{k-l}\Gamma(k+1)}{\Gamma(l+1)} \frac{F(-l, -2u+k; k-l+1; bc/ad)}{\Gamma(k-l+1)} \\ &= \frac{d^k a^{2u-l} c^{l-k}\Gamma(2u-k+1)}{\Gamma(2u-l+1)} \frac{F(-k, -2u+l; l-k+1; bc/ad)}{\Gamma(l-k+1)}. \end{aligned} \tag{5.31}$$

Finally, we have the addition theorem

$$C_{lk}(g_1 g_2) = \sum_{l=0}^{\infty} C_{lj}(g_1)C_{jk}(g_2), \qquad l, k \geqslant 0, \tag{5.32}$$

valid for $g_2 \in U$, $g_1 \in U'(g_2)$. The identities for special functions obtained from (5.32) do not differ from (5.25).

5-4 The Representation $D(2u)$

The finite-dimensional representation $D^\mu(2u)$ of $\mathscr{G}(1, 0)$ is defined for all $\mu \in \mathcal{C}$ and all nonnegative integers $2u$. The spectrum S of this representation is given by $S = \{u, u-1, ..., -u+1, -u\}$ and the vectors f_m, $m \in S$, form a basis for the $(2u + 1)$-dimensional representation space V, where

$$\mathrm{J}^3 f_m = m f_m, \qquad \mathrm{E} f_m = \mu f_m, \qquad \mathrm{J}^+ f_m = (m - u) f_{m+1},$$
$$\mathrm{J}^- f_m = -(m + u) f_{m-1}, \qquad \mathrm{C}_{1,0} f_m = (\mathrm{J}^+\mathrm{J}^- + \mathrm{J}^3\mathrm{J}^3 - \mathrm{J}^3) f_m = u(u + 1) f_m.$$

(Here, $f_{-u-1} \equiv f_{u+1} \equiv 0$.)

A realization of this representation can be constructed on the space $\mathscr{V}^{(u)}$ of all linear combinations of the functions $h_n(z) = z^n$, $n = 0, 1, \ldots, 2u$, such that the operators $J^{\pm}$, J^3, E take the form (2.35). Indeed, for $\lambda = -u$, $c_3 = -2u$ in Eqs. (2.35) we have

$$J^3 = -u + z\frac{d}{dz}, \qquad E = \mu, \qquad J^+ = -2uz + z^2\frac{d}{dz}, \qquad J^- = -\frac{d}{dz}. \tag{5.33}$$

Define the basis functions f_m, $m \in S$ by $f_m(z) = h_n(z)$ where $m = -u + n$, $0 \leqslant n \leqslant 2u$. These operators and basis functions satisfy the relations

$$\begin{aligned}
J^3 f_m &= \left(-u + z\frac{d}{dz}\right) z^n = (-u+n)z^n = mf_m,\\
J^+ f_m &= \left(-2uz + z^2\frac{d}{dz}\right) z^n = (-2u+n)z^{n+1} = (m-u)\, f_{m+1},\\
J^- f_m &= -\frac{d}{dz} z^n = -nz^{n-1} = -(u+m)\, f_{m-1},\\
E f_m &= \mu f_m,
\end{aligned}$$

and define a realization of $D^{\mu}(2u)$.

As remarked earlier, without loss of generality for special function theory we can set $E = \mu = 0$ in (5.33) and restrict ourselves to consideration of the representation $D(2u)$ of $sl(2)$ induced by the representation $D^0(2u)$ of $\mathscr{G}(1, 0)$. Thus we suppress the operator E.

The representation $D(2u)$ of $sl(2)$ on $\mathscr{V}^{(u)}$ can be extended to a multiplier representation of $SL(2)$ on $\mathscr{V}^{(u)}$. In fact, the relevant computations were carried out in Section 1-4. It was shown there that the action of $SL(2)$ is determined by operators $\mathbf{D}(g)$, such that

$$[\mathbf{D}(g)f](z) = (bz + d)^{2u} f\left(\frac{az + c}{bz + d}\right) \tag{5.34}$$

for all $z \in \mathcal{C}$, $f \in \mathscr{V}^{(u)}$, and $g \in SL(2)$. Furthermore,

$$\mathbf{D}(g_1 g_2) f = \mathbf{D}(g_1)[\mathbf{D}(g_2) f] \qquad \text{for all} \quad g_1, g_2 \in SL(2).$$

In this case the operators $\mathbf{D}(g)$ are defined for all $g \in SL(2)$, not just in a neighborhood of the identity element.

The matrix elements $D_{lk}(g)$ of $D(2u)$ with respect to the basis $h_k(z) = z^k$, $k = 0, 1, \ldots, 2u$, are defined by

$$[\mathbf{D}(g)h_k](z) = \sum_{l=0}^{2u} D_{lk}(g) h_l(z), \qquad 0 \leqslant k \leqslant 2u, \tag{5.35}$$

or

$$(bz + d)^{2u-k}(az + c)^k = \sum_{l=0}^{2u} D_{lk}(g)z^l. \tag{5.36}$$

A simple computation yields

$$\begin{aligned} D_{lk}(g) &= \frac{a^l d^{2u-k} c^{k-l} \Gamma(k+1)}{\Gamma(l+1)} \frac{F(-l, -2u+k; k-l+1; bc/ad)}{\Gamma(k-l+1)} \\ &= \frac{a^k d^{2u-l} b^{l-k} \Gamma(2u-k+1)}{\Gamma(2u-l+1)} \frac{F(-k, -2u+l; l-k+1; bc/ad)}{\Gamma(l-k+1)}, \\ &\qquad 0 \leqslant k, l \leqslant 2u, \end{aligned} \tag{5.37}$$

where the hypergeometric polynomials are defined by (A.5) when $k - l$ (or $l - k$) is a negative integer. From (5.36) and (5.37) follows the generating function

$$(1 + z)^{2u-k}(b + z)^k = \sum_{l=0}^{2u} z^l \frac{\Gamma(2u-k+1)}{\Gamma(2u-l+1)} \frac{F(-k, -2u+l; l-k+1; b)}{\Gamma(l-k+1)}, \qquad 0 \leqslant k \leqslant 2u.$$

In terms of the variables (α, β, w), Eq. (5.11), the matrix elements become

$$D_{lk}(g) = e^{\alpha(-u+l)} e^{\beta(-u+k)} \left(\frac{k!}{l!}\right) \mathfrak{B}_u^{u-l,-u+k}(\cosh w), \tag{5.38}$$

where $\mathfrak{B}_u^{r,m}$ is defined by (A.9). (See the remarks following (5.12).) The addition theorem for the matrix elements is

$$D_{lk}(g_1 g_2) = \sum_{l=0}^{2u} D_{lj}(g_1) D_{jk}(g_2), \qquad 0 \leqslant l, k \leqslant 2u, \quad g_1, g_2 \in SL(2).$$

Substituting (5.37) into this expression we obtain, after some simplification,

$$\begin{aligned} &(a+b)^l (1+ac)^{2u-k} (a+1)^{k-l} \\ &\quad \cdot \frac{F\left(-l, -2u+k; k-l+1; \dfrac{(ac+b)}{(a+b)} \dfrac{(a+1)}{(1+ac)}\right)}{\Gamma(k-l+1)} \\ &= \sum_{j=0}^{2u} a^j \frac{F(-l, -2u+j; j-l+1; b)}{\Gamma(j-l+1)} \frac{F(-j, -2u+k; k-j+1; c)}{\Gamma(k-j+1)}, \end{aligned} \tag{5.39}$$

valid for all integers l, k, $2u$ such that $2u \geqslant l, k \geqslant 0$, and all $a, b, c \in \mathbb{C}$.

5-5 The Tensor Product $D(2u) \otimes D(2v)$

We now turn to the problem of decomposing the tensor product of two finite-dimensional irreducible representations $D(2u)$ and $D(2v)$ of $sl(2)$ into a direct sum of such representations. Although the solution to this problem is well known, it will be treated again here because of the simplicity of the method and because the treatment generalizes to tensor products of infinite-dimensional representations of $sl(2)$.

Let $D(2u)$, $2u$ a nonnegative integer, be a finite-dimensional representation of $sl(2)$ defined in Section 5-4. Recall that this representation was realized on the space $\mathscr{V}^{(u)}$ generated by the functions $h_k(z) = z^k$, $k = 0, 1, \ldots, 2u$. However, we will find it more convenient to use instead of the vectors $h_n(z)$, the renormalized basis vectors

$$p_n(z) = \frac{(-z)^{u+n}}{[(u+n)!\,(u-n)!]^{1/2}}, \qquad n = -u, -u+1, \ldots, u. \tag{5.40}$$

In terms of this new basis for $\mathscr{V}^{(u)}$ the action of the differential operators (5.33) becomes

$$\begin{aligned} \mathrm{J}^3 p_n = n p_n, \qquad \mathrm{J}^+ p_n &= [(u-n)(u+n+1)]^{1/2} p_{n+1}, \\ \mathrm{J}^- p_n &= [(u-n+1)(u+n)]^{1/2} p_{n-1}. \end{aligned} \tag{5.41}$$

If we define the matrix elements $Q_{lk}^{(u)}(g)$, $g \in SL(2)$, by

$$[\mathbf{D}(g) p_{k-u}](z) = \sum_{l=0}^{2u} Q_{lk}^{(u)}(g) p_{l-u}(z), \qquad k = 0, 1, \ldots, 2u, \tag{5.42}$$

we obtain

$$Q_{lk}^{(u)}(g) = (-1)^{l+k} \left[\frac{l!\,(2u-l)!}{k!\,(2u-k)!}\right]^{1/2} D_{lk}(g), \qquad 0 \leqslant l, k \leqslant 2u, \tag{5.43}$$

where $D_{lk}(g)$ is given by (5.37). Moreover, it is easy to derive the generating function

$$\frac{1}{(2u)!}[(bz+d) + w(az+c)]^{2u} = \sum_{l,k=0}^{2u} p_{k-u}(w) Q_{lk}^{(u)}(g) p_{l-u}(z). \tag{5.44}$$

The representation $D(2u) \otimes D(2v)$ can be defined on the vector space $\mathscr{V}^{(u)} \otimes \mathscr{V}^{(v)}$ consisting of all polynomials f in the variables z, w such that

$$f(z, w) = \sum_{k=0}^{2u} \sum_{l=0}^{2v} a_{kl} z^k w^l, \qquad a_{kl} \in \mathcal{C}.$$

The vectors $p_{k,l}$,

$$p_{k,l}(z, w) = \frac{(-z)^{u+k}(-w)^{v+l}}{[(u+k)!\,(v+l)!\,(u-k)!\,(v-l)!]^{1/2}}\,;$$
$$-u \leqslant k \leqslant n, \quad -v \leqslant l \leqslant v, \tag{5.45}$$

form a basis for $\mathscr{V}^{(u)} \otimes \mathscr{V}^{(v)}$. In particular, the dimension of this space is $(2u+1)(2v+1)$. The action of $SL(2)$ is given by the operators $\mathbf{T}(g)$, $g \in SL(2)$, where

$$[\mathbf{T}(g)f](z, w) = (bz + d)^{2u}(bw + d)^{2v} f\left(\frac{az + c}{bz + d}, \frac{aw + c}{bw + d}\right) \tag{5.46}$$

for all $f \in \mathscr{V}^{(u)} \otimes \mathscr{V}^{(v)}$. Clearly, the operators $\mathbf{T}(g)$ define a multiplier representation of $SL(2)$. Furthermore, computing the generalized Lie derivatives of this representation we have

$$\mathrm{J}^3 = -u - v + z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w}, \qquad \mathrm{J}^+ = -2uz - 2vw + z^2\frac{\partial}{\partial z} + w^2\frac{\partial}{\partial w},$$
$$\mathrm{J}^- = -\frac{\partial}{\partial z} - \frac{\partial}{\partial w}. \tag{5.47}$$

$D(2u) \otimes D(2v)$ is reducible. To decompose $\mathscr{V}^{(u)} \otimes \mathscr{V}^{(v)}$ into subspaces irreducible under the action of $sl(2)$ we procede by computing the eigenvectors of J^3 which are of lowest weight, i.e., eigenvectors f of J^3 such that $\mathrm{J}^- f = 0$ where J^3 and J^- are given by (5.47). The solutions are easily seen to be the functions

$$h_{s,0}(z, w) = N_s(z - w)^s, \qquad s = 0, 1,\ldots, 2q,$$

where the N_s are arbitrary constants and $q = \min(u, v)$. Indeed,

$$\mathrm{J}^3 h_{s,0} = (s - u - v)h_{s,0}, \qquad \mathrm{J}^- h_{s,0} = 0.$$

In analogy with Section 4-5, we introduce on $\mathscr{V}^{(u)} \otimes \mathscr{V}^{(v)}$ a complex symmetric bilinear form $\mathbf{B}$ determined by the relations

$$\mathbf{B}(p_{k,l}, p_{k',l'}) = \delta_{k,k'}\,\delta_{l,l'}, \qquad 0 \leqslant k, k' \leqslant 2u,$$
$$0 \leqslant l, l' \leqslant 2v, \tag{5.48}$$

where $\delta_{l,k} = 0$ if $k \neq l$, $\delta_{k,k} = 1$. Thus, $\mathbf{B}(f, h) = \mathbf{B}(h, f) \in \mathcal{C}$ and $\mathbf{B}(af + a'f', h) = a\mathbf{B}(f, h) + a'\mathbf{B}(f', h)$ for all $f, f', h \in \mathscr{V}^{(u)} \otimes \mathscr{V}^{(v)}$, $a, a' \in \mathcal{C}$. The following lemma relating this bilinear form to the operators (5.47) is easily proved:

Lemma 5.1 For all $f, h \in \mathscr{V}^{(u)} \otimes \mathscr{V}^{(v)}$,

(i) $\mathbf{B}(J^3 f, h) = \mathbf{B}(f, J^3 h)$,

(ii) $\mathbf{B}(J^+ f, h) = \mathbf{B}(f, J^- h)$.

This bilinear form can be used to normalize the functions $h_{s,0}$, i.e., the constants N_s can be chosen so that $\mathbf{B}(h_{s,0}, h_{s,0}) = 1$. Thus,

$$\begin{aligned}\mathbf{B}(h_{s,0}, h_{s,0}) &= N_s^2 \mathbf{B}\left(\sum_{j=0}^{s} \frac{s!\, z^j(-w)^{s-j}}{j!\,(s-j)!}, \sum_{l=0}^{s} \frac{s!\, z^l(-w)^{s-l}}{l!\,(s-l)!}\right) \\ &= N_s^2 \sum_{j=0}^{s} (s!)^2 \frac{(2u-j)!\,(2v-s+j)!}{j!\,(s-j)!} = 1.\end{aligned}$$

Making use of the identity

$$\sum_{j=0}^{k} \frac{(m+k-j)!\,(n+j)!}{j!\,(k-j)!} = \frac{m!\,n!\,(m+n+k+1)!}{k!\,(m+n+1)!} \tag{5.49}$$

(Gelfand *et al.* [1], p. 149), we obtain

$$N_s = (-1)^s \left[\frac{(2u+2v-2s+1)!}{s!\,(2u-s)!\,(2v-s)!\,(2u+2v-s+1)}\right]^{1/2}, \qquad s = 0, 1, \ldots, 2q, \tag{5.50}$$

where the phase factor $(-1)^s$ has been chosen for convenience.

Now that the functions $h_{s,0}$ have been determined, we define additional vectors $h_{s,k}$ by

$$h_{s,k} = \left[\frac{(2u+2v-2s-k)!}{k!\,(2u+2v-2s)!}\right]^{1/2} (J^+)^k h_{s,0}, \qquad s = 0, 1, \ldots, 2q, \quad k = 0, 1, \ldots, 2u+2v-2s, \tag{5.51}$$

where the operator J^+ is given by (5.47). Each $h_{s,k}$ is a polynomial of order $s+k$ in z and w, and there are a total of $\sum_{s=0}^{2q} (2u+2v-2s+1) = (2u+1)(2v+1)$ such polynomials. Furthermore, it follows exactly as in the proof of Theorem 2.3, part (iii) that $J^+ h_{s,2u+2v-2s} = 0$. We will show that the $h_{s,k}$ form a basis for $\mathscr{V}^{(u)} \otimes \mathscr{V}^{(v)}$.

Lemma 5.2

(i) $J^+ h_{s,k} = [(k+1)(2u+2v-2s-k)]^{1/2} h_{s,k+1}$,

(ii) $J^- h_{s,k} = [k(2u+2v-2s-k+1)]^{1/2} h_{s,k-1}$,

(iii) $J^3 h_{s,k} = (k-(u+v-s))\, h_{s,k}$,

$s = 0, 1, \ldots, 2q, \quad q = \min(u, v), \quad k = 0, 1, \ldots, 2u+2v-2s.$

PROOF

(i) $$J^+h_{s,k} = \left[\frac{(2u+2v-2s-k)!}{k!\,(2u+2v-2s)!}\right]^{1/2} (J^+)^{k+1}h_{s,0}$$
$$= [(k+1)(2u+2v-2s-k)]^{1/2}\, h_{s,k+1}\,.$$

(iii) From the identity $[J^3, (J^+)^k] = k(J^+)^k$ there follows

$$\begin{aligned} J^3h_{s,k} &= \left[\frac{(2u+2v-2s-k)!}{k!\,(2u+2v-2s)!}\right]^{1/2} J^3(J^+)^k h_{s,0} \\ &= \left[\frac{(2u+2v-2s-k)!}{k!\,(2u+2v-2s)!}\right]^{1/2} \cdot (J^+)^k J^3 h_{s,0} \\ &\quad + \left[\frac{(2u+2v-2s-k)!}{k!\,(2u+2v-2s)!}\right]^{1/2} k(J^+)^k h_{s,0} \\ &= (k-(u+v-s))h_{s,k}\,. \end{aligned}$$

(ii) We use induction on k. Since $J^-h_{s,o} = 0$ the equation holds for $k = 0$. Assume (ii) is valid for $k \leqslant k_o$ where $0 \leqslant k_o < 2u+2v-2s$. Then

$$J^-h_{s,k_o+1} = [(k_o+1)(2u+2v-2s-k_o)]^{-1/2}\, J^-J^+h_{s,k_o}$$

from (i). The relation $J^-J^+ = J^+J^- - 2J^3$ and the induction hypothesis imply

$$J^-J^+h_{s,k_o} = J^+J^-h_{s,k_o} - 2J^3h_{s,k_o} = (k_o+1)(2u+2v-2s-k_o)h_{s,k_o}\,.$$

Therefore, $J^-h_{s,k_o+1} = [(k_o+1)(2u+2v-2s-k_o)]^{1/2}\, h_{s,k_o}$. Q.E.D.

Comparing Eq. (5.41) with Lemma 5.2 we see that for a fixed value of s the vectors $h_{s,k}$, $k = 0, 1, \ldots, 2u+2v-2s$, form a basis for the representation $D(2u+2v-2s)$ of $sl(2)$. Thus, the action of the operators $\mathbf{T}(g)$, $g \in SL(2)$, on the vectors $h_{s,k}$ is

$$[\mathbf{T}(g)h_{s,k}](z, w) = \sum_{l=0}^{2u+2v-2s} Q_{lk}^{(u+v-s)}(g)h_{s,l}(z, w), \tag{5.52}$$

$s = 0, 1, \ldots, 2q$, $k = 0, 1, \ldots, 2(u+v-s)$, where the operators $\mathbf{T}(g)$ and the matrix elements Q are defined by (5.46) and (5.43).

A simple induction argument utilizing Lemmas 5.1 and 5.2 yields:

Lemma 5.3 $\mathbf{B}(h_{s,k}, h_{s',k'}) = \delta_{s,s'}\,\delta_{k,k'}$,

$s, s' = 0, 1, \ldots, 2q, \qquad k = 0, 1, \ldots, 2(u+v-s), \quad k' = 0, 1, \ldots, 2(u+v-s').$

According to this result, the set of $(2u+1)(2v+1)$ vectors $\{h_{s,k}\}$ is linearly independent; hence, it forms a basis for $\mathscr{V}^{(u)} \otimes \mathscr{V}^{(v)}$.

Theorem 5.1 $D(2u) \otimes D(2v) \cong \sum_{k=0}^{2\min(u,v)} \oplus D(2u+2v-2k)$.

The Clebsch–Gordan coefficients for this decomposition,

$$C(u, l; v, j \mid r, k)$$

nonzero only if $l + j = k$, are defined by

$$h_{s,k} = \sum_{l,j} C(u, l; v, j \mid u+v-s, k+s-u-v) p_{l,j},$$
$$s = 0, 1, \ldots, 2\min(u, v), \quad k = 0, 1, \ldots, 2(u+v-s). \tag{5.53}$$

Thus,

$$\mathbf{B}(h_{s,k}, p_{l,j}) = C(u, l; v, j \mid u+v-s, k+s-u-v)\, \delta_{l+j, k+s-u-v}. \tag{5.54}$$

From (5.54) it is easy to invert (5.53):

$$p_{l,j} = \sum_{s=0}^{2\min(u,v)} C(u, l; v, j \mid u+v-s, l+j) h_{s, l+j+u+v-s},$$
$$-u \leqslant l \leqslant u, \quad -v \leqslant j \leqslant v. \tag{5.55}$$

To derive a generating function for the Clebsch–Gordan coefficients, set $g = \exp(-b\mathscr{J}^+)$, $k = 0$, in (5.52) and obtain

$$N_s(bz+1)^{2u-s}(bw+1)^{2v-s}(z-w)^s = \sum_{l=0}^{2(u+v-s)} (-b)^l \cdot \left[\frac{(2u+2v-2s)!}{l!\,(2u+2v-2s-l)!}\right]^{1/2} h_{s,l}(z, w).$$

or

$$\frac{N_s(bz+1)^{2u-s}(bw+1)^{2v-s}(z-w)^s}{[(2u+2v-2s)!]^{1/2}} = \sum_{l=0}^{2(u+v-s)} \sum_{j=0}^{l+s} \cdot \frac{b^l(-1)^s C(u, j-u; v, s+l-j-v \mid u+v-s, s+l-u-v) z^j w^{s+l-j}}{[l!\,(2u+2v-2s-l)!\, j!\,(2u-j)!\,(s+l-j)!\,(2v-s-l+j)!]^{1/2}}. \tag{5.56}$$

At this point it is useful to note that the Clebsch–Gordan coefficients for which we have just established a generating function are exactly the coefficients computed by Wigner for the tensor products of irreducible representations of $SU(2)$ (the group of 2×2 unitary matrices with determinant $+1$) (Wigner [3], Hamermesh [1]). This is due to the fact that the unitary irreducible representations of $SU(2)$ are obtained by restricting the representations $D(2u)$ of $SL(2)$ to the subgroup $SU(2)$. We will examine this relationship in more detail in Section 5-16.

To show the connection between (5.56) and the theory of angular momentum in quantum mechanics we cast our results into a more conventional notation. Namely, we set $l_1 = u$, $l_2 = v$, $l_3 = u + v - s$, $m_1 = j - u$, $m_2 = s + l - j - v$, $m_3 = l - u - v + s$ where $2l_i$ is a nonnegative integer and m_i takes the values $m_i = l_i, l_i - 1, \ldots, -l_i + 1, -l_i$ for $i = 1, 2, 3$. Further, we introduce the 3-j coefficients defined by

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{l_3 - m_3}}{2l_3 + 1} \mathrm{C}(l_1, m_1; l_2, m_2 \mid l_3, -m_3) \tag{5.57}$$

(Hamermesh [1]). Then, for $b^{-1} = x$, (5.56) can be written in the symmetrical form

$$\frac{(x-z)^{l_1+l_3-l_2}(x-w)^{l_2+l_3-l_1}(z-w)^{l_1+l_2-l_3}}{[(l_1+l_2-l_3)!\,(l_2+l_3-l_1)!\,(l_1+l_3-l_2)!\,(l_1+l_2+l_3+1)!]^{1/2}}$$

$$= \sum_{\substack{0 \leqslant k_i \leqslant 2l_i \\ k_1+k_2+k_3 = l_1+l_2+l_3}} \frac{z^{k_1} w^{k_2} x^{k_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ k_1 - l_1 & k_2 - l_2 & k_3 - l_3 \end{pmatrix}}{[k_1!\,(2l_1-k_1)!\,k_2!\,(2l_2-k_2)!\,k_3!\,(2l_3-k_3)!]^{1/2}}. \tag{5.58}$$

This is a well-known generating function for the 3-j coefficients (Schwinger [1]). The obvious symmetry of this expression implies the existence of corresponding symmetries among the 3-j coefficients. The reader can derive these symmetries for himself or refer to standard references on the subject (Hamermesh [1], Wigner [3], Bargmann [4]).

Equations (5.54) allow us to express the matrix elements of the representation $D(2u) \otimes D(2v)$ with respect to the $p_{l,j}$ basis in terms of the matrix elements with respect to the $h_{s,k}$ basis. A standard argument (Lyubarskii [1], p. 234) yields

$$\begin{aligned} &Q^{(u)}_{ll'}(g) Q^{(v)}_{jj'}(g) \\ &= \sum_{s=0}^{2\min(u,v)} \mathrm{C}(u, l-u; v, j-v \mid u+v-s, l+j-u-v) \\ &\quad \cdot \mathrm{C}(u, l'-u; v, j'-v \mid u+v-s, l'+j'-u-v) Q^{(u+v-s)}_{l+j-s,\, l'+j'-s}(g), \end{aligned} \tag{5.59}$$

where $l, l' = 0, 1, \ldots, 2u$; $j, j' = 0, 1, \ldots, 2v$, and $g \in SL(2)$. In terms of the $\mathfrak{B}$ functions (5.38) this relation reads

$$\mathfrak{R}_u^{-l,l'}(z)\mathfrak{R}_v^{-j,j'}(z) = \sum_{s=0}^{2\min(u,v)} C(u, l; v, j \mid u+v-s, l+j)$$
$$\cdot\, C(u, l'; v, j' \mid u+v-s, l'+j')\mathfrak{R}_{u+v-s}^{-l-j,l'+j'}(z),$$
$$-u \leqslant l, l' \leqslant u, \quad -v \leqslant j, j' \leqslant v,$$

where

$$\mathfrak{R}_u^{-l,l'}(z) = \left[\frac{(u+l')!\,(u-l)!}{(u+l)!\,(u-l')!}\right]^{1/2} \mathfrak{B}_u^{-l,l'}(z)$$

(see Section 5-14).

5-6 The Tensor Product $\uparrow_u \otimes \uparrow_v$

The local multiplier representation $\uparrow_u$, $2u$ not a nonnegative integer, can be realized on the space $\mathscr{V}_2$ with basis $h_k(z) = z^k$, $k = 0, 1, 2, \ldots$:

$$J^3 h_k = (k-u)h_k, \qquad J^+ h_k = (k-2u)h_{k+1}, \qquad J^- h_k = -kh_{k-1}, \qquad (5.60)$$

where $J^\pm$, J^3 are the differential operators (5.16). In terms of this basis the matrix elements $B_{lk}^{(u)}(g)$, $g \in P$, are defined by (5.22), (5.23) where the superscript u denotes the representation $\uparrow_u$.

The multiplier representation $\uparrow_u \otimes \uparrow_v$ of $P \subset SL(2)$ ($2u$ and $2v$ not nonnegative integers) can be defined on the vector space $\mathscr{A}_2' \otimes \mathscr{A}_2'$, consisting of all functions $f(z, w)$, analytic in a neighborhood of the point $(0, 0) \in \mathscr{C} \times \mathscr{C}$. The functions $p_{k,l}(z, w) = z^k w^l$, $k, l = 0, 1, 2, \ldots$, form an analytic basis for $\mathscr{A}_2' \otimes \mathscr{A}_2'$. The action of P on this space is given by the operators $\mathbf{T}(g)$,

$$[\mathbf{T}(g)f](z, w) = (bz+d)^{2u}(bw+d)^{2v} f\left(\frac{az+c}{bz+d}, \frac{aw+c}{bw+d}\right), \qquad (5.61)$$

where $f \in \mathscr{A}_2' \otimes \mathscr{A}_2'$ and $g \in P$ is in a suitably small neighborhood of the identity element. It is easy to check that the operators $\mathbf{T}(g)$ define a local multiplier representation of P. This action of P on $\mathscr{A}_2' \otimes \mathscr{A}_2'$ induces a representation of $sl(2)$ in terms of the generalized Lie derivatives

$$J^3 = -u - v + z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w},$$
$$J^+ = -2uz - 2vw + z^2\frac{\partial}{\partial z} + w^2\frac{\partial}{\partial w}, \qquad (5.62)$$
$$J^- = -\frac{\partial}{\partial z} - \frac{\partial}{\partial w},$$

operating on the space $\mathscr{V}_2 \otimes \mathscr{V}_2$ of all polynomials in z and w. The Lie derivatives (5.62) are formally identical with (5.47), but they operate on different spaces and are defined for different values of u and v.

As in Section 5-5 we try to decompose $\mathscr{V}_2 \otimes \mathscr{V}_2$ into subspaces irreducible under the action of $sl(2)$. Again we compute the eigenvectors f of J^3 such that $J^- f \equiv 0$, where J^3, J^- are given by (5.62). The solutions are, to within a multiplicative constant, the functions

$$f_{s,0}(z, w) = (z - w)^s, \qquad s = 0, 1, 2,\dots . \tag{5.63}$$

Thus, $J^3 f_{s,0} = (s - u - v) f_{s,0}$, $J^- f_{s,0} = 0$, for $s \geqslant 0$. Let $\mathbf{Q}$ be the symmetric bilinear form on $\mathscr{V}_2 \otimes \mathscr{V}_2$ with the property

$$\mathbf{Q}(p_{k,l}, p_{k',l'}) = \delta_{k,k'}\, \delta_{l,l'}\, \xi(k, l); \qquad k, k', l, l' = 0, 1, 2,\dots, \tag{5.64}$$

where

$$\xi(k, l) = \frac{(-1)^{k+l} k!\, l!\, \Gamma(-2u)\Gamma(-2v)}{\Gamma(k - 2u)\Gamma(l - 2v)}. \tag{5.65}$$

(The vectors $p_{k,l}$, $k, l = 0, 1, 2,\dots$, form a basis for $\mathscr{V}_2 \otimes \mathscr{V}_2$.) If the operators $J^\pm$, J^3 are defined by (5.62) it is easy to prove:

Lemma 5.4

(i) $\mathbf{Q}(J^3 p_{k,l}, p_{k',l'}) = \mathbf{Q}(p_{k,l}, J^3 p_{k',l'})$,

(ii) $\mathbf{Q}(J^+ p_{k,l}, p_{k',l'}) = \mathbf{Q}(p_{k,l}, J^- p_{k',l'})$.

The factor $\xi(k, l)$ has been chosen just so Lemma 5.4 would hold. $\mathbf{Q}$ will be used as a kind of bookkeeping device to facilitate our calculations. Direct computation using (5.62) and (5.64) proves

$$\mathbf{Q}(f_{s,0}, f_{s,0}) = \frac{(-1)^s \Gamma(-2u - 2v + 2s - 1)}{s!\, \Gamma(-2u - 2v + s - 1)}\, \xi(s, s), \qquad s = 0, 1, 2,\dots . \tag{5.66}$$

In analogy with the procedure in Section 5-5 we define vectors $f_{s,k}$ by

$$f_{s,k} = \frac{\Gamma(2s - 2u - 2v + k)}{\Gamma(2s - 2u - 2v)} (J^+)^k f_{s,0}, \qquad k, s = 0, 1, 2,\dots .$$

Each $f_{s,k}$ is a polynomial of order $s + k$ in z and w, and there are $n + 1$ such functions of order n in z and w. Thus, if the set $\{f_{s,k}\}$ is linearly independent it forms a basis for $\mathscr{V}_2 \otimes \mathscr{V}_2$.

In exact analogy with Lemma 5.2 we have:

Lemma 5.5

$$J^+f_{s,k} = (k - 2u - 2v + 2s)f_{s,k+1},$$
$$J^-f_{s,k} = -kf_{s,k-1},$$
$$J^3f_{s,k} = (k - u - v + s)f_{s,k}, \qquad k, s = 0, 1, 2,\ldots .$$

A comparison of Lemma 5.5 with (5.60) shows that for fixed s the vectors $f_{s,k}$, $k = 0, 1, 2,\ldots$, form a basis for the representation $\uparrow_{u+v-s}$ of $sl(2)$. Thus, the action of the operators $\mathbf{T}(g)$, $g \in P$, on the vectors $f_{s,k}$ is

$$[\mathbf{T}(g)f_{s,k}](z, w) = \sum_{l=0}^{\infty} B^{(u+v-s)}_{lk}(g)f_{s,l}(z, w), \qquad k, s = 0, 1, 2,\ldots, \tag{5.67}$$

where $\mathbf{T}(g)$ and the matrix elements $B_{lk}(g)$ are defined by (5.61) and (5.22).

Lemmas 5.4 and 5.5 can be used to evaluate the quantities $\mathbf{Q}(f_{s,k}, f_{s',k'})$. The result is:

Lemma 5.6 $\mathbf{Q}(f_{s,k}, f_{s',k'}) = \delta_{s,s'}\,\delta_{k,k'}\,N_{s,k}$

where

$$N_{s,k} = \frac{(-1)^{s+k}s!\,k!\,\Gamma(2s - 2u - 2v)\Gamma(-2u - 2v + 2s - 1)\Gamma(-2u)\Gamma(-2v)}{\Gamma(k - 2u - 2v + 2s)\Gamma(-2u - 2v + s - 1)\Gamma(-2u + s)\Gamma(-2v + s)},$$
$$k, k', s, s' = 0, 1, 2,\ldots.$$

According to this lemma, the set $\{f_{s,k}\,;\, s, k \geqslant 0\}$ is linearly independent; hence, it forms a basis for $\mathscr{V}_2 \otimes \mathscr{V}_2$. Thus, $\mathscr{V}_2 \otimes \mathscr{V}_2$ can be decomposed into a direct sum of subspaces, each subspace transforming according to an irreducible representation of $sl(2)$.

Theorem 5.2 $\uparrow_u \otimes \uparrow_v \cong \sum_{s=0}^{\infty} \oplus \uparrow_{u+v-s}$.

We define the Clebsch–Gordan coefficients $\mathrm{E}(u, l; v, j \mid s, k)$ nonzero only if $s + k = l + j$, by

$$f_{s,k} = \sum_{l=0}^{s+k} \mathrm{E}(u, l; v, s + k - l \mid s, k)p_{l,s+k-l}, \qquad k, s \geqslant 0. \tag{5.68}$$

Thus,

$$\mathbf{Q}(f_{s,k}, p_{l,j}) = \mathrm{E}(u, l; v, j \mid s, k)\xi(l, j)\,\delta_{j,s+k-l}\,; \qquad s, k, l, j \geqslant 0. \tag{5.69}$$

This relation allows us to invert (5.68) and obtain

$$p_{l,j} = \sum_{s=0}^{l+j} \frac{\mathrm{E}(u, l; v, j \mid s, l + j - s)\xi(l, j)}{N_{s,l+j-s}} f_{s,l+j-s}. \tag{5.70}$$

A generating function for the vectors $f_{s,k}$ and, thus, for the Clebsch–Gordan coefficients E(·) follows easily from (5.67). Evaluating this equation for the case where $k = 0$ and $g = \exp(-b\mathscr{J}^+)$, we have

$$(bz+1)^{2u-s}(bw+1)^{2v-s}(z-w)^s = \sum_{l=0}^{\infty} \binom{2u+2v-2s}{l} b^l f_{s,l}(z,w)$$

or

$$(bz+1)^{2u-s}(bw+1)^{2v-s}(z-w)^s = \sum_{k=0}^{\infty}\sum_{j=0}^{s+k} \binom{2u+2v-2s}{k} b^k \cdot \mathrm{E}(u,j;v,s+k-j\mid s,k) z^j w^{s+k-j}, \quad (5.71)$$

where $|bz| < 1$, $|bw| < 1$, $s \geqslant 0$, $2u$, $2v$ not nonnegative integers. Comparison of this expression with the generating functi)n (5.21) for the matrix elements $B_{lk}(g)$ yields the interesting relation

$$(-1)^{s+l}(1+b)^{2v-s}\frac{s!}{l!}\frac{F(-l,-2u+s;s-l+1;-b)}{\Gamma(s-l+1)} = \sum_{k=0}^{\infty}\binom{2u+2v-2s}{k} b^k \mathrm{E}(u,l;v,s+k-l\mid s,k), \quad |b|<1. \quad (5.72)$$

If $s - l + 1 \leqslant 0$, the left-hand side of (5.72) is defined by (A.5).

Equations (5.69) allow us to express matrix elements of $\uparrow_u \otimes \uparrow_v$ with respect to the $p_{l,j}$ basis in terms of the matrix elements with respect to the $f_{s,k}$ basis. The result is

$$B_{ll'}^{(u)}(g)B_{kk'}^{(v)}(g) = \sum_{j=0}^{\min[l+k,l'+k']} \frac{\xi(l',k')}{N_{j,l'+k'-j}} \mathrm{E}(u,l;v,k\mid j,l+k-j) \cdot \mathrm{E}(u,l';v,k'\mid j,l'+k'-j) B_{l+k-j,l'+k'-j}^{(u+v-j)}(g),$$
$$g \in P, \quad l,l',k,k' \geqslant 0;$$

or

$$\frac{k'!\,l'!}{k!\,l!}\mathfrak{B}_u^{u-l,-u+l'}(z)\mathfrak{B}_v^{v-k,-v+k'}(z) = \sum_{j=0}^{\min[l+k,l'+k']} \frac{\xi(l',k')}{N_{j,l'+k'-j}} \mathrm{E}(u,l;v,k\mid j,l+k-j) \cdot \mathrm{E}(u,l';v,k'\mid j,l'+k'-j)\frac{(l'+k'-j)!}{(l+k-j)!}\mathfrak{B}_{u+v-j}^{u+v-l-k,-u-v+l'+k'}(z). \quad (5.73)$$

The tensor products $\downarrow_u \otimes \downarrow_v$, $D(2u) \otimes \uparrow_v$, and $D(2u) \otimes \downarrow_v$ can also be studied by the above methods but this will be omitted since the results are similar to those we have already obtained.

5-7 Differential Relations for the Matrix Elements

The matrix elements of the representations $D(u, m_o)$, $\uparrow_u$, $\downarrow_u$, and $D(2u)$ have been shown to be analytic functions of the group coordinates (a, b, c) in a neighborhood of the identity element $\mathbf{e}$ of $SL(2)$. Thus, if $\mathscr{F}$ is the complex vector space consisting of all functions of the coordinates of $SL(2)$ analytic in some neighborhood of $\mathbf{e}$, it follows that the matrix elements $A_{jk}(g)$, $B_{jk}(g)$, $C_{jk}(g)$, $D_{jk}(g)$, determined in Sections 5-1 to 5-4, are members of $\mathscr{F}$. There is a natural action of $SL(2)$ on $\mathscr{F}$ as a local transformation group. If $g' \in SL(2)$ let $\mathbf{P}(g')$ be the linear operator from $\mathscr{F}$ to $\mathscr{F}$ defined by $[\mathbf{P}(g')f](g) = f(gg')$ for all $f \in \mathscr{F}$ and $g \in SL(2)$ such that g' and gg' are in the domain of f. From this definition there follows the relation

$$[\mathbf{P}(g_1 g_2)f](g) = [\mathbf{P}(g_1)(\mathbf{P}(g_2)f)](g)$$

for g_1, g_2, g in a sufficiently small neighborhood of $\mathbf{e}$.

The action of $\mathbf{P}$ on the matrix elements is given by

$$\begin{aligned}
&[\mathbf{P}(g')A_{jk}](g) = A_{jk}(gg') = \sum_{l=-\infty}^{\infty} A_{lk}(g')A_{jl}(g), \\
&\qquad\qquad j, k \text{ integers}; \\
&[\mathbf{P}(g')B_{jk}](g) = B_{jk}(gg') = \sum_{l=0}^{\infty} B_{lk}(g')B_{jl}(g), \\
&\qquad\qquad j, k \text{ nonnegative integers}; \\
&[\mathbf{P}(g')C_{jk}](g) = C_{jk}(gg') = \sum_{l=0}^{\infty} C_{lk}(g')C_{jl}(g), \\
&\qquad\qquad j, k \text{ nonnegative integers}; \\
&[\mathbf{P}(g')D_{jk}](g) = D_{jk}(gg') = \sum_{l=0}^{2u} D_{lk}(g')D_{jl}(g), \\
&\qquad\qquad 2u \text{ a nonnegative integer}, \quad j, k = 0, 1, \ldots, 2u;
\end{aligned} \tag{5.74}$$

defined for g', g in a sufficiently small neighborhood of $\mathbf{e}$. Thus, for fixed j, the functions $\{A_{jk}\}$ form a basis for the representation $D(u, m_o)$, the functions $\{B_{jk}\}$ form a basis for $\uparrow_u$, the functions $\{C_{jk}\}$ form a basis for $\downarrow_u$, and the functions $\{D_{jk}\}$ form a basis for $D(2u)$.

The Lie derivatives J^+, J^-, J^3 of the multiplier representation P are defined by

$$\begin{aligned} \mathrm{J}^+f(g) &= \frac{d}{dt}[\mathbf{P}(\exp t\mathscr{J}^+)f](g)\Big|_{t=0}, \\ \mathrm{J}^-f(g) &= \frac{d}{dt}[\mathbf{P}(\exp t\mathscr{J}^-)f](g)\Big|_{t=0}, \\ \mathrm{J}^3f(g) &= \frac{d}{dt}[\mathbf{P}(\exp t\mathscr{J}^3)f](g)\Big|_{t=0} \end{aligned} \tag{5.75}$$

for every $f \in \mathscr{F}$ and every g in the domain of f. It is a consequence of the discussion of Section 2-2 that these Lie derivatives satisfy the commutation relations

$$[\mathrm{J}^3, \mathrm{J}^\pm] = \pm\mathrm{J}^\pm, \qquad [\mathrm{J}^+, \mathrm{J}^-] = 2\mathrm{J}^3$$

and that they act on the matrix elements as follows:

$$\begin{aligned} \mathrm{J}^3A_{jk}(g) &= (m_o + k)A_{jk}(g) \\ \mathrm{J}^+A_{jk}(g) &= (m_o + k - u)A_{j\,k+1}(g) \\ \mathrm{J}^-A_{jk}(g) &= -(m_o + k + u)A_{j\,k-1}(g) \end{aligned} \tag{5.76}$$

$$\mathrm{C}_{1,0}A_{jk}(g) = (\mathrm{J}^+\mathrm{J}^- + \mathrm{J}^3\mathrm{J}^3 - \mathrm{J}^3)A_{jk}(g) = u(u+1)A_{jk}(g),$$

$$\begin{aligned} \mathrm{J}^3B_{jk}(g) &= (-u + k)B_{jk}(g) \\ \mathrm{J}^+B_{jk}(g) &= (-2u + k)B_{j\,k+1}(g) \\ \mathrm{J}^-B_{jk}(g) &= -kB_{j\,k-1}(g) \end{aligned} \tag{5.77}$$

$$\mathrm{C}_{1,0}B_{jk}(g) = (\mathrm{J}^+\mathrm{J}^- + \mathrm{J}^3\mathrm{J}^3 - \mathrm{J}^3)B_{jk}(g) = u(u+1)B_{jk}(g),$$

$$\begin{aligned} \mathrm{J}^3C_{jk}(g) &= (u - k)C_{jk}(g) \\ \mathrm{J}^+C_{jk}(g) &= -kC_{j\,k-1}(g) \\ \mathrm{J}^-C_{jk}(g) &= (-2u + k)C_{j\,k+1}(g) \end{aligned} \tag{5.78}$$

$$\mathrm{C}_{1,0}C_{jk}(g) = (\mathrm{J}^+\mathrm{J}^- + \mathrm{J}^3\mathrm{J}^3 - \mathrm{J}^3)C_{jk}(g) = u(u+1)C_{jk}(g),$$

$$\begin{aligned} \mathrm{J}^3D_{jk}(g) &= (-u + k)D_{jk}(g) \\ \mathrm{J}^+D_{jk}(g) &= (-2u + k)D_{j\,k+1}(g) \\ \mathrm{J}^-D_{jk}(g) &= -kD_{j\,k-1}(g) \end{aligned} \tag{5.79}$$

$$\mathrm{C}_{1,0}D_{jk}(g) = (\mathrm{J}^+\mathrm{J}^- + \mathrm{J}^3\mathrm{J}^3 - \mathrm{J}^3)D_{jk}(g) = u(u+1)D_{jk}(g).$$

The above expressions yield recursion relations and differential equations for the matrix elements, which can be evaluated by computing the Lie derivatives $J^{\pm}$, J^3 from (5.75).

Instead of the coordinates (a, b, c) for a neighborhood of $\mathbf{e}$ in $SL(2)$, however, we shall find it more convenient to adopt the coordinates $[\alpha, \beta, w]$ such that

$$a = e^{(\alpha+\beta)/2}\cosh\frac{w}{2}, \qquad b = e^{(\alpha-\beta)/2}\sinh\frac{w}{2}, \qquad c = e^{(-\alpha+\beta)/2}\sinh\frac{w}{2}$$
$$\cosh w = 2bc + 1. \tag{5.80}$$

The complex coordinates $[\alpha, \beta, w]$ are not defined for group elements such that $bc = 0$, so they cannot be extended over the entire group manifold. Furthermore, as we have shown in Section 5-1, the coordinate transformation (5.80) is not one-to-one since distinct points in the $[\alpha, \beta, w]$ coordinates correspond to the same point in the (a, b, c) coordinates. For group elements such that $bc \neq 0$, however, the Jacobian of this coordinate transformation is nonzero and the transformation is locally one-to-one and analytic.

Thus if $g \in SL(2)$ has coordinates $[\alpha, \beta, w]$ and

$$g' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

is in a sufficiently small neighborhood of $\mathbf{e}$ it is easy to verify that gg' has unique coordinates $[\alpha_1, \beta_1, w_1]$ where

$$\cosh w_1 = \cosh w\,(a'd' + b'c' + e^{\beta}a'b'\tanh w + e^{-\beta}c'd'\tanh w),$$
$$e^{\alpha_1} = e^{\alpha}\left[\frac{(e^{\beta}a'(\cosh w + 1) + c'\sinh w)(e^{-\beta}d'(\cosh w - 1) + b'\sinh w)}{(d'(\cosh w + 1) + e^{\beta}b'\sinh w)(a'(\cosh w - 1) + e^{-\beta}c'\sinh w)}\right]^{1/2},$$
$$e^{\beta_1} = e^{\beta}\left[\frac{(a'(\cosh w + 1) + e^{-\beta}c'\sinh w)(a'(\cosh w - 1) + e^{-\beta}c'\sinh w)}{(d'(\cosh w + 1) + e^{\beta}b'\sinh w)(d'(\cosh w - 1) + b'e^{\beta}\sinh w)}\right]^{1/2}. \tag{5.81}$$

From this result and the definition (5.75) of the Lie derivatives we obtain

$$J^+ = e^{\beta}\left(-(z^2-1)^{1/2}\frac{\partial}{\partial z} - \frac{1}{(z^2-1)^{1/2}}\frac{\partial}{\partial\alpha} + \frac{z}{(z^2-1)^{1/2}}\frac{\partial}{\partial\beta}\right)$$
$$J^- = e^{-\beta}\left(-(z^2-1)^{1/2}\frac{\partial}{\partial z} + \frac{1}{(z^2-1)^{1/2}}\frac{\partial}{\partial\alpha} - \frac{z}{(z^2-1)^{1/2}}\frac{\partial}{\partial\beta}\right) \tag{5.82}$$
$$J^3 = \frac{\partial}{\partial\beta}$$
$$C_{1,0} = (z^2-1)\frac{\partial^2}{\partial z^2} + 2z\frac{\partial}{\partial z} + \frac{1}{z^2-1}\left(-\frac{\partial^2}{\partial\alpha^2} - \frac{\partial^2}{\partial\beta^2} + 2z\frac{\partial^2}{\partial\alpha\,\partial\beta}\right)$$

where $z = \cosh w$. As in Section 5-1 we assume that z takes values in the cut complex plane where the cut runs along the real axis from $-\infty$ to $+1$. In particular, if $z > 1$ then $(z^2 - 1)^{1/2} > 0$.

According to the relations (5.12), (5.24), (5.31), (5.38), the matrix elements $A_{jk}(g)$, $B_{jk}(g)$, $C_{jk}(g)$, $D_{jk}(g)$ can be expressed in terms of the functions $\mathfrak{B}_u^{r,m}(z)$. Substituting (5.12) and (5.82) into Eqs. (5.76) and simplifying we obtain the relations

$$\begin{aligned}
&\left(-(z^2-1)^{1/2}\frac{\partial}{\partial z} + \frac{r}{(z^2-1)^{1/2}} + \frac{zm}{(z^2-1)^{1/2}}\right)\mathfrak{B}_u^{r,m}(z) \\
&\quad = (m+u+1)(m-u)\mathfrak{B}_u^{r,m+1}(z), \\
&\left(-(z^2-1)^{1/2}\frac{\partial}{\partial z} - \frac{r}{(z^2-1)^{1/2}} - \frac{zm}{(z^2-1)^{1/2}}\right)\mathfrak{B}_u^{r,m}(z) \\
&\quad = -\mathfrak{B}_u^{r,m-1}(z), \\
&\left((z^2-1)\frac{\partial^2}{\partial z^2} + 2z\frac{\partial}{\partial z} - \frac{2zmr + r^2 + m^2}{z^2-1}\right)\mathfrak{B}_u^{r,m}(z) \\
&\quad = u(u+1)\mathfrak{B}_u^{r,m}(z),
\end{aligned} \tag{5.83}$$

for all complex numbers u, r, m such that $u \pm m$, $u \pm r$, are not integers and $r + m$ is an integer.

Similarly, from Eqs. (5.77) we find that (5.83) remains valid for all complex numbers r, m, u such that $u + r$, $u - m$ are nonnegative integers and $2u$ is not a nonnegative integer. Finally, from (5.79) we again obtain equations (5.83), valid now for all nonnegative integers $2u$ and all numbers r, m such that $u - r$, $u + m$ are integers and $0 \leqslant u - r,\ u + m,\ \leqslant 2u$. The derivation of recursion relations from (5.78) is left to the reader.

5-8 *Type B* Realizations of $D(u, m_0)$

In this section we study realizations of $D(u, m_0)$ such that $J^\pm$, J^3 are the *type B* differential operators

$$J^3 = \frac{\partial}{\partial y}, \qquad J^\pm = e^{\pm y}\left(\pm\frac{\partial}{\partial x} - i\frac{\partial}{\partial y} + qe^{-ix}\right), \qquad q \in \mathcal{C}, \tag{5.84}$$

derived in Section 2-7. It will be more convenient, however, to introduce new variables τ, z defined by $\tau = y - i\pi/2$, $z = -ie^{-ix}$. The *type B* operators then become

$$J^3 = \frac{\partial}{\partial \tau}, \qquad J^\pm = e^{\pm\tau}\left(z\frac{\partial}{\partial z} \pm \frac{\partial}{\partial \tau} \mp qz\right). \tag{5.85}$$

To find a realization of $D(u, m_o)$, $(0 \leqslant \operatorname{Re} m_o < 1, u \pm m_o$ not integers) in terms of the operators (5.85) we look for nonzero functions $f_m(z, \tau) = Z_m(z)e^{m\tau}$ such that

$$\begin{gathered} J^3 f_m = m f_m, \qquad J^+ f_m = (m-u) f_{m+1}, \qquad J^- f_m = -(m+u) f_{m-1} \\ C_{1,0} f_m = (J^+J^- + J^3J^3 - J^3) f_m = u(u+1) f_m \end{gathered} \tag{5.86}$$

for all $m \in S$ where $S = \{m_o + n\colon n \text{ an integer}\}$. In terms of the functions $Z_m(z)$ these relations become

$$\begin{aligned} &\text{(i)} \quad \left(z\frac{d}{dz} + m - qz\right) Z_m(z) = -(m-u) Z_{m+1}(z), \\ &\text{(ii)} \quad \left(z\frac{d}{dz} - m + qz\right) Z_m(z) = -(m+u) Z_{m-1}(z), \\ &\text{(iii)} \quad \left(z^2\frac{d^2}{dz^2} + 2qzm - q^2z^2\right) Z_m(z) = u(u+1) Z_m(z). \end{aligned} \tag{5.87}$$

As shown in Section 2-7, Eq. (iii) has as solutions the functions

$$\begin{aligned} Z_m(z) &= (2qz)^{u+1} e^{-qz} {}_1F_1(u-m+1; 2u+2; 2qz), \\ Z'_m(z) &= (2qz)^{-u} e^{-qz} {}_1F_1(-u-m; -2u; 2qz). \end{aligned}$$

Expressed in terms of the generalized Laguerre functions, there are solutions

$$\begin{aligned} &(1) \quad (2qz)^{u+1} e^{-qz} L^{(2u+1)}_{m-u-1}(2qz), \\ &(2) \quad (2qz)^{-u} e^{-qz} L^{(-2u-1)}_{m+u}(2qz). \end{aligned} \tag{5.88}$$

From the form of these solutions it is clear that without loss of generality we can set $q = \frac{1}{2}$. Second, focusing attention on solution (1), we note that it is a generalized Laguerre function multiplied by the factor $(z)^{u+1}e^{-z/2}$. It would be more convenient for the study of Laguerre functions if we could find Lie derivatives whose eigenfunctions were of the form

$$f'_m(z, \tau) = L^{(2u+1)}_{m-u-1}(z) e^{m\tau}, \tag{5.89}$$

without the extraneous factor. By transforming the *type B* operators we shall bring about this situation.

Denote by $\mathscr{F}$ the space of all functions analytic in a neighborhood of the point $(z^o, \tau^o) = (1, 0)$. (The exact choice of (z^o, τ^o) is not important.) The function $f_m(z, \tau) = z^{u+1}e^{-z/2}L^{(2u+1)}_{m-u-1}(z)e^{m\tau}$ is a simultaneous eigenfunction of the operators J^3, $C_{1,0}$ and is an element of $\mathscr{F}$. Define the linear mapping φ of $\mathscr{F}$ onto $\mathscr{F}$ by $[\varphi f](z, \tau) = z^{u+1}e^{-z/2}f(z, \tau)$ for all $f \in \mathscr{F}$. The mapping φ can be considered as multiplication by the function $\varphi(z) = z^{u+1}e^{-z/2}$. Similarly, the map φ^{-1}: $\mathscr{F} \to \mathscr{F}$, the inverse of φ, is defined by $[\varphi^{-1}f](z, \tau) = z^{-u-1}e^{z/2}f(z, \tau)$. Under φ^{-1} the eigenfunction f_m is mapped into f'_m where $f'_m(z, \tau) = [\varphi^{-1}f_m](z, \tau) = L_{m-u-1}^{(2u+1)}(z)e^{m\tau}$ is the function (5.89).

As was shown in Section 4-8, Lemma 4.4, φ induces a Lie algebra isomorphism $J \to J^\varphi$ of the linear differential operators on $\mathscr{F}$ such that $(J^\varphi)(\varphi^{-1}f) = \varphi^{-1}(Jf)$ for all $f \in \mathscr{F}$. The operators $(J^+)^\varphi$, $(J^-)^\varphi$ $(J^3)^\varphi$ can be computed from Eqs. (4.82) and (5.85):

$$\begin{aligned}
(J^+)^\varphi &= t\left(z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t} - z + u + 1\right),\\
(J^-)^\varphi &= t^{-1}\left(z\frac{\partial}{\partial z} - t\frac{\partial}{\partial t} + u + 1\right),\\
(J^3)^\varphi &= t\frac{\partial}{\partial t},
\end{aligned} \tag{5.90}$$

where $t = e^\tau$. The operators (5.90) automatically satisfy the commutation relations for the generators of $sl(2)$ so they can be used to construct a realization of the representation $D(u, m_o)$ on $\mathscr{F}$. Thus, we look for nonzero functions $f_m(z, t) = Z_m(z)t^m$ in $\mathscr{F}$, defined for all $m \in S = \{m_o + n; n \text{ an integer}\}$, such that Eqs. (5.86) are satisfied, where now the differential operators are given by (5.90) and we have dropped the superscript φ. In terms of the functions $Z_m(z)$ these relations take the form

$$\begin{aligned}
\left(z\frac{d}{dz} - z + (m + u + 1)\right) Z_m(z) &= (m - u)Z_{m+1}(z),\\
\left(z\frac{d}{dz} - (m - u - 1)\right) Z_m(z) &= -(m + u)Z_{m-1}(z),\\
\left(z\frac{d^2}{dz^2} + (2(u + 1) - z)\frac{d}{dz} + (m - u - 1)\right) Z^m(z) &= 0
\end{aligned} \tag{5.91}$$

for all $m \in S$. The solutions are generalized Laguerre functions. In particular,

$$Z_m(z) = L_{m-u-1}^{(2u+1)}(z) \tag{5.92}$$

satisfies (5.91) for all $m \in S$. None of these solutions are polynomials in z since $m - u$ is not an integer.

The above remarks show that the vectors $f_m(z, t) = L^{(2u+1)}_{m-u-1}(z)\, t^m$ form a basis for a realization of $D(u, m_o)$.

This realization can be extended to a local multiplier representation T of $SL(2)$ on $\mathscr{F}$. Applying Theorem 1.10 to the generalized Lie derivatives (5.90) we can determine the action of the operators $\mathbf{T}(g)$, $g \in SL(2)$, as follows:

$$[\mathbf{T}(\exp -b\mathscr{J}^+)f](z, t) = (1 + bt)^{-u-1} e^{bzt/(1+bt)} f\left(\frac{z}{1 + bt}, \frac{t}{1 + bt}\right),$$

$$| bt | < 1,$$

$$[\mathbf{T}(\exp -c\mathscr{J}^-)f](z, t) = (1 + c/t)^{-u-1} f(z(1 + c/t)^{-1}, t + c),$$

$$| c/t | < 1,$$

$$[\mathbf{T}(\exp \tau\mathscr{J}^3)f](z, t) = f(z, e^\tau t),$$

valid for all $f \in \mathscr{F}$, (z, t) in the domain of f, and sufficiently small values of $| b |, | c |, | \tau |$.

If $g \in SL(2)$, Eq. (1.15), with $d \neq 0$ it is easy to verify the relation

$$g = (\exp -b'\mathscr{J}^+)(\exp -c'\mathscr{J}^-)(\exp \tau'\mathscr{J}^3)$$

where $b' = b/d$, $c' = cd$, $e^{\tau'/2} = d^{-1}$, $0 \leqslant \operatorname{Im} \tau' < 4\pi$. Hence, the operator $\mathbf{T}(g)$ is given by

$$\begin{aligned}[\mathbf{T}(g)f](z, t) &= [\mathbf{T}(\exp -b'\mathscr{J}^+)\, \mathbf{T}(\exp -c'\mathscr{J}^-)\, \mathbf{T}(\exp \tau'\mathscr{J}^3)f](z, t) \\ &= (d + bt)^{-u-1}\left(a + \frac{c}{t}\right)^{-u-1} e^{bzt/(d+bt)} \\ &\quad \cdot f\left(\frac{zt}{(at + c)(d + bt)}, \frac{at + c}{d + bt}\right), \qquad | c/at | < 1, \\ &\qquad | bt/d | < 1, \qquad (5.93)\end{aligned}$$

for $f \in \mathscr{F}$ and g in a small enough neighborhood of $\mathbf{e}$ so that the above expression is uniquely defined. The matrix elements of this multiplier representation with respect to the analytic basis $\{f_m\}$ are the functions $A_{lk}(g)$, Eqs. (5.10), (5.12). Therefore, we have the identity

$$[\mathbf{T}(g) f_{m_o+k}](z, t) = \sum_{l=-\infty}^{\infty} A_{lk}(g) f_{m_o+l}(z, t), \qquad k = 0, \pm 1, \pm 2, \ldots,$$

which simplifies to

$$(1+bc)^{-\mu}\left(1+\frac{bt}{d}\right)^{-\nu-\mu-1}\left(1+\frac{c}{at}\right)^{\nu}\exp\left(\frac{bzt}{d+bt}\right)$$

$$\cdot L_{\nu}^{(\mu)}\left[\frac{z}{(1+bc)(1+c/at)(1+bt/d)}\right]$$

$$=\sum_{l=-\infty}^{\infty}\left(\frac{c}{at}\right)^{l}\frac{\Gamma(\mu+\nu+1)}{\Gamma(\mu+\nu-l+1)}$$

$$\cdot\frac{F(-\mu-\nu+l,\nu+1;l+1;bc/ad)}{\Gamma(l+1)}L_{\nu-l}^{(\mu)}(z),$$

$$|bc|<1,\quad |c/a|<|t|<|d/b|,\quad d=(1+bc)/a. \tag{5.94}$$

This relation is valid for all $\mu, \nu \in \mathcal{C}$ such that ν and $\mu+\nu$ are not integers. When $l+1 \leqslant 0$ the hypergeometric function F is defined by the limit (A.5). In general the right-hand side of (5.94) converges whenever the left-hand side does.

We shall investigate two special cases of this identity. If $a=d=t=1$ and $c=0$, then

$$e^{bz/(1+b)}(1+b)^{-\nu-\mu-1}L_{\nu}^{(\mu)}\left(\frac{z}{1+b}\right)=\sum_{l=0}^{\infty}\binom{\nu+l}{l}(-b)^{l}L_{\nu+l}^{(\mu)}(z),$$

$$|b|<1.$$

If $a=d=t=1$ and $b=0$ there follows

$$(1+c)^{\nu}L_{\nu}^{(\mu)}\left(\frac{z}{1+c}\right)=\sum_{l=0}^{\infty}\binom{\mu+\nu}{l}c^{l}L_{\nu-l}^{(\mu)}(z),\qquad |c|<1.$$

Each of these expressions is valid for all $\mu, \nu \in \mathcal{C}$ such that ν and $\mu+\nu$ are not integers.

5-9 *Type B* Realizations of $\uparrow_u$

To find a realization of the representation $\uparrow_u$ of $sl(2)$, $2u$ not a non-negative integer, by *type B* operators it is most convenient to transform these operators into a new set of Lie derivatives especially adapted to solution (2) of (5.88) ($q=\frac{1}{2}$). Thus, we set $\varphi(z)=z^{-u}e^{-z/2}$ and compute the operators $(\mathrm{J}^{\pm})^{\varphi}$, $(\mathrm{J}^{3})^{\varphi}$ where $\mathrm{J}^{\pm}$, J^{3} are given by (5.85). (For this choice of $\varphi(z)$, the common eigenfunctions of $(C_{1,0})^\varphi$ and $(J^3)^\varphi$ will be constant multiples of $L_{m+u}^{(-2u-1)}(z)\,t^m$, $m \in S$.) The result is, Eq. (4.82),

$$\begin{aligned}(J^+)^\varphi &= t\left(z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t} - z - u\right),\\ (J^-)^\varphi &= t^{-1}\left(z\frac{\partial}{\partial z} - t\frac{\partial}{\partial t} - u\right),\\ (J^3)^\varphi &= t\frac{\partial}{\partial t}.\end{aligned} \tag{5.95}$$

Dropping the superscript φ, we see that to construct a realization of $\uparrow_u$ with the operators (5.95) it is necessary to find nonzero functions $f_m(z, t) = Z_m(z)\,t^m$, $m \in S = \{-u + n, n \geqslant 0\}$, with the properties

$$\begin{gathered}J^3 f_m = m f_m, \qquad J^+ f_m = (m-u) f_{m+1}, \qquad J^- f_m = -(m+u) f_{m-1},\\ C_{1,0} f_m = (J^+J^- + J^3J^3 - J^3) f_m = u(u+1) f_m\end{gathered} \tag{5.96}$$

for all $m \in S$. These conditions will be satisfied if and only if the functions $Z_m(z)$, $m \in S$, satisfy the relations

$$\begin{aligned}&\left(z\frac{d}{dz} - z + (m-u)\right) Z_m(z) = (m-u)Z_{m+1}(z),\\ &\left(z\frac{d}{dz} - (m+u)\right) Z_m(z) = -(m+u)Z_{m-1}(z),\\ &\left(z\frac{d^2}{dz^2} - (2u+z)\frac{d}{dz} + (m+u)\right) Z_m(z) = 0,\end{aligned} \tag{5.97}$$

where $Z_{-u-1}(z) \equiv 0$ on the right-hand side of these expressions. Equations (5.97) determine the functions Z_m to within an arbitrary constant. Indeed, for $m = -u$, $z\, d/dz\, Z_{-u}(z) = 0$, which has the solution $Z_{-u}(z) = c$, c a constant. We normalize this solution by setting $c = 1$. For $m = -u + n$, $n \geqslant 0$, the functions $Z_m(z)$ can be defined by the relation

$$Z_m(z)t^m = f_m(z, t) = \frac{\Gamma(-2u)}{\Gamma(n-2u)}(J^+)^n f_{-u}(z, t) = \frac{\Gamma(-2u)}{\Gamma(n-2u)}(J^+)^n(t^{-u}).$$

A straightforward induction argument using (A.13) yields

$$Z_{-u+n}(z) = \frac{\Gamma(-2u)n!}{\Gamma(n-2u)} L_n^{(-2u-1)}(z), \qquad n = 0, 1, 2, \ldots, \tag{5.98}$$

where the functions $L_n^{(-2u-1)}(z)$ are generalized Laguerre polynomials. By definition the functions

$$f_m(z, t) = \frac{\Gamma(-2u)n!}{\Gamma(n-2u)} L_n^{(-2u-1)}(z)t^{-u+n}, \qquad m = -u + n \in S,$$

satisfy the relations

$$J^3 f_m = m f_m\,, \qquad J^+ f_m = (m-u) f_{m+1}\,, \qquad J^- f_{-u} = 0.$$

Moreover, an induction argument utilizing the commutation relation $[J^+, J^-] = 2J^3$ shows that the functions f_m actually satisfy all of the relations (5.96). Thus the solutions (5.98) satisfy Eqs. (5.97) for all $m = -u + n \in S$, and we have constructed a basis for a realization of the representation $\uparrow_u$ of $sl(2)$ on $\mathscr{F}$.

This realization of $\uparrow_u$ can be extended to a local multiplier representation T' of $SL(2)$ on $\mathscr{F}$. As is easily verified, the operator $\mathbf{T}'(g)$, $g \in SL(2$ is obtained from (5.93) by replacing u with $-u-1$. Thus,

$$\begin{aligned}[\mathbf{T}'(g)f](z,t) &= (d+bt)^u (a + c/t)^u e^{bzt/(d+bt)} \\ &\quad \cdot f\left(\frac{zt}{(at+c)(d+bt)}, \frac{at+c}{(d+bt)}\right), \\ &\qquad |\, c/at \,| < 1, \quad |\, bt/d \,| < 1, \end{aligned} \tag{5.99}$$

for $f \in \mathscr{F}$ and g in a small enough neighborhood of $\mathbf{e}$ such that the right-hand side of this expression is uniquely defined.

The matrix elements of the multiplier representation T' with respect to the analytic basis $\{f_{-u+n}\}$, $n \geqslant 0$, are the functions $B_{lk}(g)$, $l, k \geqslant 0$, (5.22)–(5.23). Therefore,

$$[\mathbf{T}'(g) f_{-u+k}](z,t) = \sum_{l=0}^{\infty} B_{lk}(g) f_{-u+l}(z,t), \qquad k = 0, 1, 2, \ldots.$$

This leads to the identity

$$\begin{aligned} &\left(1 + \frac{bt}{d}\right)^{2u-k} \left(1 + \frac{c}{at}\right)^k e^{bzt/(d+bt)} L_k^{(-2u-1)} \\ &\quad \cdot \left(\frac{z}{(1+bc)(1+c/at)(1+bt/d)}\right) \\ &= \sum_{l=0}^{\infty} (-1)^{k-l} \frac{l!}{k!} \left(\frac{tb}{d}\right)^{l-k} \\ &\quad \cdot \frac{F(-k, -2u+l; l-k+1; bc/ad)}{\Gamma(l-k+1)} L_l^{(-2u-1)}(z), \\ &\qquad |\, bt/d \,| < 1, \quad d = (1+bc)/a, \end{aligned} \tag{5.100}$$

valid for all integers $k \geqslant 0$ and for all $u \in \mathcal{C}$ such that $2u$ is not a non-negative integer. When $l - k + 1 \leqslant 0$ we use the limit (A.5) to define the hypergeometric polynomial on the right-hand side of this identity.

If $a = d = t = 1$, $c = 0$, the above identity simplifies to

$$(1+b)^{2u-k}e^{bz/(1+b)}L_k^{(-2u-1)}\left(\frac{z}{1+b}\right) = \sum_{l=0}^{\infty}\binom{l+k}{l}(-b)^l L_{k+l}^{(-2u-1)}(z), \qquad |b| < 1.$$

For $k = 0$ this last expression becomes a well-known generating function for the generalized Laguerre polynomials:

$$(1-b)^{2u}e^{-bz/(1-b)} = \sum_{l=0}^{\infty} b^l L_l^{(-2u-1)}(z), \qquad |b| < 1. \tag{5.101}$$

(We have replaced b by $-b$ and used the fact that $L_0^{(-2u-1)}(z) = 1$.) A simple consequence of (5.101) is

$$L_l^{(-2u-1)}(0) = \frac{\Gamma(l-2u)}{l!\,\Gamma(-2u)}, \qquad l = 0, 1, 2, \ldots .$$

When $a = d = t = 1$, $b = 0$, (5.100) becomes

$$(1+c)^k L_k^{(-2u-1)}\left(\frac{z}{1+c}\right) = \sum_{l=0}^{k}\binom{k-2u-1}{l} c^l L_{k-l}^{(-2u-1)}(z).$$

Consideration of *type B* operator realizations of the representation $\downarrow_u$ would lead to the identity (5.100) again and give no new results. Hence, we shall not embark on this study.

Finally, as the reader can verify for himself, there are no *type B* operator realizations of the finite-dimensional representations $D(2u)$, $2u$ a nonnegative integer, of $sl(2)$. To construct a realization of $D(2u)$ operating on a space of functions of two complex variables it is necessary to use *type A* operators.

5-10 Weisner's Method for *Type B* Operators

So far the modified *type B* operators (5.95) have been used to construct identities for special functions which are simultaneous eigenfunctions of J^3 and $C_{1,0}$. However, these operators can also be used to derive identities for eigenfunctions of $C_{1,0}$ which are not eigenfunctions of J^3. We make the following observations. If $f(z, t)$ is a solution of the equation $C_{1,0}f = u(u+1)f$, i.e.,

$$\left(z\frac{\partial^2}{\partial z^2} - (2u+z)\frac{\partial}{\partial z} + t\frac{\partial}{\partial t} + u\right) f(z, t) = 0, \tag{5.102}$$

then the function $\mathbf{T}'(g)f$, $g \in SL(2)$, formally defined by

$$[\mathbf{T}'(g)f](z, t) = (d + bt)^u (a + c/t)^u e^{bzt/(d+bt)} f\left(\frac{z}{(a + c/t)(d + bt)}, \frac{at + c}{d + bt}\right),$$

is a solution of the equation $C_{1,0}[\mathbf{T}'(g)f] = u(u + 1)[\mathbf{T}'(g)f]$. This remark is true whenever the formal expression for $\mathbf{T}'(g)f$ can be interpreted as an analytic function in (z, t), and is a consequence of the fact that $C_{1,0}$ commutes with J^+, J^-, and J^3. Furthermore, if f is a solution of the differential equation

$$(x_1 J^+ + x_2 J^- + x_3 J^3) f(z, t) = \lambda f(z, t)$$

for complex constants x_1, x_2, x_3, λ, then Eq. (1.25), Section 1-2, shows that $\mathbf{T}'(g)f$ is a solution of the equation

$$\begin{aligned}\{(abx_3 - b^2x_2 + a^2x_1)J^+ &+ (-cdx_3 - c^2x_1 + d^2x_2)J^- \\ &+ ((ad + bc)x_3 - 2bdx_2 + 2acx_1)J^3\}[\mathbf{T}'(g)f] = \lambda[\mathbf{T}'(g)f], \quad (5.103)\end{aligned}$$

for $g \in SL(2)$, Eq. (1.15). Finally, if $C_{1,0}f(z, t) = u(u + 1)f(z, t)$ and f has a convergent expansion of the form $f(z, t) = \sum h_m(z)\, t^m$, then the expansion coefficients are solutions of the Laguerre differential equations

$$\left(z \frac{d^2}{dz^2} - (2u + z)\frac{d}{dz} + m + u\right) h_m(z) = 0.$$

These observations provide powerful tools for the derivation of identities satisfied by Laguerre functions. (We are now concerned with the operators (5.95) but the above comments apply as well to the other two variants of *type B* operators studied in this section.)

As an application of these remarks we choose f to be a solution of the simultaneous equations

$$C_{1,0}f = u(u + 1)f, \qquad J^-f = -f.$$

Thus,

(i) $\left(z \frac{\partial^2}{\partial z^2} - (2u + z)\frac{\partial}{\partial z} + t\frac{\partial}{\partial t} + u\right) f(z, t) = 0,$

(ii) $\left(\frac{z}{t}\frac{\partial}{\partial z} - \frac{\partial}{\partial t} - \frac{u}{t}\right) f(z, t) = -f(z, t).$

The general solution of (ii) is $f(z, t) = t^{-u}e^t h(zt)$ where h is an arbitrary function of zt. Substituting this solution into (i) we find that h must satisfy the equation

$$w\frac{d^2h(w)}{dw^2} - 2u\frac{dh(w)}{dw} + h(w) = 0.$$

This is a modification of Bessel's equation and has a solution, $h(w) = w^{(2u+1)/2} J_{-2u-1}(2w^{1/2})$, which is an entire analytic function of w. Thus, $f(z, t) = t^{-u} e^{t} (zt)^{(2u+1)/2} J_{-2u-1}(2(zt)^{1/2})$ is a solution of (i) and (ii).

According to the above remarks, the function

$$[\mathbf{T}'(g)f](z, t) = t^{-u}(d + bt)^{-1} \exp\left[\frac{(a + bz)t + c}{d + bt}\right] (zt)^{(2u+1)/2} \cdot J_{-2u-1}\left(\frac{2(zt)^{1/2}}{d + bt}\right), \qquad |bt/d| < 1,$$

satisfies the equations

$$\begin{aligned} C_{1,0}[\mathbf{T}'(g)f] &= u(u + 1)[\mathbf{T}'(g)f], \\ (-b^2 J^+ + d^2 J^- - 2bd J^3)[\mathbf{T}'(g)f] &= -[\mathbf{T}'(g)f]. \end{aligned} \tag{5.104}$$

Since $w^{-m} J_m(w)$ is an entire function of w for all $m \in \mathcal{C}$, $[\mathbf{T}'(g)f](z, t)$ has an expansion in t of the form

$$[\mathbf{T}'(g)f](z, t) = \sum_{l=0}^{\infty} h_l(g, z) t^{-u+l}, \qquad |bt/d| < 1.$$

Thus, $\mathbf{T}'(g)f$ is a generating function for the coefficients $h_l(g, z)$. We will determine these coefficients. Substituting the expansion into the first equation (5.104) we find that $h_l(g, z)$ is a solution of the Laguerre differential equation

$$\left(z \frac{d^2}{dz^2} - (2u + z) \frac{d}{dz} + l\right) h_l(g, z) = 0, \qquad l = 0, 1, 2, \ldots .$$

The function $[\mathbf{T}'(g)f](z, t)$ is regular at $z = 0$. In fact,

$$[\mathbf{T}'(g)f](0, t) = \frac{t^{-u}(d + bt)^{2u}}{\Gamma(-2u)} \exp\left(\frac{at + c}{d + bt}\right).$$

Thus, the functions $h_l(g, z)$ must be multiples of generalized Laguerre polynomials:

$$h_l(g, z) = j_l(g) L_l^{-2u-1}(z), \qquad l = 0, 1, 2, \ldots .$$

We could now use the second equation (5.104) to derive recursion relations for the $j_l(g)$. However it is simpler to proceed in a different manner. So far we have obtained the relation

$$(d + bt)^{-1} \exp\left[\frac{(a + bz)t + c}{d + bt}\right] (zt)^{(2u+1)/2} J_{-2u-1}\left(\frac{2(zt)^{1/2}}{d + bt}\right)$$

$$= \sum_{l=0}^{\infty} j_l(g) L_l^{-2u-1}(z) t^l, \qquad d = (1 + bc)/a, \quad |bt/d| < 1.$$

At $z = 0$ this relation becomes

$$\frac{(d+bt)^{2u}}{\Gamma(-2u)} \exp\left(\frac{at+c}{d+bt}\right) = \sum_{l=0}^{\infty} j_l(g) \frac{\Gamma(l-2u)}{\Gamma(-2u)l!} t^l, \qquad |bt/d| < 1.$$

The above expression is reminiscent of the generating function (5.100) for generalized Laguerre polynomials. In fact when $k = 0$, (5.100) becomes

$$\left(1+\frac{bt}{d}\right)^{2u} \exp\left(\frac{bxt}{d+bt}\right) = \sum_{l=0}^{\infty} \left(-\frac{tb}{d}\right)^l L^{(-2u-1)}(x), \qquad |bt/d| < 1.$$

Comparing these two expressions we find

$$j_l(g) = \exp\left(\frac{ac}{1+bc}\right) \frac{l!}{\Gamma(l-2u)} a^{-2u} \left(\frac{-ab}{1+bc}\right)^l L_l^{(-2u-1)}\left[\frac{a}{b(1+bc)}\right],$$
$$l = 0, 1, 2, \ldots,$$

where $2u$ is not a nonnegative integer. For $c = 0$, the result of this computation is the identity

$$(1+abt)^{-1} \exp\left[\frac{(a+bz)t}{a^{-1}+bt}\right] (a^2zt)^{(2u+1)/2} J_{-2u-1}\left(\frac{2(zt)^{1/2}}{a^{-1}+bt}\right)$$
$$= \sum_{l=0}^{\infty} \frac{l!}{\Gamma(l-2u)} (-ab)^l L_l^{(-2u-1)}(a/b) L_l^{(-2u-1)}(z) t^l,$$
$$|abt| < 1. \qquad (5.105)$$

If $a = 1$, $b = 0$, (5.105) becomes

$$e^t(zt)^{(2u+1)/2} J_{-2u-1}(2(zt)^{1/2}) = \sum_{l=0}^{\infty} L_l^{(-2u-1)}(z) \frac{t^l}{\Gamma(l-2u)},$$

while if $a = iy^{1/2}$, $b = iy^{-1/2}$, it reduces to the Hille–Hardy formula

$$(1-t)^{-1}(-yzt)^{(2u+1)/2} \exp\left[\frac{-t(y+z)}{1-t}\right] J_{-2u-1}\left(\frac{2i(yzt)^{1/2}}{1-t}\right)$$
$$= \sum_{l=0}^{\infty} \frac{l!}{\Gamma(l-2u)} L_l^{(-2u-1)}(y) L_l^{(-2u-1)}(z) t^l, \qquad |t| < 1.$$

As a final example we apply $\mathbf{T}'(g)$ to the function $f(z, t) = L_{m+u}^{-2u-1}(z)t^m$, $m, u \in \mathcal{C}$. Clearly $C_{1,0}f = u(u+1)f$, $J^3 f = mf$. In the case where g is in a small neighborhood of $\mathbf{e}$ we have already obtained identities for f from the representation theory of $sl(2)$, Eq. (5.94). However, the choice

$$g = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

violates the conditions of (5.94). In this case we can write $\mathbf{T}'(g)f$ in the form

$$t^{-u}(1-t)^{u-m}e^{-zt/(1-t)}L_{m+u}^{-2u-1}\left(\frac{zt}{1-t}\right), \qquad |t| < 1,$$

which has an expansion in terms of t^{-u+l}, $l = 0, 1, 2, \ldots$. The coefficient of t^{-u+l} is $c_l L_l^{-2u-1}(z)$ and the constants c_l can be determined from the expansion by setting $z = 0$. The result is

$$(1-t)^{u-m}e^{-zt/(1-t)}L_{m+u}^{-2u-1}\left(\frac{zt}{1-t}\right)$$
$$= \sum_{l=0}^{\infty} (-t)^l \frac{\Gamma(u-m+1)\Gamma(m-u)}{\Gamma(-2u+l)\Gamma(u-m-l+1)\Gamma(u+m+1)} L_l^{-2u-1}(z),$$
$$|t| < 1,$$

a generating function for the generalized Laguerre polynomials.

As the reader can prove using (5.103), the examples studied here are inclusive in the following sense:

Lemma 5.7 (Weisner) If f is a solution of the equation $(x_1\mathrm{J}^+ + x_2\mathrm{J}^- + x_3\mathrm{J}^3)f(z, t) = \lambda f(z, t)$ for complex constants x_1, x_2, x_3, λ such that $x^2 + 4x_1x_2 \neq 0$ then there exist $\mu_1 \in \mathcal{C}$, $g_1 \in SL(2)$, and a function $h_1(z, t)$ such that $f = \mathbf{T}'(g_1)h_1$ and $\mu_1\mathrm{J}^3h_1(z, t) = \lambda h_1(z, t)$. Moreover, if $x_3^2 + 4x_1x_2 = 0$ there exist $\mu_2 \in \mathcal{C}$, $g_2 \in SL(2)$ and a function $h_2(z, t)$ such that $f = \mathbf{T}'(g_2)h_2$ and $\mu_2\mathrm{J}^-h_2(z, t) = \lambda h_2(z, t)$.

Lemma 5.7 is also applicable to the work of Section 5-15.

5-11 *Type A* Realizations of $D(u, m_0)$

The *type A* differential operators determined in Section 2-7 form a basis for an algebra of generalized Lie derivatives isomorphic to $sl(2)$, which operates on a space of functions of the two complex variables x, y:

$$\mathrm{J}^3 = \frac{\partial}{\partial y}, \qquad \mathrm{J}^\pm = e^{\pm y}\left(\pm\frac{\partial}{\partial x} - \cot x\frac{\partial}{\partial y} + \frac{q}{\sin x}\right), \qquad q \in \mathcal{C}.$$

In terms of the new variables t, z and a new constant r defined by $t = ie^y$, $z = \cos x$, $r = -iq$, the *type A* operators take the form

$$J^3 = t\frac{\partial}{\partial t},$$
$$J^{\pm} = t^{\pm 1}\left(-(z^2-1)^{1/2}\frac{\partial}{\partial z} \pm \frac{z}{(z^2-1)^{1/2}}t\frac{\partial}{\partial t} \pm \frac{r}{(z^2-1)^{1/2}}\right). \tag{5.106}$$

To construct a realization of the representation $D(u, m_o)$ $(0 \leqslant \operatorname{Re} m_o < 1$, $u \pm m_o$ not integers) using the operators (5.106) we must find basis vectors $f_m(z, t) = Z_m(z)\,t^m$ such that

$$J^3 f_m = mf_m, \qquad J^+ f_m = (m-u)f_{m+1}, \qquad J^- f_m = -(m+u)f_{m-1},$$
$$C_{1,0}f_m = (J^+J^- + J^3J^3 - J^3)f_m = u(u+1)f_m \tag{5.107}$$

for all $m \in S = \{m_o + n\colon n \text{ an integer}\}$. These conditions are fulfilled if and only if the $Z_m(z)$ satisfy the relations

$$\text{(i)}\quad \left(-(z^2-1)^{1/2}\frac{d}{dz} + \frac{zm+r}{(z^2-1)^{1/2}}\right)Z_m(z) = (m-u)Z_{m+1}(z),$$

$$\text{(ii)}\quad \left((z^2-1)^{1/2}\frac{d}{dz} - \frac{zm+r}{(z^2-1)^{1/2}}\right)Z_m(z) = (m+u)Z_{m-1}(z), \tag{5.108}$$

$$\text{(iii)}\quad \left((z^2-1)\frac{d^2}{dz^2} + 2z\frac{d}{dz} - \frac{(2rmz + r^2 + m^2)}{z^2-1}\right)Z_m(z) = u(u+1)Z_m(z)$$

for all $m \in S$. As shown by Eqs. (5.83), the functions

$$Z_m(z) = \Gamma(u+m+1)\mathfrak{B}_u^{r,m}(z), \qquad m \in S,$$

satisfy these relations for $m + r$ an integer. However, it is easy to verify directly that the above functions satisfy (5.108) even if $m + r$ is not an integer. (As usual we assume these functions are defined in the z-plane cut along the real axis from $-\infty$ to $+1$. In (5.108) the branch of the square root is chosen so that $(z^2-1)^{1/2} > 0$ if $z > 1$.) Thus, for a fixed complex number r the vectors $f_m(z, t) = \Gamma(u+m+1)\,\mathfrak{B}_u^{r,m}(z)\,t^m$ form a basis for a realization of the representation $D(u, m_o)$ of $sl(2)$. We can use this realization by *type A* operators to induce a local multiplier representation T of $SL(2)$ on $\mathscr{F}$. Here $\mathscr{F}$ is the space of all functions of the complex variables z, t, analytic in a neighborhood of the point (say) $(z^o, t^o) = (2, 1)$. (The exact choice of (z^o, t^o) is not important.)

According to Theorem 1.10 and (5.106) the operators $\mathbf{T}(g)$ on $\mathscr{F}$ are determined by

$$[\mathbf{T}(\exp -b\mathscr{J}^+)f](z,t) = \left[\frac{(z^2-1)^{1/2}+bt(z+1)}{(z^2-1)^{1/2}+bt(z-1)}\right]^{r/2}$$

$$\cdot f\left[z+bt(z^2-1)^{1/2}, t\left(1+b^2t^2+\frac{2btz}{(z^2-1)^{1/2}}\right)^{-1/2}\right],$$

$$[\mathbf{T}(\exp -c\mathscr{J}^-)f](z,t) = \left[\frac{(z^2-1)^{1/2}+ct^{-1}(z-1)}{(z^2-1)^{1/2}+ct^{-1}(z+1)}\right]^{r/2}$$

$$\cdot f\left[z+ct^{-1}(z^2-1)^{1/2}, t\left(1+c^2t^{-2}+\frac{2ct^{-1}z}{(z^2-1)^{1/2}}\right)^{1/2}\right],$$

$$[\mathbf{T}(\exp \tau\mathscr{J}^3)f](z,t) = f(z, e^\tau t),$$

valid for $f \in \mathscr{F}$, (z,t) in the domain of f, and $|b|$, $|c|$, $|\tau|$ sufficiently small.

If $g \in SL(2)$, $d \neq 0$, then

$$g = (\exp -b'\mathscr{J}^+)(\exp -c'\mathscr{J}^-)(\exp \tau'\mathscr{J}^3)$$

for $b' = b/d$, $c' = cd$, $e^{\tau'/2} = d^{-1}$, $0 \leqslant \operatorname{Im}\tau' < 4\pi$. Thus, $\mathbf{T}(g)$ is defined by

$$[\mathbf{T}(g)f](z,t)$$

$$= [\mathbf{T}(\exp -b'\mathscr{J}^+)\mathbf{T}(\exp -c'\mathscr{J}^-)\mathbf{T}(\exp \tau'\mathscr{J}^3)f](z,t)$$

$$= \left[\left(\frac{d(z^2-1)^{1/2}+bt(z+1)}{d(z^2-1)^{1/2}+bt(z-1)}\right)\left(\frac{a(z^2-1)^{1/2}+ct^{-1}(z-1)}{a(z^2-1)^{1/2}+ct^{-1}(z+1)}\right)\right]^{r/2}$$

$$\cdot f\Bigg\{z+2bcz+abt(z^2-1)^{1/2}+cdt^{-1}(z^2-1)^{1/2},$$

$$t\left[\frac{(a(z^2-1)^{1/2}+ct^{-1}(z-1))(a(z^2-1)^{1/2}+ct^{-1}(z+1))}{(d(z^2-1)^{1/2}+bt(z+1))(d(z^2-1)^{1/2}+bt(z-1))}\right]^{1/2}\Bigg\},$$

$$\left|\frac{bt(z\pm 1)}{d(z^2-1)^{1/2}}\right| < 1, \qquad \left|\frac{ct^{-1}(z\pm 1)}{a(z^2-1)^{1/2}}\right| < 1, \qquad f \in \mathscr{F}. \tag{5.109}$$

With respect to the analytic basis $\{f_m(z,t)\}$ the matrix elements of this local representation are the functions $A_{lk}(g)$ given by Eqs. (5.10), (5.12). Thus,

$$[\mathbf{T}(g)f_{m_0+k}](z,t) = \sum_{l=-\infty}^{\infty} A_{lk}(g)f_{m_0+l}(z,t),$$

$$k = 0, \pm 1, \pm 2, \ldots . \tag{5.110}$$

Rather than attempt to work out the detailed consequences of (5.110) we consider only a few special cases. From Eqs. (A.9), (iii) it is clear that the function $\mathfrak{B}_u^{r,m}(z)$ is analytic and single valued in the right-half plane $\operatorname{Re} z > 1$. (If $r + m = 0$, $\mathfrak{B}_u^{r,m}(z)$ is analytic and single valued for $\operatorname{Re} z > 0$.) For simplicity we shall restrict ourselves to the half plane $\operatorname{Re} z > 1$.

If $a = d = t = 1$, $c = 0$, Eq. (5.110) can be written in the form

$$\left(1 + \frac{b(z+1)}{(z^2-1)^{1/2}}\right)^{(r-m)/2} \left(1 + \frac{b(z-1)}{(z^2-1)^{1/2}}\right)^{(-r-m)/2} \mathfrak{B}_u^{r,m}[z + b(z^2-1)^{1/2}]$$

$$= \sum_{l=0}^{\infty} \binom{u+m+l}{l} \frac{\Gamma(-u+m+l)}{\Gamma(-u+m)} (-b)^l \mathfrak{B}_u^{r,m+l}(z),$$

valid for $\operatorname{Re} z > 1$, $|\, b^2(z+1)/(z-1)| < 1$. If $a = d = t = 1$, $b = 0$, we have

$$\left(1 + \frac{c(z-1)}{(z^2-1)^{1/2}}\right)^{(r+m)/2} \left(1 + \frac{c(z+1)}{(z^2-1)^{1/2}}\right)^{(-r+m)/2} \mathfrak{B}_u^{r,m}[z + c(z^2-1)^{1/2}]$$

$$= \sum_{l=0}^{\infty} \frac{c^l}{l!} \mathfrak{B}_u^{r,m-l}(z),$$

for $\operatorname{Re} z > 1$, $|\, c^2(z+1)/(z-1)| < 1$. Finally, for $a = d = \cosh w/2$, $b = c = \sinh w/2$, $\rho = \cosh w$, $t = e^{i\alpha}$, there follows the identity

$$\left(1 + e^{-i\alpha}\left[\frac{(\rho-1)(z-1)}{(\rho+1)(z+1)}\right]^{1/2}\right)^{(r+m)/2} \left(1 + e^{i\alpha}\left[\frac{(\rho-1)(z-1)}{(\rho+1)(z+1)}\right]^{1/2}\right)^{(-r-m)/2}$$

$$\cdot \left(1 + e^{-i\alpha}\left[\frac{(\rho-1)(z+1)}{(\rho+1)(z-1)}\right]^{1/2}\right)^{(-r+m)/2}$$

$$\cdot \left(1 + e^{i\alpha}\left[\frac{(\rho-1)(z+1)}{(\rho+1)(z-1)}\right]^{1/2}\right)^{(r-m)/2}$$

$$\cdot\, \mathfrak{B}_u^{r,m}[z\rho + (z^2-1)^{1/2}(\rho^2-1)^{1/2}\cos\alpha]$$

$$= \sum_{l=-\infty}^{\infty} \mathfrak{B}_u^{-m-l,m}(\rho)\mathfrak{B}_u^{r,m+l}(z)e^{il\alpha},$$

where $\operatorname{Re} z > 1$, $\operatorname{Re} \rho > 1$, $|(\rho-1)(z+1)(\rho+1)^{-1}(z-1)^{-1}| < 1$, α real. If $m = r = 0$ this last formula reduces to the well-known addition theorem

$$\mathfrak{B}_u[z\rho + (z^2-1)^{1/2}(\rho^2-1)^{1/2}\cos\alpha] = \sum_{l=-\infty}^{\infty} \mathfrak{B}_u^l(\rho)\mathfrak{B}_u^{-l}(z)e^{il\alpha}$$

for generalized spherical harmonics, valid for $\operatorname{Re} z > 0$, $\operatorname{Re} \rho > 0$, $|\arg(z-1)| < \pi$, $|\arg(\rho - 1)| < \pi$. Here, $\mathfrak{B}_u^l(z) \equiv \mathfrak{B}_u^{0,-l}(z)$, $\mathfrak{B}_u(z) \equiv \mathfrak{B}_u^{0,0}(z)$. In all of the above equations $u \pm m$ are not integers.

Another interesting variant of the identities (5.110) is obtained by restricting $z = x$ to the real interval $-1 \leqslant x \leqslant 1$. We define functions $P_u^{r,m}(x)$ on this interval by

$$\begin{aligned} P_u^{r,m}(x) &= e^{-i\pi[(m+r)/2]}\mathfrak{B}_u^{r,m}(x + i\cdot 0) \\ &= \left(\frac{1+x}{2}\right)^{(m-r)/2}\left(\frac{1-x}{2}\right)^{(m+r)/2} \\ &\quad \cdot\frac{F(u+m+1, -u+m; m+r+1; (1-x)/2)}{\Gamma(m+r+1)} \end{aligned} \tag{5.111}$$

where $\mathfrak{B}_u^{r,m}(x + i\cdot 0)$ is the limit value of $\mathfrak{B}_u^{r,m}(z)$ as z approaches x on the upper side of the cut between -1 and $+1$. By inspection these functions have the properties

$$P_u^{r,m}(x) = P_u^{m,r}(x), \qquad P_u^{r,m}(x) = P_{-u-1}^{r,m}(x).$$

Moreover, $P_u^{0,-m}(x) = P_u^m(x)$, $P_u^{0,0}(x) = P_u(x)$, where $P_u^m(x)$, $P_u(x)$ are Legendre functions (Erdélyi *et al.* [1], Vol. I). The reader can derive identities for the functions $P_u^{r,m}(x)$ directly from (5.110).

Up to now we have used *type A* operators to obtain identities for the $\mathfrak{B}$ functions. However, these operators can also be employed to derive identities directly for the hypergeometric functions. The simplest way to proceed is to return to the original form of *type A* operators established in Section 2-7:

$$\mathrm{J}^3 = \frac{\partial}{\partial y}, \qquad \mathrm{J}^{\pm} = e^{\pm y}\left(\pm\frac{\partial}{\partial x} - \cot x\frac{\partial}{\partial y} + \frac{q}{\sin x}\right),$$

$$\mathrm{C}_{1,0} = -\frac{\partial^2}{\partial x^2} - \cot x\frac{\partial}{\partial x} + \frac{1}{\sin^2 x}\left(q^2 + \frac{\partial^2}{\partial y^2} - 2q\cos x\frac{\partial}{\partial y}\right).$$

The function

$$h_m(z, y) = (1-z)^{(m-q)/2}z^{-u}F(m-u, -q-u; -2u; z)e^{my};$$

$$z = \sec^2(x/2), \qquad 1 > |z| > 0,$$

is a simultaneous eigenfunction of J^3 and $\mathrm{C}_{1,0}$:

$$\mathrm{J}^3 h_m = m h_m, \qquad \mathrm{C}_{1,0}h_m = u(u+1)h_m.$$

We will transform these *type A* operators into new operators with simultaneous eigenfunctions of the form $h_m'(z, t) = F(m-u, -q-u; -2u; z)\, t^m$.

The procedure for carrying out this transformation is clear. We introduce a new variable t defined by $t = (1 - z)^{1/2}e^y$ and the function $\varphi(z) = z^{-u}(1 - z)^{-q/2}$. Then

$$[\varphi^{-1}h_m](z, y) = F(m - u, -q - u; -2u; z)t^m = h'_m(z, t).$$

The transformed operators $(J^3)^\varphi$, $(J^+)^\varphi$, $(J^-)^\varphi$ in the variables z, t are defined by Eq. (4.82) and are easily computed to be

$$\begin{aligned}(J^3)^\varphi &= t\frac{\partial}{\partial t}, \qquad (J^+)^\varphi = t\left(z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t} - u\right),\\ (J^-)^\varphi &= t^{-1}\left(z(1 - z)\frac{\partial}{\partial z} - t\frac{\partial}{\partial t} + z(q + u) - u\right).\end{aligned} \tag{5.112}$$

By construction these operators satisfy the usual commutation relations for the generators of $sl(2)$ so they can be used to form a realization of the representation $D(u, m_o)$.

To construct such a realization we must determine nonzero functions $f_m(z, t) = Z_m(z)\, t^m$, $m \in S = \{m_o + k\colon k \text{ an integer}\}$, such that

$$\left(z\frac{d}{dz} + m - u\right) Z_m(z) = (m - u)Z_{m+1}(z)$$

$$\left(z(1 - z)\frac{d}{dz} + z(q + u) - (m + u)\right) Z_m(z) = -(m + u)Z_{m-1}(z)$$

$$\left(z(1 - z)\frac{d^2}{dz^2} + [-2u - z(m - q - 2u + 1)]\frac{d}{dz} - (u - m)(q + u)\right) Z_m(z) = 0,$$

for all $m \in S$. These equations have the following solution, regular at $z = 0$, (A.7):

$$Z_m(z) = F(m - u, -u - q; -2u; z).$$

Thus, the functions $f_m(z, t) = F(m - u, -u - q; -2u; z)\, t^m$, $m \in S$, form a basis for a realization of the representation $D(u, m_o)$ by the generalized Lie derivatives (5.112). Clearly these operators induce a local multiplier representation T' of $SL(2)$ on the space $\mathcal{O}\!\ell$ of functions $f(z, t)$ analytic for (z, t) in some neighborhood of $(z^o, t^o) = (0, 1)$. The operators $\mathbf{T}'(g)$, $g \in SL(2)$, Eq. (1.15), acting on $\mathcal{O}\!\ell$ can be computed from the results of Theorem 1.10:

$$\begin{aligned}[\mathbf{T}'(g)f](z, t) &= (d + bt)^u\left(a + \frac{c}{t}\right)^{-q}\left(a - \frac{c(z - 1)}{t}\right)^{q+u}\\ &\quad \cdot f\left[\frac{zt}{(d + bt)(at - c(z - 1))}, \frac{c + at}{d + bt}\right],\\ |bt/d| < 1, \quad |c/at| &< 1, \quad |c(z - 1)/at| < 1, \quad d = (1 + bc)/a.\end{aligned} \tag{5.113}$$

With respect to the analytic basis $f_m(z, t)$ computed above, the operators $\mathbf{T}'(g)$ have matrix elements $A_{lk}(g)$, Eq. (5.10). Thus,

$$[\mathbf{T}'(g) f_{m_0+k}](z, t) = \sum_{l=-\infty}^{\infty} A_{lk}(g) f_{m_0+l}(z, t),$$

which reduces to the identity

$$\begin{aligned} F&\left[m, \mu; \nu; \frac{z(1-y)}{(1-(z-1)\tau)(1+\tau^{-1}y)}\right] \\ &\cdot (1+\tau^{-1}y)^{-m}(1+\tau)^{m+\mu-\nu}(1-(z-1)\tau)^{-\mu} \\ &= \sum_{l=-\infty}^{\infty} \tau^{-l} \frac{\Gamma(-\nu+m+1)}{\Gamma(-\nu+m+l+1)} \frac{F(\nu-m-l, m; -l+1; y)}{\Gamma(-l+1)} \\ &\cdot F(m+l, \mu; \nu; z), \end{aligned}$$

$$|\tau^{-1}y| < 1, \quad |\tau| < 1, \quad |(z-1)\tau| < 1, \quad z \neq 1, \quad |\tau^{-1}y - z| < 1, \tag{5.114}$$

valid for all $m, \mu, \nu, \in \mathcal{C}$ such that $m, -\nu + m$, are not integers and $-\nu$ is not a nonnegative integer. (The last restriction on ν can be avoided if we divide both sides of the identity by $\Gamma(\nu)$.) If $y = 0$ this identity becomes

$$\begin{aligned} F&\left(m, \mu; \nu; \frac{z}{1-(z-1)\tau}\right)(1+\tau)^{m+\mu-\nu}(1-(z-1)\tau)^{-\mu} \\ &= \sum_{l=0}^{\infty} \tau^{l} \binom{m-\nu}{l} F(m-l, \mu; \nu; z), \\ &\qquad z \neq 1, \quad |(z-1)\tau| < 1, \quad |\tau| < 1. \end{aligned}$$

If, on the other hand, we set $y = x\tau$ and let $\tau \to 0$, we obtain

$$\begin{aligned} F\left(m, \mu; \nu; \frac{z}{1+x}\right)(1+x)^{-m} &= \sum_{l=0}^{\infty} (-x)^l \binom{m+l-1}{l} F(m+l, \mu; \nu; z), \\ &\qquad |x| < 1, \quad |x| < |1-z|. \end{aligned}$$

5-12 *Type A* Realizations of $\uparrow_u$

To construct realizations of the representation $\uparrow_u$, $2u$ not a nonnegative integer, using the *type A* operators (5.106) we must find nonzero functions $f_m(z, t) = Z_m(z)\, t^m$, $m = -u, -u+1, \ldots$, such that

$$\begin{aligned} J^3 f_m = m f_m, \qquad J^+ f_m = (m-u) f_{m+1}, \qquad J^- f_m = -(m+u) f_{m-1}, \\ C_{1,0} f_m = (J^+J^- + J^3J^3 - J^3) f_m = u(u+1) f_m. \end{aligned}$$

The above conditions are equivalent to Eqs. (5.108) for the $Z_m(z)$ where, on the left-hand sides of these equations, m takes the values $-u$, $-u+1$, $-u+2,\ldots$. From (5.83) it follows that these relations are satisfied by

$$Z_m(z) = \frac{(-1)^{u+m}}{\Gamma(m-u)} \mathfrak{B}_u^{-r,-m}(z), \qquad m = -u, -u+1, \ldots, \tag{5.115}$$

for all $r \in \mathcal{C}$ such that $u - r$ is a nonnegative integer. Moreover, the functions (5.115) actually satisfy the required relations for all $r \in \mathcal{C}$, as can be verified by explicit computation. Thus the vectors

$$f_{-u+k}(z, t) = \frac{(-1)^k}{\Gamma(k-2u)} \mathfrak{B}_u^{-r,u-k}(z) t^{-u+k}, \qquad k \geqslant 0,$$

form a basis for a realization of the representation $\uparrow_u$ by the *type A* operators (5.106).

These Lie derivatives determine a local multiplier representation T of $SL(2)$ on $\mathcal{F}$ by operators $\mathbf{T}(g)$, $g \in SL(2)$, Eq. (5.109). With respect to the analytic basis $\{f_{-u+k}(z, t), k \geqslant 0\}$, the matrix elements of this local representation are the functions $B_{lk}(g)$, Eqs. (5.22)–(5.24). Hence,

$$[\mathbf{T}(g) f_{-u+k}](z, t) = \sum_{l=0}^{\infty} B_{lk}(g) f_{-u+l}(z, t), \qquad k = 0, 1, 2, \ldots. \tag{5.116}$$

Since $\mathfrak{B}_u^{r,m}(z)$ is analytic and single valued for $\operatorname{Re} z > 1$ the following identities are easily obtained from (5.116).

If $a = d = t = 1$, $c = 0$, then

$$\left(1 + \frac{b(z+1)}{(z^2-1)^{1/2}}\right)^{(r-m)/2} \left(1 + \frac{b(z-1)}{(z^2-1)^{1/2}}\right)^{(-r-m)/2} \mathfrak{B}_u^{-r,-m}(z + b(z^2-1)^{1/2})$$
$$= \sum_{l=0}^{\infty} \frac{b^l}{l!} \mathfrak{B}_u^{-r,-m-l}(z),$$

valid for $\operatorname{Re} z > 1$, $|\, b^2(z+1)/(z-1)| < 1$ and $u + m$ a nonnegative integer. If $a = d = t = 1$, $b = 0$, there follows

$$\left(1 + \frac{c(z-1)}{(z^2-1)^{1/2}}\right)^{(r+m)/2} \left(1 + \frac{c(z+1)}{(z^2-1)^{1/2}}\right)^{(-r+m)/2} \mathfrak{B}_u^{-r,-m}(z + c(z^2-1)^{1/2})$$
$$= \sum_{l=0}^{m+u} (-c)^l \binom{m+u}{l} \frac{\Gamma(m-u)}{\Gamma(m-u-l)} \mathfrak{B}_u^{-r,-m+l}(z),$$
$$\operatorname{Re} z > 1, \quad |\, c^2(z+1)/(z-1)\,| < 1,$$

where $u + m$ is a nonnegative integer. For $a = d = \cosh(w/2)$, $b = c = \sinh(w/2)$, $\rho = \cosh w$, $t = e^{i\alpha}$, there follows

$$\left(1 + e^{-i\alpha}\left[\frac{(\rho-1)(z-1)}{(\rho+1)(z+1)}\right]^{1/2}\right)^{(r+m)/2}\left(1 + e^{i\alpha}\left[\frac{(\rho-1)(z-1)}{(\rho+1)(z+1)}\right]^{1/2}\right)^{(-r-m)/2}$$

$$\cdot\left(1 + e^{-i\alpha}\left[\frac{(\rho-1)(z+1)}{(\rho+1)(z-1)}\right]^{1/2}\right)^{(-r+m)/2}\left(1 + e^{i\alpha}\left[\frac{(\rho-1)(z+1)}{(\rho+1)(z-1)}\right]^{1/2}\right)^{(r-m)/2}$$

$$\cdot\,\mathfrak{B}_u^{-r,-m}[z\rho + (z^2-1)^{1/2}(\rho^2-1)^{1/2}\cos\alpha]$$

$$= \sum_{l=0}^{\infty} \mathfrak{B}_u^{-u+l,-m}(\rho)\mathfrak{B}_u^{-r,u-l}(z)e^{i(-u-m+l)\alpha},$$

valid for $\operatorname{Re} z > 1$, $\operatorname{Re}\rho > 1$, $|(\rho-1)(z+1)(\rho+1)^{-1}(z-1)^{-1}| < 1$, α real, and $u + m$ a nonnegative integer.

There is a special case of the identity (5.116) which is worthy of consideration in its own right. If $r = 0$, $v = z(z^2-1)^{-1/2}$ the basis functions $\mathfrak{B}_u^{0,u-k}$, considered as functions of v, are multiples of $(1-v^2)^{-u/2}C_k^{-u}(v)$ where the C_k^{-u} are Gegenbauer polynomials, (A.9). Rather than obtain identities for Gegenbauer polynomials by manipulating (5.116) we will start at the beginning and derive the identities from a realization of the representation $\uparrow_u$ of $sl(2)$. Thus we introduce the new variable $v = z(z^2-1)^{-1/2}$ into expressions (5.106) for the *type A* operators. In terms of the variables (v, t) the differential operators become

$$\mathrm{J}^3 = t\frac{\partial}{\partial t}, \qquad \mathrm{J}^\pm = t^{\pm 1}\left((v^2-1)\frac{\partial}{\partial v} \pm vt\frac{\partial}{\partial t}\right), \tag{5.117}$$

where we have set $r = 0$. To realize the representation $\uparrow_u$, $2u \neq 0, 1, 2, \ldots$, with these operators, we must construct basis vectors $f_{-u+k}(v, t) = W_{-u+k}(v)\, t^{-u+k}$, $k \geqslant 0$, such that

$$\begin{aligned} \mathrm{J}^3 f_{-u+k} &= (-u+k)f_{-u+k}, \qquad & \mathrm{J}^+ f_{-u+k} &= (-2u+k)f_{-u+k+1}, \\ \mathrm{J}^- f_{-u+k} &= -kf_{-u+k-1}, \qquad & \mathrm{C}_{1,0} f_{-u+k} &= u(u+1)f_{-u+k}. \end{aligned} \tag{5.118}$$

The functions $W_{-u+k}(v)$ are determined to within a nonzero multiplicative constant by these conditions. Indeed, $\mathrm{J}^- f_{-u} = 0$ implies

$$(v^2-1)\frac{dW_{-u}}{dv} - uvW_{-u} = 0.$$

This equation has the solution $W_{-u}(v) = (v^2 - 1)^{-u/2}$, unique to within a multiplicative constant. The functions $W_{-u+k}(v)$, $k \geqslant 0$, can now be defined recursively by the formula

$$\begin{aligned} W_{-u+k}(v)t^{-u+k} &= (-2u + k - 1)^{-1}\mathrm{J}^+(W_{-u+k-1}(v)t^{-u+k-1}) \\ &= \frac{\Gamma(-2u)}{\Gamma(-2u+k)}(\mathrm{J}^+)^k((v^2-1)^{-u/2}t^{-u}). \end{aligned} \tag{5.119}$$

From the explicit expression for the differential operator J^+ it is easy to verify that the functions $W_{-u+k}(v)$ can be written in the form

$$W_{-u+k}(v) = \frac{\Gamma(-2u)k!}{\Gamma(-2u+k)}(v^2-1)^{-u/2}C_k^{-u}(v),$$

where C_k^{-u} is a polynomial in v of order k. Furthermore, it follows from the commutation relations of the operators $\mathrm{J}^{\pm}$, J^3 that the vectors $f_{-u+k}(v, t) = W_{-u+k}(v)\, t^{-u+k}$, $k \geqslant 0$, defined by (5.119) do in fact satisfy all of the conditions (5.118) and form a basis for a realization of $\uparrow_u$. Writing conditions (5.118) in terms of the polynomials $C_k^{-u}(v)$ we find

$$\left[(1-v^2)\frac{d}{dv} - v(k-2u)\right] C_k^{-u}(v) = -(k+1)C_{k+1}^{-u}(v),$$

$$\left[(1-v^2)\frac{d}{dv} + vk\right] C_k^{-u}(v) = (k-2u-1)C_{k-1}^{-u}(v),$$

$$\left[(1-v^2)\frac{d^2}{dv^2} - (1-2u)v\frac{d}{dv} + k(k-2u)\right] C_k^{-u}(v) = 0.$$

This realization of $\uparrow_u$ can be extended to a local multiplier representation T of $SL(2)$. The action of the operator $\mathbf{T}(g)$ on a function f, analytic in v and t, can be obtained formally from (5.109) by setting $r = 0$ and introducing the new variable $v = z(z^2-1)^{-1/2}$. The result is

$$\begin{aligned} [\mathbf{T}(g)f](v,t) = f\Bigg[&\frac{v(1+2bc) + abt + cdt^{-1}}{([2v(1+bc) + abt + cdt^{-1}][2vbc + abt + cdt^{-1}] + 1)^{1/2}}, \\ &\frac{at}{d}\left(\frac{1 + (c/at)^2 + 2(c/at)v}{1 + (tb/d)^2 + 2(tb/d)v}\right)^{1/2}\Bigg], \qquad d = (1+bc)/a, \end{aligned} \tag{5.120}$$

convergent for g in a sufficiently small neighborhood of $\mathbf{e}$. As before, the functions $B_{lk}(g)$ are the matrix elements of $\mathbf{T}(g)$ with respect to the analytic basis $\{f_{-u+k}\}$:

$$[\mathbf{T}(g)f_{-u+k}](v,t) = \sum_{l=0}^{\infty} B_{lk}(g)f_{-u+l}(v,t).$$

After simplification, this expression leads to the identity

$$(1 + y^2\tau^{-2} - 2y\tau^{-1}v)^{k/2}(1 + \tau^2 - 2\tau v)^{-\mu-k/2}$$

$$\cdot\, C_k^\mu\left[\frac{v(1+y) - \tau - y\tau^{-1}}{(1 + y^2\tau^{-2} - 2y\tau^{-1}v)^{1/2}(1 + \tau^2 - 2\tau v)^{1/2}}\right]$$

$$= \sum_{l=0}^{\infty} \tau^{l-k}\,\frac{l!}{k!}\,\frac{F(-k, 2\mu + l; l - k + 1; y)}{\Gamma(l - k + 1)}\, C_l^\mu(v), \qquad k \geqslant 0, \quad (5.121)$$

valid for $|\tau^2 - 2\tau v| < 1$, $|\tau^{-2}y^2 - 2\tau^{-1}yv| < 1$, and $2\mu \neq 0, -1, -2,\ldots$. If $y = 0$ the identity reduces to

$$(1 + \tau^2 - 2\tau v)^{-\mu-(k/2)} C_k^\mu\left[\frac{v - \tau}{(1 + \tau^2 - 2\tau v)^{1/2}}\right] = \sum_{l=0}^{\infty} \tau^l \binom{l+k}{l} C_{k+l}^\mu(v).$$

By construction, $C_0^\mu(v) \equiv 1$, so for $k = 0$ the above equation yields the well-known generating function

$$(1 + \tau^2 - 2\tau v)^{-\mu} = \sum_{l=0}^{\infty} \tau^l C_l^\mu(v).$$

For $v = 0$ it follows that

$$(1 + \tau^2)^{-\mu} = \sum_{l=0}^{\infty} \tau^l C_l^\mu(0),$$

whence

$$C_l^\mu(0) = \begin{cases} 0 & \text{if } l \text{ is odd,} \\ \dbinom{-\mu}{k} & \text{if } l = 2k \text{ is even.} \end{cases}$$

Another useful relation for the Gegenbauer polynomials can be obtained by setting $\tau = -y/t$ in (5.121) and going to the limit as $y \to 0$. The result is

$$(1 + t^2 + 2tv)^{k/2} C_k^\mu\left[\frac{v + t}{(1 + t^2 + 2tv)^{1/2}}\right] = \sum_{l=0}^{k} \binom{2\mu + k - 1}{l} t^l C_{k-l}^\mu(v).$$

If $v = 0$, $x = t(1 + t^2)^{-1/2}$, this expression takes the form

$$C_k^\mu(x) = \sum_{l=0}^{k} x^l (1 - x^2)^{(k-l)/2} \binom{2\mu + k - 1}{l} C_{k-l}^\mu(0),$$

or

$$C^{\mu}_{2k}(x) = \sum_{l=0}^{k} x^{2l}(1 - x^2)^{k-l} \binom{2\mu + 2k - 1}{2l}\binom{-\mu}{k - l},$$

$$C^{\mu}_{2k+1}(x) = \sum_{l=0}^{k} x^{2l+1}(1 - x^2)^{k-l} \binom{2\mu + 2k}{2l + 1}\binom{-\mu}{k - l}.$$

5-13 *Type A* Realizations of $\downarrow_u$

Fundamental identities for the Jacobi polynomials can be derived from realizations of $\downarrow_u$, $2u$ not a nonnegative integer, by the *type A* operators (5.112). According to the usual procedure, in order to construct such realizations we must find basis vectors $f_m(z, t) = Z_m(z)\, t^m$; $m = u - k$, $k = 0, 1, 2, \ldots$, such that

$$\left(z \frac{d}{dz} - k\right) Z_{u-k}(z) = -kZ_{u-k+1}(z),$$

$$\left(z(1 - z) \frac{d}{dz} + z(q + u) - (2u - k)\right) Z_{u-k}(z) = -(2u - k)Z_{u-k-1}(z), \quad (5.122)$$

$$\left(z(1 - z) \frac{d^2}{dz^2} + [-2u + z(k + q + u - 1)] \frac{d}{dz} - k(q + u)\right) Z_{u-k}(z) = 0.$$

These relations determine the functions $Z_{u-k}(z)$, $k \geqslant 0$, to within an arbitrary constant. Indeed, the relation $z\, d/dz\, Z_u(z) = 0$ implies $Z_u(z)$ is a constant function. If we set $Z_u(z) \equiv 1$ and use the second of Eqs. (5.122) to define the functions $Z_{u-k}(z)$ recursively, we easily obtain the result that the eigenfunctions are Jacobi polynomials,

$$Z_{u-k}(z) = F(-k, -q - u; -2u; z), \qquad k = 0, 1, 2, \ldots,$$

and that these polynomials satisfy all of the relations (5.122). The Lie derivatives (5.112) induce a local multiplier representation T' on $\mathcal{A}$. With respect to the analytic basis $\{f_{u-k}(z, t) = Z_{u-k}(z)\, t^{u-k}\}$ computed here, the operators $\mathbf{T}'(g)$ defined by (5.113) have matrix elements $C_{lk}(g)$, (5.31):

$$[\mathbf{T}'(g) f_{u-k}](z, t) = \sum_{l=0}^{\infty} C_{lk}(g) f_{u-l}(z, t), \qquad k = 0, 1, 2, \ldots .$$

This expression is equivalent to the identity

$$(1 + y\tau^{-1})^k(1 - (z - 1)\tau)^{-\mu}(1 + \tau)^{\mu-\nu-k}$$
$$\cdot F\left(-k, \mu; \nu; \frac{z(1-y)}{(1-(z-1)\tau)(1+y\tau^{-1})}\right)$$
$$= \sum_{l=0}^{\infty} \tau^{l-k} \frac{\Gamma(-\nu-k+1)}{\Gamma(-\nu-l+1)} \frac{F(-k, \nu+l; l-k+1; y)}{\Gamma(l-k+1)}$$
$$\cdot F(-l, \mu; \nu; z),$$
$$|(z-1)\tau| < 1, \quad |\tau| < 1, \quad k = 0, 1, 2, \ldots, \tag{5.123}$$

valid for all $\mu, \nu \in \mathcal{C}$ such that ν is not a nonpositive integer. For $y = 0$, this identity simplifies to

$$(1 - (z-1)\tau)^{-\mu}(1+\tau)^{\mu-\nu-k} F\left(-k, \mu; \nu; \frac{z}{1-(z-1)\tau}\right)$$
$$= \sum_{l=0}^{\infty} \tau^l \binom{-\nu-k}{l} F(-k-l, \mu; \nu; z),$$
$$|(z-1)\tau| < 1, \quad |\tau| < 1.$$

In particular if $k = 0$ we have a simple generating function for the hypergeometric polynomials:

$$(1 - (z-1)\tau)^{-\mu}(1+\tau)^{\mu-\nu} = \sum_{l=0}^{\infty} \tau^l \binom{-\nu}{l} F(-l, \mu; \nu; z). \tag{5.124}$$

Finally, if we set $y = w\tau$ in (5.123) and take the limit as $\tau \to 0$, we obtain

$$(1+w)^k F\left(-k, \mu; \nu; \frac{z}{1+w}\right) = \sum_{l=0}^{k} w^l \binom{k}{l} F(-k+l, \mu; \nu; z).$$

5-14 *Type A* Realizations of $D(2u)$

We shall now construct realizations of the finite-dimensional representation $D(2u)$, $2u$ a nonnegative integer, in terms of the generalized Lie derivatives (5.106). In order to compare the results more easily with the computations in Section 5-16, we choose basis vectors p_m, $m = -u, -u+1, \ldots, u-1, u$, for the representation space such that the action of the infinitesimal operators is

$$J^3 p_m = m p_m, \qquad J^+ p_m = [(u-m)(u+m+1)]^{1/2} p_{m+1},$$
$$J^- p_m = [(u-m+1)(u+m)]^{1/2} p_{m-1}, \qquad C_{0,1} p_m = u(u+1) p_m,$$

see (5.41). Hence, to construct a realization of $D(2u)$ with the operators

$$\mathrm{J}^3 = t\frac{\partial}{\partial t}, \qquad \mathrm{J}^\pm = t^{\pm 1}\left(-(z^2-1)^{1/2}\frac{\partial}{\partial z} \pm \frac{z}{(z^2-1)^{1/2}}\, t\frac{\partial}{\partial t} \pm \frac{r}{(z^2-1)^{1/2}}\right)$$

we must find basis vectors $p_m(z, t) = W_m(z)\, t^m$, $m = -u, \ldots, u$, such that

$$\begin{aligned}
&\left(-(z^2-1)^{1/2}\frac{d}{dz} + \frac{zm}{(z^2-1)^{1/2}} + \frac{r}{(z^2-1)^{1/2}}\right) W_m(z) \\
&\qquad = [(u-m)(u+m+1)]^{1/2} W_{m+1}(z), \\
&\left(-(z^2-1)^{1/2}\frac{d}{dz} - \frac{zm}{(z^2-1)^{1/2}} - \frac{r}{(z^2-1)^{1/2}}\right) W_m(z) \\
&\qquad = [(u+m)(u-m+1)]^{1/2} W_{m-1}(z), \\
&\left((z^2-1)\frac{d^2}{dz^2} + 2z\frac{d}{dz} - \frac{2rmz + r^2 + m^2}{z^2-1}\right) W_m(z) \\
&\qquad = u(u+1) W_m(z).
\end{aligned} \tag{5.125}$$

(We assume the functions $W_m(z)$ are defined in the z-plane cut along the real axis from $-\infty$ to $+1$.) These relations determine the $W_m(z)$ to within a multiplicative constant. In fact, from the first of the above equations we have

$$\left(-(z^2-1)^{1/2}\frac{d}{dz} + \frac{uz + r}{(z^2-1)^{1/2}}\right) W_u(z) = 0;$$

whence

$$W_u(z) = c_r\left(\frac{z+1}{2}\right)^{(u-r)/2}\left(\frac{z-1}{2}\right)^{(u+r)/2} = c_r \Gamma(u+r+1)\mathfrak{B}_u^{r,u}(z),$$

where c_r is a constant. Using the second of Eqs. (5.125) we can define the functions $W_m(z)$ recursively from $W_u(z)$. A straightforward computation gives

$$\begin{aligned}
W_m(z) &= c_r \frac{(-1)^{u-m}}{\prod_{n=m+1}^{u}[(u+n)(u-n+1)]^{1/2}} (z-1)^{-(m+r)/2}(z+1)^{-(m-r)/2} \\
&\quad \cdot \frac{d^{u-m}}{dz^{u-m}}\left[\left(\frac{z-1}{2}\right)^{(u+r)/2}\left(\frac{z+1}{2}\right)^{(u-r)/2}\right] \\
&= c_r(-1)^{u-m}(u+m)!\,\frac{\Gamma(u+r+1)}{(2u)!}\prod_{n=m+1}^{u}\left[\frac{(u+n)}{(u-n+1)}\right]^{1/2}\mathfrak{B}_u^{r,m}(z),
\end{aligned} \tag{5.126}$$

for $m = -u, -u+1, \ldots, u$. However, we must still satisfy the condition $J^- p_{-u} = 0$ or

$$\left((z^2-1)^{1/2}\frac{d}{dz} + \frac{r-uz}{(z^2-1)^{1/2}}\right) W_{-u}(z) = 0.$$

According to the first expression (5.126), this condition can be satisfied if and only if $u \pm r$ are both nonnegative integers, i.e., if and only if $r = -u, -u+1, \ldots, u$. For convenience we fix the constant c_r by the requirement

$$c_r = \frac{(-1)^{-u-r}}{(u+r)!}\left[\frac{(u+r)!\,(2u)!}{(u-r)!}\right]^{1/2}.$$

Thus,

$$W_m(z) = (-1)^{m+r}\left[\frac{(u+m)!\,(u+r)!}{(u-m)!\,(u-r)!}\right]^{1/2} \mathfrak{B}_u^{r,m}(z),$$

$$r, m = -u, -u+1, \ldots, u.$$

The normalization constant c_r has been chosen so that the eigenfunctions $W_m(z)$ will be readily comparable with the matrix elements of irreducible representations of $SU(2)$ (Section 5-16). From the commutation relations of the operators $J^{\pm}$, J^3, it is easy to check that the functions $W_m(z)$, $m = -u, \ldots, u$, satisfy all of the equations (5.125). Thus, the vectors $p_m(z, t) = W_m(z)\, t^m$ form a basis for a realization of the representation $D(2u)$.

As we have shown earlier, the *type A* operators (5.106) determine a local multiplier representation T of $SL(2)$ in terms of operators $\mathbf{T}(g)$, $g \in SL(2)$, defined by Eq. (5.109). With respect to the basis vectors $p_m(z, t)$, the matrix elements of this representation are $Q_{lk}(g)$, Eqs. (5.43), (5.37). Therefore,

$$[\mathbf{T}(g)p_{-u+k}](z, t) = \sum_{l=0}^{2u} Q_{lk}(g) p_{-u+l}(z, t), \qquad k = 0, 1, \ldots, 2u. \tag{5.127}$$

The following consequences of this identity are easily derived: If $a = d = t = 1$, $b = 0$, then

$$\left(1 + \frac{c(z-1)}{(z^2-1)^{1/2}}\right)^{(r-u+k)/2}\left(1 + \frac{c(z+1)}{(z^2-1)^{1/2}}\right)^{(-r-u+k)/2} \mathfrak{B}_u^{r,-u+k}(z + c(z^2-1)^{1/2})$$

$$= \sum_{l=0}^{k} \frac{c^{k-l}}{(k-l)!} \mathfrak{B}_u^{r,-u+l}(z),$$

$$\operatorname{Re} z > 1, \quad |c^2(z+1)/(z-1)| < 1, \quad k = 0, 1, \ldots, 2u;$$

while if $a = d = \cosh(w/2)$, $b = c = \sinh(w/2)$, $\rho = \cosh w$, $t = e^{i\alpha}$, there follows

$$\left(1 + e^{-i\alpha}\left[\frac{(\rho-1)(z-1)}{(\rho+1)(z+1)}\right]^{1/2}\right)^{(r+m)/2}\left(1 + e^{i\alpha}\left[\frac{(\rho-1)(z-1)}{(\rho+1)(z+1)}\right]^{1/2}\right)^{(-r-m)/2}$$

$$\cdot\left(1 + e^{-i\alpha}\left[\frac{(\rho-1)(z+1)}{(\rho+1)(z-1)}\right]^{1/2}\right)^{(-r+m)/2}\left(1 + e^{i\alpha}\left[\frac{(\rho-1)(z+1)}{(\rho+1)(z-1)}\right]^{1/2}\right)^{(r-m)/2}$$

$$\cdot\,\mathfrak{B}_u^{r,m}[z\rho + (z^2-1)^{1/2}(\rho^2-1)^{1/2}\cos\alpha] = \sum_{l=0}^{2u}\mathfrak{B}_u^{u-l,m}(\rho)\mathfrak{B}_u^{r,-u+l}(z)e^{i(-u-m+l)\alpha},$$

$$\operatorname{Re} z > 1,\quad \operatorname{Re}\rho > 1,\quad |(\rho-1)(z+1)(\rho+1)^{-1}(z-1)^{-1}| < 1,$$

$$\alpha \text{ real},\quad m = -u, -u+1, \ldots, u.$$

In terms of the functions $P_u^{r,m}(z)$ defined on the real interval $-1 \leqslant x \leqslant 1$ by $P_u^{r,m}(x) = e^{-i\pi[(m+r)/2]}\mathfrak{B}_u^{r,m}(x + i0)$ (see (5.111)) (5.127) implies the relations

$$\left(1 - \tau\tan\frac{\theta}{2}\right)^{(r-u+k)/2}\left(1 + \tau\cot\frac{\theta}{2}\right)^{(-r-u+k)/2} P_u^{r,-u+k}(\cos\theta - \tau\sin\theta)$$

$$= \sum_{l=0}^{k}\frac{\tau^{k-l}}{(k-l)!}P_u^{r,-u+l}(\cos\theta),$$

$$\left|\tau\cot\frac{\theta}{2}\right| < 1,\quad \left|\tau\tan\frac{\theta}{2}\right| < 1,\quad 0 < \theta < \pi,\quad k = 0, 1, \ldots, 2u;$$

$$\left(1 - e^{-i\alpha}\tan\frac{\varphi}{2}\tan\frac{\theta}{2}\right)^{(r+m)/2}\left(1 + e^{-i\alpha}\tan\frac{\varphi}{2}\cot\frac{\theta}{2}\right)^{(-r+m)/2}$$

$$\cdot\left(1 + e^{i\alpha}\tan\frac{\varphi}{2}\cot\frac{\theta}{2}\right)^{(r-m)/2}\left(1 - e^{i\alpha}\tan\frac{\varphi}{2}\tan\frac{\theta}{2}\right)^{(-r-m)/2}$$

$$\cdot\,P_u^{r,m}(\cos\theta\cos\varphi - \sin\theta\sin\varphi\cos\alpha)$$

$$= \sum_{l=0}^{2u}P_u^{u-l,m}(\cos\varphi)P_u^{r,-u+l}(\cos\theta)e^{i(-u-m+l)\alpha},$$

$$\left|\tan\frac{\varphi}{2}\tan\frac{\theta}{2}\right| < 1,\quad \left|\tan\frac{\varphi}{2}\cot\frac{\theta}{2}\right| < 1,$$

$$0 < \varphi, \theta < \pi,\quad \alpha \text{ real},\quad m = -u, -u+1, \ldots, +u.$$

In all of these equations $2u$ is a nonnegative integer and $r = -u, -u+1, \ldots, u$.

5-15 Weisner's Method for *Type A* Operators

The identities for the hypergeometric functions derived so far have all been related to local multiplier representations of $SL(2)$. We can follow Weisner [1], however, and use the operators (5.112) to derive identities which are not associated with local representations. The basic fact to be noted is: If $f(z, t)$ is an eigenfunction of the operator $C_{1,0}$, i.e., $C_{1,0}f(z, t) = u(u+1)f(z, t)$, where

$$C_{1,0} = z^2(1-z)\frac{\partial^2}{\partial z^2} - z^2 t\frac{\partial^2}{\partial t\,\partial z} + z[-2u + z(q+2u-1)]\frac{\partial}{\partial z}$$

$$+ z(q+u)t\frac{\partial}{\partial t} - zu(u+q) + u(u+1),$$

then $C_{1,0}[\mathbf{T}'(g)f] = u(u+1)[\mathbf{T}'(g)f]$ where

$$[\mathbf{T}'(g)f](z, t) = (d+bt)^u\left(a+\frac{c}{t}\right)^{-q}\left(a-\frac{c(z-1)}{t}\right)^{q+u}$$

$$\cdot f\left[\frac{zt}{(d+bt)(at-c(z-1))}, \frac{c+at}{d+bt}\right], \qquad \text{Eq. (5.113).}$$

This is true because the operators $(J^{\pm})$, J^3 commute with

$$C_{1,0} = J^+J^- + J^3J^3 - J^3,$$

and is valid for all $g \in SL(2)$, and all (z, t) in the domain of f such that $[\mathbf{T}'(g)f](z, t)$ can be defined. It is not necessary that the convergence conditions following (5.113) be satisfied. Moreover, if the function f has a convergent expansion of the form

$$f(z, t) = \sum_m h_m(z)t^m$$

then $h_m(z)$ must be a solution of the hypergeometric equation

$$\left[z(1-z)\frac{d^2}{dz^2} + [-2u - z(m-q-2u+1)]\frac{d}{dz} - (u-m)(q+u)\right]h_m(z) = 0.$$

Hence, if $2u$ is not an integer, $h_m(z)$ is a linear combination of the linearly independent solutions $F(m-u, -u-q; -2u; z)$ and $z^{1+2u}F(m+u+1, u-q+1; 2+2u; z)$. If $h_m(z)$ is regular at $z = 0$ it is a multiple of the first of these solutions.

As an application of these remarks we apply the operator $\mathbf{T}'(g)$ to the basis function $f_m(z, t) = F(m-u, -u-q; -2u; z)\, t^m$ and consider the resulting expression in the domain where $\tau = t^{-1} \neq 0$ ranges over a neighborhood of zero:

$$a^{u+m}b^{u-m}\left(1+\frac{(1+bc)\tau}{ab}\right)^{u-m}\left(1-\frac{c\tau(z-1)}{a}\right)^{q+u}\left(1+\frac{c\tau}{a}\right)^{-q+m}$$

$$\cdot F\left[m-u, -u-q; -2u; \frac{z\tau}{ab(1-c\tau(z-1)/a)(1+(1+bc)\tau/ab)}\right]\tau^{-u}$$

$$= \sum_{n=0}^{\infty} h_{u-n}(g, z)\tau^{-u+n} \tag{5.128}$$

valid for

$$|\tau| < \min\left(\left|\frac{ab}{1+bc}\right|, \left|\frac{a}{c(z-1)}\right|, \left|\frac{a}{c}\right|, \left|\frac{a^2b}{(1+bc)c(z-1)}\right|\right),$$

if $c \neq 0$. The expansion on the right-hand side of this expression follows because the left-hand side is equal to τ^{-u} times a function of τ analytic at $\tau = 0$. Since $|\tau| < |ab/(1+bc)|$ the group element g is bounded away from $\mathbf{e}$ and we cannot use the representation theory of $SL(2)$ to obtain the expansion coefficients h_{u-n}. However, these coefficients can be obtained directly. The left-hand side of (5.128) is regular at $z = 0$, which implies

$$h_{u-n}(g, z) = h'_n(g)F(-n, -u-q; -2u; z), \qquad n = 0, 1, 2, \ldots .$$

Setting $z = 0$ in (5.128) and comparing the resulting expansion with the generating function (5.124) for Jacobi polynomials, we have

$$h'_n(g) = a^{u+m-n}b^{u-m}c^nF(-n, m-u; -2u; -(bc)^{-1})\binom{2u}{n}.$$

There are two interesting consequences of this computation: If $a = b = 1$, $c = 0$ then

$$h'_n(g) = \binom{u-m}{n}$$

and

$$(1+\tau)^{-\mu}F\left(\mu, \rho; \nu; \frac{z\tau}{1+\tau}\right) = \sum_{n=0}^{\infty}\binom{-\mu}{n}F(-n, \rho; \nu; z)\tau^n,$$

$$|\tau| < \min(1, |1-z|^{-1}),$$

where $\mu = m - u$, $\rho = -u - q$, $\nu = -2u$. For $a = c = 1$, $b = -w^{-1}$ we have the symmetric relation

$$(1 + (1 - w)\tau)^{-\mu}(1 + (1 - z)\tau)^{-\rho}(1 + \tau)^{\mu+\rho-\nu}$$

$$\cdot F\left[\mu, \rho; \nu; \frac{-z\tau w}{(1 + (1 - z)\tau)(1 + (1 - w)\tau)}\right]$$

$$= \sum_{n=0}^{\infty} \binom{2\mu}{n} F(-n, \mu; \nu; w)F(-n, \rho; \nu; z)\tau^n,$$

$$|\tau| < \min(1, |1 - z|^{-1}, |1 - w|^{-1}, |1 - z|^{-1}|1 - w|^{-1}).$$

This is a bilinear generating function for the Jacobi polynomials.

As another example consider a solution $f(x, t)$ of the simultaneous equations

$$\mathrm{C}_{1,0} f = u(u + 1) f, \qquad \mathrm{J}^+ f = -f,$$

which is regular at $z = 0$. Thus, f satisfies the equations

$$\text{(i)}\quad \left[z(1 - z)\frac{\partial^2}{\partial z^2} - zt\frac{\partial^2}{\partial t\,\partial z} + [-2u + z(q + 2u - 1)]\frac{\partial}{\partial z}\right.$$

$$\left. + (q + u)t\frac{\partial}{\partial t} - u(u + q)\right] f(z, t) = 0,$$

$$\text{(ii)}\quad t\left[z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t} - u\right] f(z, t) = -f(z, t).$$

Equation (ii) has the general solution $f(z, t) = t^u \exp(t^{-1})\, h(zt^{-1})$ where h is arbitrary. Substituting this expression into (i), we find $h(x)$ satisfies the confluent hypergeometric equation

$$x\frac{d^2h}{dx^2} + (2u - x)\frac{dh}{dx} + (q + u)h = 0,$$

(A.11). Since f is regular at $z = 0$, the solution is (unique to within a multiplicative constant)

$$f(z, t) = t^u \exp(t^{-1})\, {}_1F_1(q + u; 2u; zt^{-1}).$$

Note that $t^{-u}f(z, t)$ has a power series expansion in $\tau = t^{-1}$ about $\tau = 0$. Similarly $[\mathbf{T}'(g)f](z, \tau^{-1})$ is a solution of the equation

$C_{1,0}[\mathbf{T}'(g)f] = u(u+1)[\mathbf{T}'(g)f]$ and has an expansion in τ whose coefficients are multiples of Jacobi polynomials:

$$\begin{aligned}
&[\mathbf{T}'(g)f](z, \tau^{-1}) \\
&\quad = \tau^{-u}(a - c(z-1)\tau)^{q+u}(a + c\tau)^{-q+u} \exp\left[\frac{ab + (1+bc)\tau}{a(a+c\tau)}\right] \\
&\quad\quad \cdot {}_1F_1\left[q+u; 2u; \frac{z\tau}{(a - c(z-1)\tau)(a+c\tau)}\right] \\
&\quad = \sum_{n=0}^{\infty} j_{u-n}(g) F(-n, -u-q; -2u; z)\tau^{-u+n}, \\
&\qquad\qquad\qquad |\tau| < \min\left(\left|\frac{a}{c(z-1)}\right|, \left|\frac{a}{c}\right|\right). \qquad (5.129)
\end{aligned}$$

We can compute the coefficients $j_{u-n}(g)$ by setting $z = 0$ and comparing the resulting expression with the generating function (5.101) for the generalized Laguerre polynomials:

$$j_{u-n}(g) = a^{2u}e^{b/a}(-c/a)^n L_n^{(-2u-1)}(1/ac).$$

For the special case $b = c = 0$, $a = 1$, this result simplifies to $j_{u-n}(\mathbf{e}) = (n!)^{-1}$, whence

$$e^{\tau}\,{}_1F_1(-\mu; -\nu; z\tau) = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} F(-n, \mu; \nu; z), \qquad \mu, \nu \in \mathcal{C}. \qquad (5.130)$$

If $a = c = w^{-1/2}$, $b = 0$, we obtain

$$\begin{aligned}
&(1 - (z-1)\tau)^{-\mu}(1+\tau)^{\mu-\nu} \exp\left[\frac{w\tau}{1+\tau}\right] {}_1F_1\left[-\mu; -\nu; \frac{z\tau w}{(1-(z-1)\tau)(1+\tau)}\right] \\
&\quad = \sum_{n=0}^{\infty} (-\tau)^n L_n^{(\nu-1)}(w) F(-n, \mu; \nu; z), \\
&\qquad\qquad\qquad |\tau| < \min(1, |z-1|^{-1}).
\end{aligned}$$

Finally, we can make sense of (5.129) even as $a \to 0$ if we assume $c = -b \to 1$, $(1 + bc)/a = d \to 0$:

$$t^{-u}(1-z)^{q+u}e^t\,{}_1F_1\left(q+u; 2u; \frac{zt}{1-z}\right), \qquad t = \tau^{-1}.$$

This function has an expansion in terms of t^{-u+n}, $n \geq 0$, which leads to

$$(1-z)^{-\mu}e^t\,{}_1F_1\left(-\mu; -\nu; \frac{zt}{1-z}\right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} F(\nu+n, \mu; \nu; z),$$
$$z \neq 1. \qquad (5.131)$$

The above expression can also be easily obtained from (5.130) and the transformation formula

$$F(\rho, \mu; \nu; z) = (1 - z)^{-\mu} F\left(\nu - \rho, \mu; \nu; \frac{z}{z - 1}\right), \qquad \text{(A. 8)}.$$

Viewed another way, (5.130) and (5.131) constitute a proof of this transformation formula. Group theoretic methods can be used to derive many more such identities which are equivalent, by means of a transformation formula (A.8), to identities already given, but this will not be carried out here.

5-16 The Group $SU(2)$

At this point we begin a study of the representation theory of real 3-parameter Lie groups whose Lie algebras are real forms of $sl(2)$. Our treatment will be brief since the representation theory of these groups is well known and since many of the results for special functions obtained from such a study are special cases of results derived earlier in this chapter.

$SU(2)$ is the group of all 2×2 unitary unimodular matrices, i.e., the group of all matrices of the form

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \qquad |a|^2 + |b|^2 = 1. \tag{5.132}$$

where $a, b \in \mathcal{C}$ and $\bar{a}$ is the complex conjugate of a (Hamermesh [1], Chapter 9). The identity element of $SU(2)$ is the 2×2 identity matrix and the inverse of a group element is its conjugate transpose:

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}^{-1} = \begin{pmatrix} \bar{a} & -b \\ \bar{b} & a \end{pmatrix} \qquad \text{if} \quad |a|^2 + |b|^2 = 1.$$

Clearly $SU(2)$ is a real 3-parameter Lie group; furthermore, it is a subgroup of $SL(2)$. The Lie algebra $su(2)$ of $SU(2)$ can be identified with the space of all 2×2 complex skew-Hermitian matrices of trace zero. In particular, as a basis for $su(2)$ we can choose the elements

$$\mathcal{J}_1 = \begin{pmatrix} 0 & i/2 \\ i/2 & 0 \end{pmatrix}, \quad \mathcal{J}_2 = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}, \quad \mathcal{J}_3 = \begin{pmatrix} -i/2 & 0 \\ 0 & i/2 \end{pmatrix} \tag{5.133}$$

with commutation relations

$$[\mathcal{J}_1, \mathcal{J}_2] = \mathcal{J}_3, \quad [\mathcal{J}_3, \mathcal{J}_1] = \mathcal{J}_2, \quad [\mathcal{J}_2, \mathcal{J}_3] = \mathcal{J}_1, \tag{5.134}$$

where $[\alpha, \beta] = \alpha\beta - \beta\alpha$ for $\alpha, \beta \in su(2)$. Comparing these commutation relations with expressions (2.6), Chapter 2, we see that $su(2)$ is isomorphic to the Lie algebra $L(O_3)$ of the 3×3 rotation group. The isomorphism of $su(2)$ and $L(O_3)$ can be extended to a local isomorphism of the corresponding groups. In fact $SU(2)$ is the simply connected covering group of O_3 and there is a group homomorphism R of $SU(2)$ onto O_3 such that

$$A\left(\sum_{j=1}^{3} x_j \sigma^j\right) A^{-1} = \sum_{j=1}^{3} (R(A)\mathbf{x})_j \sigma^j \tag{5.135}$$

where $\mathbf{x} = (x_1, x_2, x_3)$ is a column vector, $A \in SU(2)$,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

are the Pauli matrices, and $R(A) \in O_3$ (see Gel'fand *et al.* [1]). This equation defines the 3×3 matrix $R(A)$ uniquely. Here $R(A) = R(-A)$ so the homomorphism is not an isomorphism.

The 2×2 matrices $\mathcal{J}^+$, $\mathcal{J}^-$, $\mathcal{J}^3$ defined in terms of the matrices (5.133) by

$$\mathcal{J}^{\pm} = \mp \mathcal{J}_2 + i\mathcal{J}_1, \qquad \mathcal{J}^3 = i\mathcal{J}_3$$

satisfy the commutation relations

$$[\mathcal{J}^3, \mathcal{J}^{\pm}] = \pm\mathcal{J}^{\pm}, \qquad [\mathcal{J}^+, \mathcal{J}^-] = 2\mathcal{J}^3$$

and generate a complex Lie algebra isomorphic to $sl(2)$. Thus, the complexification of $su(2)$ is isomorphic to $sl(2)$ and $su(2)$ is a real form of $sl(2)$.

It follows from this relationship between the two Lie algebras that the abstract irreducible representations $D(u, m_o)$, $\uparrow_u$, $\downarrow_u$, $D(2u)$, of $sl(2)$ induce irreducible representations of $su(2)$. We shall determine which of these induced representations of $su(2)$ can be extended to a unitary irreducible representation of $SU(2)$. To proceed with this determination we follow the technique initiated in Section 3-6. Consider a unitary irreducible representation U of $SU(2)$ on a Hilbert space $\mathcal{H}$ and define the infinitesimal operators J_1, J_2, J_3 by

$$\mathrm{J}_k f = \frac{d}{dt} \mathbf{U}(\exp t\mathcal{J}_k) f \Big|_{t=0}, \qquad k = 1, 2, 3, \tag{5.136}$$

for all $f \in \mathscr{D}$. ($\mathscr{D}$ is a dense subspace of $\mathscr{H}$ which satisfies the properties (3.45), (3.46).) By definition, on the domain $\mathscr{D}$ these operators satisfy the commutation relations

$$[J_1, J_2] = J_3, \qquad [J_3, J_1] = J_2, \qquad [J_2, J_3] = J_1.$$

Hence, the operators $J^\pm$, J^3 defined by $J^\pm = \mp J_2 + iJ_1$, $J^3 = iJ_3$ satisfy the relations

$$[J^3, J^\pm] = \pm J^\pm, \qquad [J^+, J^-] = 2J^3$$

and determine a representation ρ of the complex Lie algebra $sl(2)$ on $\mathscr{D}$.

To begin with, we shall investigate which of the finite-dimensional representations $D(2u)$, $2u$ a nonnegative integer, can be obtained in this manner from a unitary representation U of $SU(2)$. That is, we shall determine the conditions under which ρ can be identified with $D(2u)$. In this case $\mathscr{H}$ is finite-dimensional so $\mathscr{D} = \mathscr{H}$. From Lemma 3.1 we require

$$\langle J_k f, h\rangle = -\langle f, J_k h\rangle, \qquad k = 1, 2, 3,$$

for all $f, h \in \mathscr{H}$, where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathscr{H}$. In terms of the operators $J^\pm$, J^3 these conditions can be expressed as

$$\langle J^3 f, h\rangle = \langle f, J^3 h\rangle, \qquad \langle J^+ f, h\rangle = \langle f, J^- h\rangle. \tag{5.137}$$

The $(2u + 1)$-dimensional representation $D(2u)$ is determined by the conditions

$$\begin{aligned} J^3 p_m = m p_m, \qquad J^+ p_m &= [(u - m)(u + m + 1)]^{1/2} p_{m+1}, \\ J^- p_m &= [(u + m)(u - m + 1)]^{1/2} p_{m-1} \end{aligned} \tag{5.138}$$

where m ranges over the spectrum $S = \{-u, -u + 1, \ldots, +u\}$ on the left-hand sides of these equations. (For convenience the basis vectors p_m have been normalized in accordance with (5.41) rather than (2.27).) As usual we apply conditions (5.137) to the operators $J^\pm$, J^3. Thus,

$$m\langle p_m, p_n\rangle = \langle J^3 p_m, p_n\rangle = \langle p_m, J^3 p_n\rangle = n\langle p_m, p_n\rangle,$$

or $(m - n)\langle p_m, p_n\rangle = 0$ for all $m, n \in S$. This implies $\langle p_m, p_n\rangle = 0$ for $m \neq n$. The relation

$$\begin{aligned} [(u - m + 1)(u + m)]^{1/2}\langle p_m, p_m\rangle &= \langle J^+ p_{m-1}, p_m\rangle = \langle p_{m-1}, J^- p_m\rangle \\ &= [(u - m + 1)(u + m)]^{1/2}\langle p_{m-1}, p_{m-1}\rangle \end{aligned}$$

implies $|p_m| = |p_n|$ for all $m, n \in S$. Without loss of generality we can assume that the basis vectors are of length 1. Thus, conditions (5.137) merely require that the vectors p_m form an orthonormal basis for $\mathscr{H}$ and in no way restrict the value of the nonnegative integer $2u$.

Conversely, we will show that every finite-dimensional representation (5.138) induces an irreducible unitary representation U of $SU(2)$. First, every element A of $SU(2)$ can be written in the form

$$\begin{aligned} A &= (\exp \varphi_1 \mathscr{J}_3)(\exp \theta \mathscr{J}_1)(\exp \varphi_2 \mathscr{J}_3) \\ &= \begin{pmatrix} e^{-i\varphi_1/2} & 0 \\ 0 & e^{+i\varphi_1/2} \end{pmatrix} \begin{pmatrix} \cos(\theta/2) & i\sin(\theta/2) \\ i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} e^{-i\varphi_2/2} & 0 \\ 0 & e^{+i\varphi_2/2} \end{pmatrix} \\ &= \begin{pmatrix} e^{-i(\varphi_1+\varphi_2)/2}\cos(\theta/2) & ie^{-i(\varphi_1-\varphi_2)/2}\sin(\theta/2) \\ ie^{+i(\varphi_1-\varphi_2)/2}\sin(\theta/2) & e^{+i(\varphi_1+\varphi_2)/2}\cos(\theta/2) \end{pmatrix}. \end{aligned} \tag{5.139}$$

In fact, if $A \in SU(2)$ is given by (5.132) the parameters φ_1, θ, φ_2 can be defined uniquely by

$$\cos(\theta/2) = |a|, \qquad \sin(\theta/2) = |b|, \qquad 0 \leqslant \theta \leqslant \pi,$$

and

$$-\frac{\varphi_1 + \varphi_2}{2} = \arg a, \qquad \frac{\varphi_2 - \varphi_1}{2} + \frac{\pi}{2} = \arg b,$$

whenever $ab \neq 0$. If $ab = 0$, an infinite number of values of φ_1, φ_2 will satisfy (5.139). Expression (5.139) shows that the elements of $SU(2)$ are completely determined by the elements $\mathscr{J}_1$, $\mathscr{J}_2$, $\mathscr{J}_3$ of $su(2)$; hence, that the operators $\mathbf{U}(A)$, $A \in SU(2)$ are determined by the infinitesimal operators J_1, J_2, J_3, Eq. (5.136). Thus, the irreducible representation (5.138) of $su(2)$ uniquely determines the unitary representation U of $SU(2)$ from which it is derived.

The actual construction of the unitary irreducible representations of $SU(2)$ is easy; we merely restrict the representations $D(2u)$ of $SL(2)$ to $SU(2)$. For every nonnegative integer $2u$, denote by $\mathscr{K}_u$ the Hilbert space consisting of all complex linear combinations of the basis vectors $p_m(z) = (-z)^{u+m}/[(u+m)!\,(u-m)!]^{1/2}$; $m \in S = \{-u, -u+1, \ldots, +u\}$, with inner product determined by $\langle p_m, p_n \rangle = \delta_{m,n}$, $m, n \in S$. The elements of $\mathscr{K}_u$ are polynomials of order $\leqslant 2u$ in the complex variable z. We construct a representation D_u of $SU(2)$ on $\mathscr{K}_u$ in terms of linear operators $\mathbf{U}^u(A)$:

$$[\mathbf{U}^u(A)f](z) = (bz + \bar{a})^{2u} f\left(\frac{az - \bar{b}}{bz + \bar{a}}\right) \tag{5.140}$$

for all $f \in \mathscr{H}_u$. For $A \in SU(2)$, (5.132), the operator $\mathbf{D}(g)$ given by (5.34) is formally identical with $\mathbf{U}^u(A)$. Thus we immediately obtain the relation

$$\mathbf{U}^u(A_1)\mathbf{U}^u(A_2) = \mathbf{U}^u(A_1A_2), \qquad A_1, A_2 \in SU(2),$$

which proves D_u is a representation of $SU(2)$. The infinitesimal operators of this representation corresponding to the elements $\mathscr{J}_1, \mathscr{J}_2, \mathscr{J}_3$ of $su(2)$ are given by

$$\mathrm{J}_1 = iuz + \frac{i}{2}\frac{d}{dz} - \frac{i}{2}z^2\frac{d}{dz},$$
$$\mathrm{J}_2 = uz - \frac{1}{2}\frac{d}{dz} - \frac{1}{2}z^2\frac{d}{dz},$$
$$\mathrm{J}_3 = iu - iz\frac{d}{dz}.$$

Applying these operators to the basis vectors $p_m(z)$, $m \in S$, we find

$$\langle \mathrm{J}_k f, h\rangle = -\langle f, \mathrm{J}_k h\rangle, \qquad k = 1, 2, 3,$$

for all $f, h \in \mathscr{H}_u$. Thus, $\langle(\exp \theta\mathrm{J}_k) f, h\rangle = \langle f, (\exp -\theta\mathrm{J}_k)\, h\rangle$ where $\exp \theta\mathrm{J}_k = \sum_{l=0}^{\infty} \theta^l(\mathrm{J}_k)^l/l!$. (Since $\mathscr{H}_u$ is finite-dimensional the operators J_k are bounded and there are no convergence difficulties.) This proves that the operators $\exp \theta\mathrm{J}_k$, $k = 1, 2, 3$, on $\mathscr{H}_u$ are unitary. Moreover, we have

$$\mathbf{U}^u(A) = \exp(\varphi_1\mathrm{J}_3)\exp(\theta\mathrm{J}_1)\exp(\varphi_2\mathrm{J}_3)$$

where $A \in SU(2)$ is given by (5.139). Hence, the operator $\mathbf{U}^u(A)$ is unitary.

Theorem 5.3 D_u is a unitary representation of $SU(2)$ for all nonnegative integers $2u$.

To compute the matrix elements

$$U^u_{nm}(A) = \langle p_n, \mathbf{U}^u(A)p_m\rangle, \qquad n, m \in S, \quad A \in SU(2),$$

apply $\mathbf{U}^u(A)$ to the basis vector p_m:

$$\mathbf{U}^u(A)p_m = \sum_{n=-u}^{u} U^u_{nm}(A)p_n.$$

Thus,

$$(-1)^{u+m}\frac{(bz+\bar{a})^{u-m}(az-\bar{b})^{u+m}}{[(u+m)!\,(u-m)!]^{1/2}} = \sum_{n=-u}^{u} U^u_{nm}(A)\frac{(-z)^{u+n}}{[(u+n)!\,(u-n)!]^{1/2}}. \quad (5.141)$$

From this formula it is easy to derive the symmetrical expression

$$\frac{1}{(2u)!}[(bz+\bar{a})+w(az-\bar{b})]^{2u}=\sum_{m,n=-u}^{u} p_m(w)U_{nm}^u(A)p_n(z).$$

If this last expression is multiplied by s^{2u} and summed over all nonnegative integers $2u$ one obtains the generating function

$$\begin{aligned} Q(A;s,w,z) &= \exp s[(bz+\bar{a})+w(az-\bar{b})] \\ &= \sum_{2u=0}^{\infty} s^{2u}\sum_{m,n=-u}^{u} p_m(w)U_{nm}^u(A)p_n(z). \end{aligned} \tag{5.142}$$

Explicitly, the matrix elements are given by

$$\begin{aligned} U_{nm}^u(A) &= \left[\frac{(u+m)!\,(u-n)!}{(u+n)!\,(u-m)!}\right]^{1/2} a^{u+n}\bar{a}^{u-m}\bar{b}^{m-n} \\ &\quad\cdot\frac{1}{\Gamma(m-n+1)}F(-u-n;m-u;m-n+1;-|b/a|^2) \\ &= (i)^{n-m}\left[\frac{(u+m)!\,(u-n)!}{(u+n)!\,(u-m)!}\right]^{1/2} e^{-i(n\varphi_1+m\varphi_2)}P_u^{-n,m}(\cos\theta), \end{aligned} \tag{5.143}$$

where $P_u^{r,m}(x)=e^{-i\pi[(m+r)/2]}\mathfrak{P}_u^{r,m}(x+i0)$, and the coordinates φ_1, θ, φ_2 for A are defined by (5.139). The addition theorem for the matrix elements is

$$U_{nm}^u(A_1A_2)=\sum_{j=-u}^{u} U_{nj}^u(A_1)U_{jm}^u(A_2), \qquad n,m=-u,\ldots,u.$$

We shall not work out the implications of this identity for special functions since all of the results are merely special cases of Eqs. (5.39). The fact that D_u is unitary does lead to additional information, however. Indeed,

$$U_{nm}^u(A^{-1})=\overline{U_{mn}^u(A)},$$

which implies

$$(-1)^{m-n}P_u^{-n,m}(\cos\theta)=\frac{(u+n)!\,(u-m)!}{(u-n)!\,(u+m)!}P_u^{-m,n}(\cos\theta).$$

Furthermore, $|\,U_{nm}^u(A)|\leqslant 1$, or

$$|\,P_u^{-n,m}(\cos\theta)\,|\leqslant\left[\frac{(u+n)!\,(u-m)!}{(u+m)!\,(u-n)!}\right]^{1/2}.$$

The Haar measure on $SU(2)$, suitably normalized, is given by $dA = (1/16\pi^2) \sin\theta \, d\varphi_1 \, d\theta \, d\varphi_2$ where φ_1, θ, φ_2, range over the values $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \varphi_1 < 4\pi$, $0 \leqslant \varphi_2 < 2\pi$ (Gel'fand *et al.* [1]). The matrix elements $U^u_{nm}(A)$ satisfy orthogonality relations with respect to this measure. To compute these relations we evaluate the integral

$$\frac{1}{16\pi^2}\int_0^{4\pi}\int_0^{\pi}\int_0^{2\pi} \overline{Q(A; \bar{s}', \bar{w}', \bar{z}',)} Q(A; s, w, z) \sin\theta \, d\varphi_1 \, d\theta \, d\varphi_2$$

$$= \sum_{2u,2v=0}^{\infty} \sum_{q,r=-v}^{v} \sum_{m,n=-u}^{u} s'^{2v} s^{2u} p_q(w') p_m(w) p_r(z') p_n(z) \int \overline{U^v_{qr}(A)} U^u_{nm}(A) \, dA$$

where the function $Q(\cdot)$ is defined by (5.142). The integral on the left-hand side of this equation can be evaluated explicitly by power series expansion with the result

$$\int \overline{Q(A; \bar{s}', \bar{w}', \bar{z}',)} Q(A; s, w, z) \, dA = {}_0F_1(2; ss'(zz' + 1)(ww + 1)).$$

Thus, we obtain the relations

$$\int \overline{U^v_{qr}(A)} U^u_{nm}(A) \, dA = \frac{\delta_{q,n} \, \delta_{r,m} \, \delta_{v,u}}{2u + 1}. \tag{5.144}$$

In terms of the functions $P^{n,m}_u(\cos\theta)$ the orthogonality relations are

$$\int_0^{\pi} P^{n,m}_u(\cos\theta) P^{n,m}_v(\cos\theta) \sin\theta \, d\theta = \frac{2}{2u+1} \delta_{u,v} \frac{(u-n)!\,(u-m)!}{(u+n)!\,(u+m)!}.$$

NOTE Since $SU(2)$ is a compact group, these orthogonality relations could also have been derived from the Peter–Weyl theorem (Pontrjagin [1], Chapter 4). Moreover, the Peter–Weyl theorem can be used to show that the matrix elements $\{U^u_{mn}(A)\}$; $n, m = -u, -u+1, \ldots, u$; $2u = 0, 1, 2, \ldots$; form an orthogonal basis for the Hilbert space of all complex functions on the group manifold $SU(2)$, square integrable with respect to the Haar measure dA.

The irreducibility of the representation D_u can be demonstrated exactly as in the proof of Lemma 3.2.

Lemma 5.8 D_u is irreducible.

The procedure for reducing the tensor product representation $D_u \otimes D_v$ of $SU(2)$ into a direct sum of irreducible representations is almost word-for-word the same as the procedure carried out for representations of $SL(2)$ in Section 5-5. This follows from the fact that the restriction of the representation $D(2u)$ of $SL(2)$ to $SU(2)$ is the irreducible representation D_u. In particular we obtain

$$D_u \otimes D_v \cong \sum_{k=0}^{2\min(u,v)} \oplus D_{u+v-k}, \qquad k \text{ an integer.}$$

The Clebsch–Gordan coefficients C(·) for this decomposition are given by (5.57) and (5.58). Thus, we find

$$U^v_{rs}(A)U^u_{nm}(A) = \sum_{l=|u-v|}^{u+v} \mathrm{C}(v, r; u, n \mid l, r+n) \cdot \mathrm{C}(v, s; u, m \mid l, s+m)U^l_{r+n,s+m}(A),$$

which is a special case of (5.59).

So far we have examined only those unitary irreducible representations of $SU(2)$ which are induced by the representations $D\,(2u)$ of $sl(2)$. We still have to consider the possibility of unitary representations of $SU(2)$ induced by the infinite-dimensional representations $\uparrow_u$, $\downarrow_u$, $D(u, m_o)$ of $sl(2)$. However, as the reader can verify for himself, none of these infinite-dimensional representations induces a unitary representation of $SU(2)$. In fact, since $SU(2)$ is a compact group, it follows from the Peter–Weyl theorem (Naimark [1], chapter 6) that all of its unitary irreducible representations are finite-dimensional. Furthermore, it can easily be shown that the representations D_u, $2u$ a nonnegative integer, which we have constructed here are the only irreducible unitary representations of $SU(2)$. For proofs of these remarks and a more detailed study of the representation theory of $SU(2)$ the reader should consult the standard texts (Gelfand *et al.* [1], Hamermesh [1]).

5-17 The Group G_3

G_3 is the real 3-parameter matrix group with elements

$$g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \qquad a, b \in \mathcal{C}, \quad |a|^2 - |b|^2 = 1.$$

(The condition $|a|^2 - |b|^2 = 1$ means g has determinant $+1$.) These matrices do indeed form a group as is clear from the relation

$$g_1g_2 = \begin{pmatrix} a_1 & b_1 \\ \bar{b}_1 & \bar{a}_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ \bar{b}_2 & \bar{a}_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + b_1\bar{b}_2 & a_1b_2 + b_1\bar{a}_2 \\ \bar{b}_1a_2 + \bar{a}_1\bar{b}_2 & \bar{b}_1b_2 + \bar{a}_1\bar{a}_2 \end{pmatrix}.$$

Since the elements of G_3 are unimodular matrices it follows that G_3 is a subgroup of $SL(2)$. As real coordinates for G_3 in a neighborhood of the identity element $\mathbf{e}$ we can choose (x_2, y_1, y_2) where $a = 1 + x_1 + ix_2$, $b = y_1 + iy_2$, $x_1 = -1 + (1 + y_1^2 + y_2^2 - x_2^2)^{1/2}$. Using the standard procedure for the computation of the Lie algebra of a local Lie group, we find $L(G_3)$ can be identified with the Lie algebra of matrices

$$\begin{pmatrix} ix & y + iz \\ y - iz & -ix \end{pmatrix}, \qquad x, y, z \text{ real.}$$

Clearly, the elements

$$\mathscr{J}_1 = \begin{pmatrix} 0 & i/2 \\ -i/2 & 0 \end{pmatrix}, \qquad \mathscr{J}_2 = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix},$$
$$\mathscr{J}_3 = \begin{pmatrix} -i/2 & 0 \\ 0 & +i/2 \end{pmatrix} \tag{5.145}$$

with commutation relations

$$[\mathscr{J}_1, \mathscr{J}_2] = -\mathscr{J}_3, \qquad [\mathscr{J}_3, \mathscr{J}_1] = \mathscr{J}_2, \qquad [\mathscr{J}_3, \mathscr{J}_2] = -\mathscr{J}_1, \tag{5.146}$$

form a basis for $L(G_3)$. The exponential map from $L(G_3)$ to G_3 has the properties

$$\exp \theta\mathscr{J}_1 = \begin{pmatrix} \cosh \theta/2 & i \sinh \theta/2 \\ -i \sinh \theta/2 & \cosh \theta/2 \end{pmatrix},$$
$$\exp \theta\mathscr{J}_2 = \begin{pmatrix} \cosh \theta/2 & \sinh \theta/2 \\ \sinh \theta/2 & \cosh \theta/2 \end{pmatrix}, \tag{5.147}$$
$$\exp \theta\mathscr{J}_3 = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}, \qquad \theta \text{ real.}$$

To give an explicit relation between $L(G_3)$ and $sl(2)$ we define 2×2 matrices $\mathscr{J}^+$, $\mathscr{J}^-$, $\mathscr{J}^3$ by $\mathscr{J}^\pm = -\mathscr{J}_2 \pm i\mathscr{J}_1$, $\mathscr{J}^3 = i\mathscr{J}_3$. As is easily seen, these matrices satisfy the commutation relations

$$[\mathscr{J}^+, \mathscr{J}^-] = 2\mathscr{J}^3, \qquad [\mathscr{J}^3, \mathscr{J}^\pm] = \pm\mathscr{J}^\pm;$$

hence, they generate a complex Lie algebra isomorphic to $sl(2)$. This shows that $L(G_3)$ is a real form of the complex Lie algebra $sl(2)$.

The irreducible unitary representations of G_3 were first constructed by Bargmann [1], and this group has been the subject of numerous investigations ever since (Gel'fand *et al.* [2], Pukánszky [1], Vilenkin [3]). Here, we shall give a brief treatment of the representation theory of G_3

to show the relationship between this theory and the identities for hypergeometric functions derived earlier in Chapter 5. The technique to be used should be familiar to the reader by now: Since $L(G_3)$ is a real form of $sl(2)$, the abstract irreducible representations $D(u, m_o)$, $\uparrow_u$, $\downarrow_u$, $D(2u)$ of $sl(2)$ induce irreducible representations of $L(G_3)$. We shall determine which of these irreducible representations can be extended to an irreducible unitary representation of G_3 on a Hilbert space. It will turn out that **all** of the irreducible unitary representations of G_3 can be obtained in this way.

Following the procedure introduced in Section 3-6, we consider a unitary irreducible representation U of G_3 on a Hilbert space $\mathscr{H}$ and define the infinitesimal operators J_k by

$$\mathrm{J}_k f = \frac{d}{dt} \mathbf{U}(\exp t\mathscr{J}_k)f\Big|_{t=0}, \qquad k = 1, 2, 3, \tag{5.148}$$

for all $f \in \mathscr{D}$. The $\mathscr{D}$ is a dense subspace of $\mathscr{H}$ satisfying properties (3.45), (3.46). On $\mathscr{D}$ we have

$$[\mathrm{J}_1, \mathrm{J}_2] = -\mathrm{J}_3, \qquad [\mathrm{J}_3, \mathrm{J}_1] = \mathrm{J}_2, \qquad [\mathrm{J}_3, \mathrm{J}_2] = -\mathrm{J}_1,$$

so the operators $\mathrm{J}^{\pm} = -\mathrm{J}_2 \pm i\mathrm{J}_1$, $\mathrm{J}^3 = i\mathrm{J}_3$ satisfy the relations

$$[\mathrm{J}^+, \mathrm{J}^-] = 2\mathrm{J}^3, \qquad [\mathrm{J}^3, \mathrm{J}^{\pm}] = \pm \mathrm{J}^{\pm}$$

and, thus, determine a representation ρ of $sl(2)$ on $\mathscr{D}$.

We shall first investigate under what conditions ρ could be isomorphic to the representation $D(u, m_o)$ on some dense subspace $\mathscr{D}'$ of $\mathscr{D}$. Recall that u and m_o are complex numbers such that $m_o \pm u$ are not integers, and $0 \leqslant \operatorname{Re} m_o < 1$. There is a basis $\{f_m\}$, $m \in S = \{m_o + n\colon n$ an integer$\}$, for the representation space of $D(u, m_o)$ with the properties

$$\mathrm{J}^3 f_m = m f_m, \qquad \mathrm{J}^+ f_m = (m - u) f_{m+1}, \qquad \mathrm{J}^- f_m = -(m + u) f_{m-1},$$
$$\mathrm{C}_{1,0} f_m = (\mathrm{J}^+\mathrm{J}^- + \mathrm{J}^3\mathrm{J}^3 - \mathrm{J}^3) f_m = u(u + 1) f_m .$$

Thus, if ρ is isomorphic to $D(u, m_o)$ on $\mathscr{D}'$ we can assume $\{f_m\}$ is a basis for the space $\mathscr{D}'$. (Since $D(u, m_o) \cong D(-u - 1, m_o)$, without loss of generality we assume either $\operatorname{Im} u > 0$ or $\operatorname{Im} u = 0$, $\operatorname{Re} u \geqslant -\frac{1}{2}$.)

From (5.147), $\exp 4\pi\mathscr{J}_3 = \mathbf{e}$. Thus we must have $\mathbf{U}(\exp 4\pi\mathscr{J}_3) = \exp 4\pi\mathrm{J}_3 = \mathbf{I}$, where $\mathbf{I}$ is the identity operator on $\mathscr{H}$. However, from the definition of the representation $D(u, m_o)$ it follows that $(\exp 4\pi\mathrm{J}_3) f_m = e^{-4\pi i m} f_m$ for all $m \in S$. This is possible only if $2m$ is an integer. Therefore, if $D(u, m_o)$ is induced by the representation U of G_3 then either $m_o = 0$ or $m_o = \frac{1}{2}$.

According to Lemma 3.1 the unitarity of U implies the conditions

$$\langle \mathrm{J}_k f, h\rangle = -\langle f, \mathrm{J}_k h\rangle, \qquad k = 1, 2, 3,$$

for all $f, h \in \mathscr{D}$ where $\langle \cdot , \cdot \rangle$ is the inner product on $\mathscr{H}$. In terms of the operators $\mathrm{J}^{\pm}$, J^3, the conditions become

$$\langle \mathrm{J}^3 f, h\rangle = \langle f, \mathrm{J}^3 h\rangle, \qquad \langle \mathrm{J}^+ f, h\rangle = -\langle f, \mathrm{J}^- h\rangle.$$

From the first of these relations we find

$$m\langle f_m , f_n\rangle = \langle \mathrm{J}^3 f_m , f_n\rangle = \langle f_m , \mathrm{J}^3 f_n\rangle = n\langle f_m , f_n\rangle$$

for all $m, n \in S$. Thus $\langle f_m , f_n\rangle = 0$ unless $m = n$. The second relation implies

$$\begin{aligned}(m - 1 - \bar{u})\langle f_m , f_m\rangle = \langle \mathrm{J}^+ f_{m-1} , f_m\rangle &= -\langle f_{m-1} , \mathrm{J}^- f_m\rangle \\ &= (m + u)\langle f_{m-1} , f_{m-1}\rangle,\end{aligned}$$

or

$$\frac{m - 1 - \bar{u}}{m + u} = \frac{|\, f_{m-1}\,|^2}{|\, f_m\,|^2} > 0 \qquad \text{for all} \quad m \in S. \tag{5.149}$$

If $m_o = \frac{1}{2}$, it follows that $u = -\frac{1}{2} + iq$, where $q > 0$. Substituting this value for u in the above relation we find $|\, f_{m-1}\,| = |\, f_m\,|$ for all $m = \frac{1}{2} + n$, n an integer. Thus, without loss of generality we can assume $|\, f_m\,| = 1$.

If $m_o = 0$ the situation is more complicated. In this case (5.149) implies either (i) $u = -\frac{1}{2} + iq$, $q > 0$; or (ii) $u = q$, $-\frac{1}{2} < q < 0$. Corresponding to solution (i) we have $|\, f_{m-1}\,| = |\, f_m\,|$ for all integers m, so that we can assume $|\, f_m\,| = 1$. However, for solution (ii) $(m - q - 1)|\, f_m\,|^2 = (m + q)|\, f_{m-1}\,|^2$, all integers m. To get an orthonormal basis for $\mathscr{H}$, define new vectors j_m by

$$j_m = \left[\frac{\Gamma(q + 1)\Gamma(m - q)}{\Gamma(-q)\Gamma(m + q + 1)}\right]^{1/2} f_m , \qquad m = 0, \pm 1, \pm 2, \ldots, \tag{5.150}$$

where we take the positive square root of the positive quantity inside the brackets. Then $|\, j_m\,| = |\, j_{m-1}\,|$ and without loss of generality it can be assumed that $|\, j_m\,| = 1$ for all integers m.

At this point we have obtained the following possibilities for a representation of $sl(2)$ on $\mathscr{D}$ induced by a unitary irreducible representation U of G_3 on $\mathscr{H}$:

(I) $A^{0,q}$ $(q > 0)$. There is an orthonormal basis $\{f_m\}$, $m \in S = \{0, \pm 1, \pm 2, \ldots\}$, for $\mathscr{H}$ such that

$$\begin{aligned} J^3 f_m &= m f_m, \qquad J^+ f_m = (m + \tfrac{1}{2} - iq) f_{m+1}, \\ J^- f_m &= -(m - \tfrac{1}{2} + iq) f_{m-1}, \qquad C_{1,0} f_m = -(q^2 + \tfrac{1}{4}) f_m. \end{aligned} \tag{5.151}$$

(II) $A^{1/2,q}$ $(q > 0)$. There is an orthonormal basis $\{f_m\}$, $m \in S = \{\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots\}$, for $\mathscr{H}$ such that

$$\begin{aligned} J^3 f_m &= m f_m, \qquad J^+ f_m = (m + \tfrac{1}{2} - iq) f_{m+1}, \\ J^- f_m &= -(m - \tfrac{1}{2} + iq) f_{m-1}, \qquad C_{1,0} f_m = -(q^2 + \tfrac{1}{4}) f_m. \end{aligned} \tag{5.152}$$

(III) $A^{0,q}$ $(-\frac{1}{2} < q < 0)$. There exists an orthonormal basis $\{j_m\}$, $m \in S = \{0, \pm 1, \pm 2, \ldots\}$, for $\mathscr{H}$ such that

$$\begin{aligned} J^3 j_m &= m j_m, \qquad J^+ j_m = \epsilon_m[(m - q)(m + q + 1)]^{1/2} j_{m+1}, \\ J^- j_m &= -\epsilon_{m-1}[(m + q)(m - q - 1)]^{1/2} j_{m-1}, \qquad C_{1,0} j_m = q(q + 1) j_m, \end{aligned} \tag{5.153}$$

where $\epsilon_m = 1$ for $m \geqslant 0$, $\epsilon_m = -1$ for $m < 0$.

Conversely, in the following section it will be shown that each of the Lie algebra representations listed above does arise from an irreducible unitary representation of G_3 on $\mathscr{H}$.

Next we investigate under what conditions a representation $\uparrow_u$ of $sl(2)$ could be induced on $\mathscr{D}'$ by a unitary representation of G_3 on $\mathscr{H}$. Here, $2u$ is not a nonnegative integer, $S = \{-u, -u + 1, -u + 2, \ldots\}$, and there is a basis $\{f_m\}$, $m \in S$, for the representation space $\mathscr{D}'$ such that

$$J^3 f_m = m f_m, \qquad J^+ f_m = (m - u) f_{m+1}, \qquad J^- f_m = -(m + u) f_{m-1}.$$

Exactly as in the treatment of the representation $D(u, m_o)$ given above, we find in order for the representation $\uparrow_u$ of $sl(2)$ to be induced on $\mathscr{D}'$ by U it is necessary that (i) $2u$ is a negative integer; (ii) $\langle f_m, f_n \rangle = 0$ for $m \neq n$; and (iii) $(m - u)/(m + u + 1) = |f_m|^2/|f_{m+1}|^2 > 0$ for all $m \in S$. This leads to the following possibilities:

(IV) D_n^+ $(n = \frac{1}{2}, 1, \frac{3}{2}, \ldots)$. There is an orthonormal basis $\{j_m\}$, $m \in S = \{n, n + 1, n + 2, \ldots\}$, for $\mathscr{H}$ such that

$$\begin{aligned} J^3 j_m &= m j_m, \qquad J^+ j_m = [(m + n)(m - n + 1)]^{1/2} j_{m+1}, \\ J^- j_m &= -[(m - n)(m + n - 1)]^{1/2} j_{m-1}, \qquad C_{1,0} j_m = n(n - 1) j_m \end{aligned} \tag{5.154}$$

for all $m \in S$ on the left-hand sides of these equations. Here

$$j_m = \left[\frac{(m+n-1)!}{(2n-1)!\,(m-n)!}\right]^{1/2} f_m, \qquad u = -n.$$

Similarly, the possible representations of the form $\downarrow_u$ on $\mathscr{D}'$ induced by unitary representations of G_3 can be denoted by

(V) D_n^- $(n = \frac{1}{2}, 1, \frac{3}{2}, \ldots)$. There is an orthonormal basis $\{j_m\}$, $m \in S = \{-n, -n-1, -n-2, \ldots\}$, for $\mathscr{H}$ such that

$$\begin{aligned} \mathrm{J}^3 j_m = m j_m, \qquad \mathrm{J}^+ j_m &= -[(m+n)(m-n+1)]^{1/2} j_{m+1}, \\ \mathrm{J}^- j_m = [(m-n)(m+n-1)]^{1/2} j_{m-1}, &\qquad \mathrm{C}_{1,0} j_m = n(n-1)\, j_m. \end{aligned} \tag{5.155}$$

Finally, as the reader can verify for himself, the only possible finite-dimensional representation $D(2u)$ of $sl(2)$ on $\mathscr{H}$ induced by a unitary representation of G_3 is the trivial 1-dimensional representation $D(0)$.

5-18 Unitary Representations of G_3

Now that we have classified all of the possible irreducible Lie algebra representations of $sl(2)$ induced by unitary irreducible representations of G_3, we will show, conversely, that each of these Lie algebra representations uniquely defines a unitary irreducible representation of G_3. This will prove to be an easy task.

To begin with, it is convenient to adopt a new coordinate system for the elements of G_3. From the identity

$$g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} = \begin{pmatrix} e^{-i\mu/2} & 0 \\ 0 & e^{i\mu/2} \end{pmatrix} \begin{pmatrix} \cosh \rho/2 & \sinh \rho/2 \\ \sinh \rho/2 & \cosh \rho/2 \end{pmatrix} \begin{pmatrix} e^{-i\nu/2} & 0 \\ 0 & e^{i\nu/2} \end{pmatrix} \tag{5.156}$$

valid for

$$a = e^{-i(\mu+\nu)/2} \cosh \rho/2, \qquad b = e^{i(\nu-\mu)/2} \sinh \rho/2, \qquad \mu, \nu, \rho \text{ real}, \tag{5.157}$$

we can conclude that every $g \in G_3$ can be expressed in the form

$$g = \exp(\mu \mathscr{J}_3) \exp(\rho \mathscr{J}_2) \exp(\nu \mathscr{J}_3). \tag{5.158}$$

In fact, for every $a, b \in \mathscr{C}$ such that $|a|^2 - |b|^2 = 1$, it is possible to find real numbers μ, ρ, ν satisfying (5.157). The coordinates (μ, ρ, ν) for G_3 are analogous to the complex coordinates α, w, β for $SL(2)$, Eqs. (5.11). Since G_3 is a subgroup of $SL(2)$ we can readily verify: If $g \in G_3$ has coordinates (μ, ρ, ν), i.e., if (5.158) is valid, then considered as an element of $SL(2)$, g has coordinates α, w, β, where $\alpha = -i\mu$, $w = \rho$, $\beta = -i\nu$.

Equations (5.157) do not determine the parameters μ, ρ, ν uniquely. Indeed we can restrict ρ to take values in the range $0 \leqslant \rho < \infty$ in which case these parameters can be defined by

$$|a| = \cosh\rho, \qquad |b| = \sinh\rho,$$

$$\mu = -\arg a - \arg b, \qquad \nu = \arg b - \arg a.$$

As an immediate consequence of the validity of the decomposition (5.157) for all $g \in G_3$ we have the result that an irreducible unitary representation U of G_3 is uniquely determined by the infinitesimal operators J_3 and J_2. Indeed, if g has coordinates (μ, ρ, ν) then

$$\mathbf{U}(g) = \exp(\mu \mathrm{J}_3)\exp(\rho \mathrm{J}_2)\exp(\nu \mathrm{J}_3).$$

This result shows that the Lie algebra representations listed in Section 5-16 uniquely determine the group representations U from which they are derived.

We now proceed to the actual computation of the unitary irreducible representations of G_3. Since G_3 is a subgroup of $SL(2)$ we will be able to construct these representations as restrictions to G_3 of corresponding local multiplier representations of $SL(2)$.

(I) $A^{0,q}$ $(q > 0)$. The Lie algebra representation $A^{0,q}$, (5.151), of $sl(2)$ on $\mathscr{D}'$ is isomorphic to the representation $D(-\frac{1}{2} + iq, 0)$ on the abstract vector space V. In Section 5-1 it was shown that this abstract Lie algebra representation induced a local multiplier representation of $SL(2)$ defined by operators $\mathbf{A}(g)$ acting on the space $\mathcal{O}_1$ of all functions analytic in a neighborhood of $z = 1$. With respect to the basis $\{f_m\}$, $m = 0, \pm 1, \pm 2, \ldots$, the matrix elements $A^{0,q}_{lk}(g)$ of this multiplier representation restricted to G_3 are given by the equivalent expressions

$$A^{0,q}_{lk}(g) = \frac{|a|^{-1+2iq} a^l \bar{a}^{-k} \bar{b}^{k-l}}{\Gamma(\frac{1}{2} + iq + l)} \Gamma(\tfrac{1}{2} + iq + k)$$

$$\cdot \frac{F(\frac{1}{2} - iq - l, \frac{1}{2} - iq + k; k - l + 1; |b/a|^2)}{\Gamma(k - l + 1)}$$

$$= e^{-i(\mu l + \nu k)} \frac{\Gamma(\frac{1}{2} + iq + k)}{\Gamma(\frac{1}{2} + iq + l)} \mathfrak{P}^{-l,k}_{-\frac{1}{2}+iq}(\cosh\rho),$$

$$l, k \text{ integers}, \tag{5.159}$$

where $g = \exp(\mu\mathscr{J}_3)\exp(\rho\mathscr{J}_2)\exp(\nu\mathscr{J}_3)$. Since $|a|^2 = 1 + |b|^2$ these matrix elements are defined for all $g \in G_3$, not just in a neighborhood of the identity element. The addition theorem, (5.8), for the matrix elements becomes (on restriction to G_3)

$$A_{lk}^{0,q}(g_1g_2) = \sum_{j=-\infty}^{\infty} A_{lj}^{0,q}(g_1)A_{jk}^{0,q}(g_2), \qquad g_1, g_2 \in G_3. \tag{5.160}$$

In particular, the addition theorem is valid over the entire group manifold G_3, not just in a neighborhood of the identity. (This last remark follows from the facts: $|b/a| < 1$ for all $g \in G_3$, and $m_o = 0$.)

From expressions (5.159) and the transformation formulas (A.8), it is easy to verify the equality

$$A_{lk}^{0,q}(g) = \overline{A_{kl}^{0,q}(g^{-1})}, \qquad k, l \text{ integers}, \quad g \in G_3. \tag{5.161}$$

This equality implies that the infinite matrix $[A_{lk}^{0,q}(g)]$, $-\infty < l$, $k < \infty$, is unitary for all $g \in G_3$. Thus, we have constructed a representation of G_3 by unitary matrices.

We can shed more light on the unitary property of the matrix elements by returning to Eqs. (5.151) which define the representation $D(-\frac{1}{2} + iq, 0)$ of $sl(2)$ on the pre-Hilbert space $\mathscr{D}'$. The operators $\mathrm{J}^{\pm}$, J^3 given there satisfy the relations

$$\langle \mathrm{J}^+f, h\rangle = -\langle f, \mathrm{J}^-h\rangle, \qquad \langle \mathrm{J}^3f, h\rangle = \langle f, \mathrm{J}^3h\rangle$$

for all $f, h \in \mathscr{D}'$, where $\langle\cdot,\cdot\rangle$ is the inner product on $\mathscr{H}$. From the first of these relations we can obtain the identity

$$A_{lk}(\exp b\mathscr{J}^+) = \overline{A_{kl}(\exp -\bar{b}\mathscr{J}^-)}, \qquad b \in \mathcal{C}, \tag{5.162}$$

where the matrix elements $A_{lk}(g)$ of $D(-\frac{1}{2} + iq, 0)$ are defined by (5.9). Indeed, if $l \geqslant k$, then

$$A_{lk}(\exp b\mathscr{J}^+) = \left\langle f_l, \frac{(b\mathrm{J}^+)^{l-k}}{(l-k)!} f_k\right\rangle = \left\langle \frac{(-\bar{b}\mathrm{J}^-)^{l-k}}{(l-k)!} f_l, f_k\right\rangle$$

$$= \overline{\left\langle f_k, \frac{(-\bar{b}\mathrm{J}^-)^{l-k}}{(l-k)!} f_l\right\rangle} = \overline{A_{kl}(\exp -\bar{b}\mathscr{J}^-)},$$

where f_l, f_k are elements of the orthonormal basis $\{f_m\}$ for $\mathscr{H}$. If $k > l$, both sides of (5.162) are zero. Similarly, the relation involving J^3 implies the identity

$$A_{lk}(\exp \alpha\mathscr{J}^3) = \overline{A_{kl}(\exp \bar{\alpha}\mathscr{J}^3)}, \qquad \alpha \in \mathcal{C}.$$

If

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2)$$

is in a sufficiently small neighborhood of the identity element then

$$g = \exp(-[c/a]\mathscr{J}^-) \exp(-ab\mathscr{J}^+) \exp(\alpha\mathscr{J}^3)$$

where $e^{\alpha/2} = a$. Thus,

$$A_{lk}(g) = \sum_{j,n=-\infty}^{\infty} A_{lj}(\exp -[c/a]\mathscr{J}^-)A_{jn}(\exp -ab\mathscr{J}^+)A_{nk}(\exp \alpha\mathscr{J}^3)$$

$$= \overline{A_{kl}[(\exp \bar{\alpha}\mathscr{J}^3)(\exp \bar{a}\bar{b}\mathscr{J}^-)(\exp[\bar{c}/\bar{a}]\mathscr{J}^+)]} = \overline{A_{kl}(\hat{g})},$$

where

$$\hat{g} = \begin{pmatrix} \bar{a} & -\bar{c} \\ -\bar{b} & \bar{d} \end{pmatrix}$$

We have derived the identity

$$A_{lk}(g) = \overline{A_{kl}(\hat{g})}, \qquad k, l \text{ integers}, \quad g \in SL(2), \tag{5.163}$$

only for g in a sufficiently small neighborhood of the identity element. However, by analytic continuation its validity can be established for all $g \in SL(2)$ such that both sides of the equation are defined. In particular, if $g \in G_3$ we find $\hat{g} = g^{-1} \in G_3$ and (5.163) reduces to the unitarity relation (5.161).

Our results can be phrased in terms of unitary operators on the Hilbert space $\mathscr{H}$. The vectors $\{f_m\}$, $m = 0, \pm 1, \pm 2, \ldots$, form an orthonormal basis for $\mathscr{H}$, and the elements of $\mathscr{H}$ are vectors $f = \sum_{m=-\infty}^{\infty} c_m f_m$, such that $\sum_{m=-\infty}^{\infty} | c_m |^2 < \infty$. If $f = \sum c_m f_m$, $h = \sum d_m f_m$ are elements of $\mathscr{H}$. the inner product $\langle f, h \rangle$ is given by $\langle f, h \rangle = \sum_{m=-\infty}^{\infty} \bar{c}_m d_m$. The operators $\mathbf{U}(g)$, $g \in G_3$, defined on $\mathscr{H}$ by

$$\mathbf{U}(g) f_k = \sum_{l=-\infty}^{\infty} A_{lk}^{0,q}(g) f_l, \qquad k = 0, \pm 1, \pm 2, \ldots,$$

are obviously unitary and satisfy the representation property

$$\mathbf{U}(g_1 g_2) = \mathbf{U}(g_1)\mathbf{U}(g_2), \qquad g_1, g_2 \in G_3.$$

These operators thus define a unitary representation $A^{0,q}$ of G_3. Furthermore, by proceeding exactly as in the proof of Lemma 3.2, we can show that $A^{0,q}$ is irreducible. (Recall that the proof depends on the fact that none of the matrix elements $A^{0;q}_{lk}(g)$ is identically zero.)

The realization of the representation $A^{0,q}$ given here is purely abstract. However, it is not difficult to construct concrete function space realizations. For example $A^{0,q}$ can be defined on the Hilbert space $L_2(\varphi)$ of functions $f(e^{i\varphi})$ square integrable on the unit circle. The inner product is given by

$$\langle f, h\rangle = (2\pi)^{-1}\int_0^{2\pi} \overline{f(e^{i\varphi})}h(e^{i\varphi})\, d\varphi; \qquad f, h \in L_2(\varphi).$$

The functions $f_m(\varphi) = e^{im\varphi}$, $m = 0, \pm 1, \pm 2, \ldots$, form an orthonormal basis for $L_2(\varphi)$. With every $g \in G_3$ we associate the operator $\mathbf{U}(g)$:

$$[\mathbf{U}(g)f](e^{i\varphi}) = |\, \bar{a} + be^{i\varphi}\,|^{-1+2iq} f\left(\frac{ae^{i\varphi} + \bar{b}}{be^{i\varphi} + \bar{a}}\right), \qquad f \in L_2(\varphi).$$

(These operators can be obtained formally from (5.4) by setting $z = e^{i\varphi}$, $m_0 = 0$, $u = -\frac{1}{2} + iq$.) Since $|(ae^{i\varphi} + \bar{b})/(be^{i\varphi} + \bar{a})| = 1$, $\mathbf{U}(g)$ is well defined. The reader can verify for himself that the operators $\mathbf{U}(g)$ define a unitary representation of G_3 on $L_2(\varphi)$ with matrix elements

$$\langle f_l, \mathbf{U}(g) f_k\rangle = A^{0;q}_{lk}(g),$$

where $A^{0;q}_{lk}(g)$ is given by (5.159). For more details see Bargmann [1].

It is not worthwhile to work out identities for the matrix elements $A^{0;q}_{lk}(g)$ implied by the addition theorem (5.160) since these identities are merely special cases of (5.110). Similarly, the differential relations obeyed by the matrix elements are special cases of the results of Section 5-7. However, the fact that the matrices $[A^{0;q}_{lk}(g)]$ are unitary does give us one bit of new information: The matrix elements satisfy the inequality $|\, A^{0;q}_{lk}(g)| \leqslant 1$ or

$$\left| \frac{\Gamma(\frac{1}{2} + iq + k)}{\Gamma(\frac{1}{2} + iq + l)} \mathfrak{B}^{-l,k}_{-\frac{1}{2}+iq}(\cosh \rho) \right| \leqslant 1,$$
$$l, k \text{ integers}, \quad q > 0, \quad \rho \geqslant 0.$$

(II) $A^{1/2,q}$ $(q > 0)$. From (5.152) it follows that the Lie algebra representation $A^{1/2,q}$ of $sl(2)$ on $\mathscr{D}'$ is associated with the abstract representation $D(-\frac{1}{2} + iq, \frac{1}{2})$. As in (I) above, this abstract Lie algebra representation induces a local multiplier representation of $SL(2)$ defined by operators $\mathbf{A}(g)$ acting on $\mathcal{O}\!\mathcal{l}_1$, Eq. (5.4). The matrix elements $A_{lk}^{1/2,q}(g)$ of the multiplier representation, where g is restricted to G_3, are given by

$$A_{lk}^{1/2,q}(g) = |\, a\,|^{2iq} a^l \bar{a}^{-1-k} \bar{b}^{k-l} \frac{\Gamma(iq+k+1)}{\Gamma(iq+l+1)}$$

$$\cdot \frac{F(-iq-l, -iq+k+1; k-l+1; |\, b/a\,|^2)}{\Gamma(k-l+1)}$$

$$= e^{-i[\mu(l+\frac{1}{2})+\nu(k+\frac{1}{2})]} \frac{\Gamma(iq+k+1)}{\Gamma(iq+l+1)} \mathfrak{B}_{-\frac{1}{2}+iq}^{-\frac{1}{2}-l,\, \frac{1}{2}+k}(\cosh \rho),$$

$$l, k \text{ integers}, \tag{5.164}$$

where $g = \exp(\mu \mathcal{J}_3) \exp(\rho \mathcal{J}_2) \exp(\nu \mathcal{J}_3)$. These matrix elements are defined for all $g \in G_3$, rather than just in a neighborhood of the identity. The addition theorem for the matrix elements, Eq. (5.8), becomes

$$A_{lk}^{1/2,q}(g_1 g_2) = \sum_{j=-\infty}^{\infty} A_{lj}^{1/2,q}(g_1) A_{jk}^{1/2,q}(g_2), \tag{5.165}$$

valid for all $g_1, g_2 \in G_3$.

From (5.164) and the transformation formulas (A.8) we can readily verify the equality

$$A_{lk}^{1/2,q}(g) = \overline{A_{kl}^{1/2,q}(g^{-1})}, \qquad k, l \text{ integers}, \quad g \in G_3, \tag{5.166}$$

which proves that the infinite matrices $[A_{lk}^{1/2,q}(g)]$ are unitary. Moreover, just as in the proof of the identity (5.163) we can show that the matrix elements $A_{lk}(g)$ of the representation $D(-\frac{1}{2} + iq, \frac{1}{2})$ of $sl(2)$ obey the relation

$$A_{lk}(g) = \overline{A_{kl}(\hat{g})}. \tag{5.167}$$

For $g \in G_3$ this relation reduces to (5.166). To construct the unitary representation $A^{1/2,q}$ using these matrices we consider a Hilbert space $\mathcal{H}$ with orthonormal basis $\{f_m\}$, $m = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$. Then the operators $\mathbf{U}(g)$, $g \in G_3$, defined on $\mathcal{H}$ by

$$\mathbf{U}(g) f_{1/2+k} = \sum_{l=-\infty}^{\infty} A_{lk}^{1/2,q}(g) f_{1/2+l}, \qquad k = 0, \pm 1, \pm 2, \ldots,$$

form a unitary irreducible representation of G_3.

The representation $A^{1/2,q}$ of G_3 can also be given a concrete realization in terms of unitary operators on the Hilbert space $L_2(\varphi)$. This time we label the orthonormal basis vectors for $L_2(\varphi)$ as $f_{\frac{1}{2}+k}(\varphi) = e^{ik\varphi}$. Corresponding to each $g \in G_3$ we associate the operator $\mathbf{U}'(g)$:

$$[\mathbf{U}'(g)f](e^{i\varphi}) = |\,\bar{a} + be^{i\varphi}\,|^{2iq}(\bar{a} + be^{i\varphi})^{-1} f\left(\frac{ae^{i\varphi} + \bar{b}}{be^{i\varphi} + \bar{a}}\right), \qquad f \in L_2(\varphi).$$

(This operator can be obtained formally from (5.4) by setting $z = e^{i\varphi}$, $m_0 = \frac{1}{2}$, $u = -\frac{1}{2} + iq$.) The operators $\mathbf{U}'(g)$ define a unitary representation of G_3 on $\mathrm{L}_2(\varphi)$ with matrix elements

$$\langle f_{\frac{1}{2}+l}, \mathbf{U}'(g) f_{\frac{1}{2}+k}\rangle = A_{lk}^{1/2,q}(g),$$

where $A_{lk}^{1/2,q}(g)$ is given by (5.164). For the details of the construction see Bargmann [1].

The addition theorem for the matrix elements is again a special case of (5.110), while the recursion relations obeyed by the matrix elements are obtained from Section 5-7. The unitarity condition(5.166) implies $|\,A_{lk}^{1/2,q}(g)| \leqslant 1$ or

$$\left|\frac{\Gamma(iq + k + 1)}{\Gamma(iq + l + 1)}\,\mathfrak{B}_{-\frac{1}{2}+iq}^{-\frac{1}{2}-l,\,\frac{1}{2}+k}\,(\cosh\rho)\right| \leqslant 1, \qquad l, k \text{ integers}, \quad q > 0,$$

(III) $A^{0,q}$ $(-\frac{1}{2} < q < 0)$. The Lie algebra representation $A^{0,q}$, (5.153), of $sl(2)$ on $\mathscr{D}'$ is isomorphic to the abstract representation $D(q, 0)$ on the vector space V. With respect to the basis $\{f_m\}$, $m = 0, \pm 1, \pm 2, \ldots$, the matrix elements of $D(q, 0)$, restricted to G_3, are given by

$$A_{lk}(g) = \frac{|\,a\,|^{2q} a^l \bar{a}^{-k} \bar{b}^{k-l}\,\Gamma(q + k + 1)}{\Gamma(q + l + 1)}\,\frac{F(-q - l, -q + k; k - l + 1; |\,b/a\,|^2)}{\Gamma(k - l + 1)}$$

$$= e^{-i(\mu l + \nu k)}\,\frac{\Gamma(q + k + 1)}{\Gamma(q + l + 1)}\,\mathfrak{B}_q^{-l,k}(\cosh\rho), \qquad k, l \text{ integers},$$

With respect to the basis $\{j_m\}$, $m = 0, \pm 1, \pm 2, \ldots$, where

$$j_m = \left[\frac{\Gamma(q + 1)\Gamma(m - q)}{\Gamma(-q)\Gamma(m + q + 1)}\right]^{1/2} f_m\,,$$

see (5.150), the matrix elements of $D(q, 0)$ can be expressed in the equivalent forms

$$A_{lk}^{0,q}(g) = \left[\frac{\Gamma(k-q)\Gamma(l+q+1)}{\Gamma(k+q+1)\Gamma(l-q)}\right]^{1/2} A_{lk}(g)$$

$$= |a|^{2q} a^l \bar{a}^{-k} \bar{b}^{k-l} \left[\frac{\Gamma(l+q+1)}{\Gamma(k+q+1)} \frac{\Gamma(k-q)}{\Gamma(l-q)}\right]^{1/2} \frac{\Gamma(k+q+1)}{\Gamma(l+q+1)}$$

$$\cdot \frac{F(-q-l, -q+k; k-l+1; |b/a|^2)}{\Gamma(k-l+1)}$$

$$= e^{-i(\mu l+\nu k)} \frac{\Gamma(k+q+1)}{\Gamma(l+q+1)} \left[\frac{\Gamma(l+q+1)\Gamma(k-q)}{\Gamma(k+q+1)\Gamma(l-q)}\right]^{1/2} \mathfrak{P}_q^{-l,k}(\cosh\rho), \tag{5.168}$$

where $g = \exp(\mu\mathscr{J}_3)\exp(\rho\mathscr{J}_2)\exp(\nu\mathscr{J}_3) \in G_3$ is given by (5.156). The addition theorem (5.8) becomes

$$A_{lk}^{0,q}(g_1 g_2) = \sum_{j=-\infty}^{\infty} A_{lj}^{0,q}(g_1) A_{jk}^{0,q}(g_2), \tag{5.169}$$

valid for all $g_1, g_2 \in G_3$. From the expressions (5.168) for the matrix elements it is easy to verify the identity

$$\overline{A_{lk}^{0,q}(g)} = A_{kl}^{0,q}(g^{-1}), \tag{5.170}$$

which proves that the matrices $[A_{kl}^{0,q}(g)]$ are unitary. Moreover, if we mimic the derivation of the identity (5.163), case (I), we can derive the relation

$$A_{lk}(g) = \frac{\Gamma(l-q)\Gamma(k+q+1)}{\Gamma(l+q+1)\Gamma(k-q)} \overline{A_{kl}(\hat{g})}$$

for the matrix elements of $D(q, 0)$, where $g \in SL(2)$. When $g \in G_3$ this relation reduces to (5.170).

Let $\mathscr{H}$ be a Hilbert space with orthonormal basis $\{j_m\}$, $m = 0, \pm 1, \pm 2, \ldots$. It is easy to show that the operators $\mathbf{U}(g)$,

$$\mathbf{U}(g) j_k = \sum_{l=-\infty}^{\infty} A_{lk}^{0,q}(g) j_l, \qquad k \text{ an integer}, \quad g \in G_3, \quad -\tfrac{1}{2} < q < 0,$$

define a unitary irreducible representation of G_3 on $\mathscr{H}$.

The problem of constructing a concrete function space realization of the representation $A^{0,q}$ for $-\frac{1}{2} < q < 0$ is somewhat more difficult than the previous cases and will not be treated here. See Bargmann [1] for details of the construction.

Again the addition theorem and recursion relations for the matrix elements are special cases of equations (5.110), so it is pointless to re-derive them. The unitarity condition (5.170) implies

$$\left| \left[\frac{\Gamma(k+q+1)\Gamma(k-q)}{\Gamma(l+q+1)\Gamma(l-q)}\right]^{1/2} \mathfrak{B}_q^{-l,k}(\cosh\rho) \right| \leqslant 1, \qquad -\tfrac{1}{2} < q < 0.$$

(IV) D_n^+ $(n = \frac{1}{2}, 1, \frac{3}{2},\ldots)$. According to Eqs. (5.154) the Lie algebra representation D_n^+ of $sl(2)$ on $\mathscr{D}'$ is isomorphic to the representation $\uparrow_{-n}$ on the abstract vector space V. This abstract representation of $sl(2)$ induces a local multiplier representation of $SL(2)$ defined by the operators $\mathbf{B}(g)$ acting on the space $\mathscr{O}'_2$ of all functions $f(z)$ analytic in a neighborhood of $z = 0$, (5.18). With respect to the basis $\{f_{n+k}\}$, $k = 0, 1, 2,\ldots$, the matrix elements restricted to G_3 take the form

$$B_{lk}(g) = a^l\bar{a}^{-2n-k}\bar{b}^{k-l}\,\frac{k!}{l!}\,\frac{F(-l, 2n+k; k-l+1; |b/a|^2)}{\Gamma(k-l+1)}, \qquad l, k \geqslant 0$$

(see Eq. (5.23)). However, in terms of the orthonormal basis $\{j_m\}$ for $\mathscr{D}$,

$$j_m = \left[\frac{(m+n-1)!}{(2n-1)!\,(m-n)!}\right]^{1/2} f_m, \qquad m = n, n+1, n+2,\ldots,$$

the matrix elements of $\mathbf{B}(g)$ are

$$\begin{aligned} B_{lk}^n(g) &= \left[\frac{(2n+k-1)!\,l!}{k!\,(2n+l-1)!}\right]^{1/2} B_{lk}(g) \\ &= \left[\frac{(2n+k-1)!\,k!}{(2n+l-1)!\,l!}\right]^{1/2} a^l\bar{a}^{-2n-k}\bar{b}^{k-l}\frac{F(-l, 2n+k; k-l+1; |b/a|^2)}{\Gamma(k-l+1)} \\ &= \left[\frac{(2n+k-1)!\,k!}{(2n+l-1)!\,l!}\right]^{1/2} e^{-i[\mu(n+l)+\nu(n+k)]}\,\mathfrak{B}_{-n}^{-n-l,n+k}(\cosh\rho), \end{aligned} \tag{5.171}$$

where $g = \exp(\mu\mathscr{J}_3)\exp(\rho\mathscr{J}_2)\exp(\nu\mathscr{J}_3) \in G_3$ is given by (5.156). Since $2n$ is an integer, these matrix elements are well defined over the entire group manifold G_3. The addition theorem is

$$B_{lk}^n(g_1g_2) = \sum_{j=0}^{\infty} B_{lj}^n(g_1)B_{jk}^n(g_2), \tag{5.172}$$

valid for all $g_1, g_2 \in G_3$.

The infinite matrix $[B^n_{lk}(g)]$, $l, k \geqslant 0$, is unitary, i.e.,

$$B^n_{lk}(g) = \overline{B^n_{kl}(g^{-1})}. \tag{5.173}$$

This property can be verified directly from the explicit expressions (5.171). Alternatively, one could mimic the proof of the identity (5.163) and show that the restrictions on the operators $\mathbf{J}^{\pm}$, $\mathbf{J}^3$ implied by Lemma 3.1 lead to the following relation for the matrix elements of $\uparrow_{-n}$:

$$(2n + k - 1)!\, l!\, B_{lk}(g) = (2n + l - 1)!\, k!\, \overline{B_{kl}(\hat{g})},$$
$$l, k \geqslant 0, \quad g \in SL(2). \tag{5.174}$$

For $g \in G_3$ this identity reduces to the unitarity condition (5.173).

Using the unitary matrices $[B^n_{lk}(g)]$ we can now construct the unitary representation D^+_n of G_3 on a Hilbert space $\mathscr{H}$ with orthonormal basis $\{j_m\}$, $m = n, n + 1, n + 2, \ldots$. In fact the operators $\mathbf{U}(g), g \in G_3$, defined on $\mathscr{H}$ by

$$\mathbf{U}(g)\, j_{n+k} = \sum_{l=0}^{\infty} B^n_{lk}(g)\, j_{n+l}, \qquad k = 0, 1, 2, \ldots,$$

clearly form a unitary representation of G_3. Exactly as in the proof of Lemma 3.2, one can show that D^+_n is irreducible.

The representation D^+_n can also be given concrete function space realizations. We mention Bargmann's construction. The Hilbert space $\mathscr{H}_{2n}$, $n = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$, is the space of all functions $f(z)$ analytic on the open unit circle $\mathscr{M}$ in the complex plane, $\mathscr{M} = \{z\bar{z} < 1\}$, such that

$$\lim_{l \to 2n} \frac{l-1}{\pi} \int_{\mathscr{M}} |f(z)|^2 (1 - z\bar{z})^{l-2}\, dx\, dy < \infty, \qquad l > 1.$$

Here $z = x + iy$ and the integral is taken over $\mathscr{M}$. The limit $l \to 2n$ is essential only for $n = \frac{1}{2}$; otherwise it is redundant. The inner product $\langle \cdot, \cdot \rangle_{2n}$ on $\mathscr{H}_{2n}$ is defined by

$$\langle f, h \rangle_{2n} = \lim_{l \to 2n} \frac{l-1}{\pi} \int_{\mathscr{M}} \overline{f(z)} h(z) (1 - z\bar{z})^{l-2}\, dx\, dy, \qquad f, h \in \mathscr{H}_{2n}.$$

Set $f_{n+k}(z) = z^k$, $k = 0, 1, 2, \ldots$. By introducing polar coordinates $z = re^{i\theta}$, it is easy to show

$$\langle f_{n+l}, f_{n+k} \rangle_{2n} = \langle z^l, z^k \rangle_{2n} = \delta_{l,k} \frac{k!\,(2n-1)!}{(2n+k-1)!}, \qquad l, k \geqslant 0. \tag{5.175}$$

Therefore, $f(z) = \sum_{k=0}^{\infty} c_k z^k$ is an element of $\mathscr{H}_{2n}$ if and only if

$$\sum_{k=0}^{\infty} \frac{k!\,(2n-1)!}{(2n+k-1)!} \mid c_k \mid^2 < \infty.$$

According to (5.175) the functions

$$j_{n+k}(z) = \left[\frac{(2n+k-1)!}{(2n-1)!\,k!}\right]^{1/2} z^k, \qquad k = 0, 1, 2,...,$$

form an orthonormal basis for $\mathscr{H}_{2n}$ (compare with Eqs. (5.154)). On $\mathscr{H}_{2n}$ we define the operators $\mathbf{U}'(g)$ by

$$[\mathbf{U}'(g)f](z) = (bz + \bar{a})^{-2n} f\left(\frac{az + \bar{b}}{bz + \bar{a}}\right), \qquad g \in G_3, \quad f \in \mathscr{H}_{2n}. \tag{5.176}$$

Formally these operators are identical with the multiplier representation (5.18), when $g \in G_3$. Expression (5.176) is well defined since

$$1 - \left|\frac{az + \bar{b}}{bz + \bar{a}}\right|^2 = \frac{1 - z\bar{z}}{\mid bz + \bar{a} \mid^2}, \qquad \text{for} \quad a\bar{a} - b\bar{b} = 1.$$

Thus $|(az + \bar{b})/(bz + \bar{a})| < 1$ if $\mid z \mid < 1$, and the manifold $\mathscr{M}$ is mapped into itself. The reader can verify that the operators $\mathbf{U}'(g)$ define a unitary representation of G_3 which is isomorphic to D_n^+. With respect to the orthonormal basis $\{j_{n+k}\}$ the matrix elements are

$$\langle j_{n+l}\,, \mathbf{U}'(g)\, j_{n+k}\rangle_{2n} = B_{lk}^n(g).$$

For more details on this representation see Bargmann [1].

Identities for special functions implied by the addition theorem (5.172) are special cases of (5.116). The unitarity condition on the matrix elements yields the inequality

$$\left| \left[\frac{(2n+k-1)!\,k!}{(2n+l-1)!\,l!}\right]^{1/2} \mathfrak{B}_{-n}^{-n-l,n+k}(\cosh \rho) \right| \leqslant 1,$$

$$l, k = 0, 1, 2,..., \quad n = \tfrac{1}{2}, 1, \tfrac{3}{2},....$$

(V) D_n^- $(n = \tfrac{1}{2}, 1, \tfrac{3}{2},...)$. According to (5.155) the Lie algebra representation D_n^- of $sl(2)$ on $\mathscr{D}'$ is isomorphic to $\downarrow_{-n}$. In Section 5-3 we showed that $\downarrow_{-n}$ induces a local multiplier representation of $SL(2)$ defined by operators $\mathbf{C}(g)$ acting on the space $\mathscr{A}_2'$ of all functions $f(z)$ analytic in a neighborhood of $z = 0$, (5.28). With respect to the basis

$\{f_{-n-k}\}$, $k = 0, 1, 2,\ldots$, the matrix elements of the operators $\mathbf{C}(g)$, for g restricted to G_3, are

$$C_{lk}(g) = \bar{a}^l a^{-2n-k} b^{k-l} \frac{k!}{l!} \frac{F(-l, 2n+k; k-l+1; |b/a|^2)}{\Gamma(k-l+1)}, \qquad l, k \geqslant 0,$$

Eq. (5.31). In terms of the orthonormal basis $\{j_m\}$, for $\mathscr{H}$,

$$j_m = \left[\frac{(n-m-1)!}{(2n+1)!\,(-n-m)!}\right]^{1/2} f_m, \qquad m = -n, -n-1,\ldots,$$

the matrix elements of $\mathbf{C}(g)$ are

$$\begin{aligned} C^n_{lk}(g) &= \left[\frac{(2n+k-1)!\, l!}{k!\,(2n+l-1)!}\right]^{1/2} C_{lk}(g) \\ &= \left[\frac{(2n+k-1)!\, k!}{(2n+l-1)!\, l!}\right]^{1/2} \bar{a}^l a^{-2n-k} b^{k-l} \frac{F(-l, 2n+k; k-l+1, |b/a|^2)}{\Gamma(k-l+1)} \\ &= \left[\frac{(2n+k-1)!\, k!}{(2n+l-1)!\, l!}\right]^{1/2} e^{i[\mu(n+l)+\nu(n+k)]} \mathfrak{B}_{-n}^{-n-l, n+k}(\cosh \rho), \end{aligned} \tag{5.177}$$

where $g = \exp(\mu \mathscr{J}_3) \exp(\rho \mathscr{J}_2) \exp(\nu \mathscr{J}_3) \in G_3$.

Comparing these expressions with the matrix elements $B^n_{lk}(g)$ of the representation D^+_n, (5.171), we find

$$C^n_{lk}(g) = \overline{B^n_{lk}(g)}, \qquad l, k \geqslant 0, \quad g \in G_3. \tag{5.178}$$

Thus, from Eqs. (5.172), (5.173) we immediately obtain the addition theorem

$$C^n_{lk}(g_1 g_2) = \sum_{j=0}^{\infty} C^n_{lj}(g_1) C^n_{jk}(g_2), \qquad g_1, g_2 \in G_3,$$

and the unitarity relation

$$C^n_{lk}(g) = \overline{C^n_{kl}(g^{-1})}.$$

Exactly as in the treatment of the representation D^+_n, we can show that the unitary matrices $[C^n_{lk}(g)]$ define a unitary irreducible representation D^-_n of G_3.

This representation has a concrete realization on the Hilbert space $\mathscr{H}_{2n}$ described in case (IV). To construct it define operators $\mathbf{U}''(g)$:

$$[\mathbf{U}''(g)f](z) = (\bar{b}z + a)^{-2n} f\left(\frac{\bar{a}z + b}{\bar{b}z + a}\right), \qquad z \in \mathscr{M}, \quad f \in \mathscr{H}_{2n},$$

obtained formally from the multiplier representation (5.28) by restricting g to G_3. As the reader can verify, these operators define a unitary irreducible representation of G_3 on $\mathscr{H}_{2n}$, isomorphic to D_n^-. With respect to the orthonormal basis

$$j_{-n-k}(z) = \left[\frac{(2n+k-1)!}{(2n+1)!\,k!}\right]^{1/2} z^k, \qquad k \geqslant 0,$$

for $\mathscr{H}_{2n}$ the matrix elements of these operators are

$$\langle j_{-n-l}, \mathbf{U}''(g) j_{-n-k}\rangle_{2n} = C_{lk}^n(g).$$

As shown by Bargmann, the above five classes together with the trivial identity representation constitute all the irreducible unitary representations of G_3. A study of the orthogonality and completeness relations obeyed by the matrix elements of the unitary irreducible representations with respect to the Haar measure on G_3 is beyond the scope of this book. For these results see Bargmann [1]. Likewise we will not consider the important problem of decomposing tensor products of unitary irreducible representations of G_3 into irreducible representations. The decomposition

$$D_n^+ \otimes D_{n'}^+ \cong \sum_{k=0}^{\infty} \oplus\, D_{n+n'+k}^+$$

is a special case of our results on the representation $\uparrow_u \otimes \uparrow_{u'}$ in Section 5.6. However, a study of all possible tensor products of representations in classes (I)–(V) is rather involved. Indeed there are few published results on this problem which are explicit enough for application to special function theory (Pukánszky [1], Romm[1]).

5-19 Contractions of $\mathscr{G}(1, 0)$

In Section 2-5, it was shown that the Lie algebra $\mathscr{G}(0, 0) \cong \mathscr{T}_3 \oplus (\mathscr{E})$ was a contraction of $\mathscr{G}(1, 0) \cong sl(2) \oplus (\mathscr{E})$. Indeed, $\mathscr{T}_3$ is a contraction of $sl(2)$. Similarly $\mathscr{G}(0, 1)$ was proved to be a contraction of $\mathscr{G}(1, 0)$. These relations between Lie algebras suggest that the matrix elements of irreducible representations of $\mathscr{T}_3$ and $\mathscr{G}(0, 1)$ may in some way be obtainable as limits of matrix elements of irreducible representations $sl(2)$. This is the case, as will be seen from the following example.

Here we follow Wigner and Inönu [1], and derive a relation between Jacobi polynomials and Bessel functions. According to Section 5-16, the irreducible unitary representation D_u of $SU(2)$, $2u$ a nonnegative integer, can be realized on the $(2u + 1)$-dimensional Hilbert space $\mathscr{K}_u$ consisting of all polynomials of order not greater than $2u$ in the complex variable z. The inner product $\langle .,.\rangle$ on $\mathscr{K}_u$ (conjugate linear in the first argument, linear in the second) is uniquely determined by the requirement

$$\langle p_m^u, p_n^u\rangle = \delta_{m,n}, \qquad -u \leqslant m, n \leqslant u,$$

where

$$p_m^u(z) = \frac{(-z)^{u+m}}{[(u+m)!\,(u-m)!]^{1/2}}, \qquad m = -u, -u+1, ..., +u.$$

Thus, the $2u + 1$ vectors p_m^u form an orthonormal basis for $\mathscr{K}_u$. The infinitesimal operators J_k, $k = 1, 2, 3$, on $\mathscr{K}_u$ are defined by

$$\begin{aligned} J_1 &= -iuz + \frac{i}{2}\frac{d}{dz} - \frac{i}{2} z^2 \frac{d}{dz}, \\ J_2 &= uz - \frac{1}{2}\frac{d}{dz} - \frac{z^2}{2}\frac{d}{dz}, \\ J_3 &= -iu - iz\frac{d}{dz}, \end{aligned} \tag{5.179}$$

and obey the commutation relations

$$[J_1, J_2] = J_3, \qquad [J_3, J_1] = J_2, \qquad [J_2, J_3] = J_1. \tag{5.180}$$

In terms of the basis $\{p_m^u\}$, the matrix elements of these operators are

$$\begin{aligned} (J_3^u)_{n,m} &= \langle p_n^u, J_3 p_m^u\rangle = -in\,\delta_{n,m}, \\ (J_1^u)_{n,m} &= \langle p_n^u, J_1 p_m^u\rangle = -\frac{i}{2}[(u+n+1)(u-n)]^{1/2}\,\delta_{n+1,m} \\ &\qquad -\frac{i}{2}[(u-n+1)(u+n)]^{1/2}\,\delta_{n-1,m}, \qquad (5.181) \\ (J_2^u)_{n,m} &= \langle p_n^u, J_2 p_m^u\rangle = \frac{1}{2}[(u+n+1)(u-n)]^{1/2}\,\delta_{n+1,m} \\ &\qquad -\frac{i}{2}[(u-n+1)(u+n)]^{1/2}\,\delta_{n-1,m}, \\ &\qquad n, m = -u, -u+1, ..., +u. \end{aligned}$$

According to Eq. (3.64),

$$K_1 = -i\rho\cos\alpha, \qquad K_2 = -i\rho\sin\alpha, \qquad K_3 = -\frac{d}{d\alpha}, \qquad \rho > 0, \quad \alpha \text{ real},$$

are infinitesimal operators induced by the irreducible representation (ρ) of E_3 on the Hilbert space $\mathscr{H}$. Clearly,

$$[K_1, K_2] = 0, \qquad [K_3, K_1] = K_2, \qquad [K_3, K_2] = -K_1, \tag{5.182}$$

and the infinitesimal operators generate a Lie algebra isomorphic to $\mathscr{E}_3$, a real form of $\mathscr{T}_3$. With respect to the orthonormal basis $h_n(\alpha) = e^{in\alpha}$, $\pm n = 0, 1, 2,\ldots$, for $\mathscr{H}$ the matrix elements of these operators are

$$\begin{aligned}
(K_3^\rho)_{n,m} &= \langle h_n, K_3 h_m\rangle^* = -in\,\delta_{n,m},\\
(K_1^\rho)_{n,m} &= \langle h_n, K_1 h_m\rangle^* = -\frac{i}{2}\rho(\delta_{n-1,m} + \delta_{n+1,m}),\\
(K_2^\rho)_{n,m} &= \langle h_n, K_2 h_m\rangle^* = \frac{\rho}{2}(\delta_{n+1,m} - \delta_{n-1,m}),\\
&\qquad n, m = 0, \pm 1, \pm 2,\ldots.
\end{aligned} \tag{5.183}$$

The inner product $\langle .,. \rangle^*$ refers to the Hilbert space $\mathscr{H}$.

Corresponding to the parameter $\epsilon > 0$ we can define a new set of generators for the Lie algebra of $SU(2)$:

$$G_1 = \epsilon J_1, \qquad G_2 = \epsilon J_2, \qquad G_3 = J_3. \tag{5.184}$$

The structure constants for the commutation relations of the G operators are functions of ϵ. Indeed,

$$[G_1, G_2] = \epsilon^2 G_3, \qquad [G_3, G_1] = G_2, \qquad [G_2, G_3] = G_1,$$

As $\epsilon \to 0$ the structure constants approach limits which are the structure constants of a new 3-dimensional Lie algebra. In the limit the commutation relations become

$$[G_1, G_2] = 0, \qquad [G_3, G_1] = G_2, \qquad [G_2, G_3] = G_1,$$

which defines a Lie algebra isomorphic to $\mathscr{E}_3$. We have rederived the fact that $\mathscr{E}_3$ is a contraction of $su(2)$.

Using this relationship we can obtain the irreducible representation (ρ) of $\mathscr{E}_3$ as a limit of a sequence of representations of $su(2)$. Consider the representation D_u of $SU(2)$, u a nonnegative integer. The matrix elements of the G operators corresponding to this representation can be determined from (5.181) and (5.184). Computing the limit of these matrix elements as $\epsilon \to 0$ and $u \to +\infty$ in such a manner that $\epsilon u \to \rho$, we find

$$\begin{aligned}
&\lim_{\epsilon\to 0} (G_1^{\rho/\epsilon})_{n,m} = \lim_{\epsilon\to 0} \langle p^{\rho/\epsilon}, \epsilon J_1 p_m^{\rho/\epsilon}\rangle = -\frac{i\rho}{2}(\delta_{n+1,m} + \delta_{n-1,m}) = (K_1^\rho)_{n,m}, \\
&\lim_{\epsilon\to 0} (G_2^{\rho/\epsilon})_{n,m} = \frac{\rho}{2}(\delta_{n+1,m} - \delta_{n-1,m}) = (K_2^\rho)_{n,m}, \qquad (5.185) \\
&\lim_{\epsilon\to 0} (G_3^{\rho/\epsilon})_{n,m} = -in\,\delta_{n,m} = (K_3^\rho)_{n,m}, \qquad n, m = 0, \pm 1, \pm 2, \ldots .
\end{aligned}$$

Thus, we have obtained the irreducible representation (ρ) as a limit of irreducible representations of $su(2)$. From this result it is easy to compute the effect of the limiting process on matrix elements of the representation D_u of $SU(2)$. For our purpose it will be sufficient to consider the matrix elements of the unitary operator $e^{\theta J_1} = \mathbf{U}^u(\exp \theta \mathscr{J}_1)$. From (5.143) we have

$$(e^{\theta J_1})_{n,m} = (i)^{n-m}\left[\frac{(u+m)!\,(u-n)!}{(u-m)!\,(u+n)!}\right]^{1/2} P_u^{-n,m}(\cos\theta).$$

Similarly, from (3.57) it follows that the matrix elements of the unitary operator $e^{\theta K_1}$, corresponding to the representation (ρ) of E_3, are given by

$$(e^{\theta K_1})_{n,m} = i^{m-n} J_{n-m}(\rho\theta).$$

Thus the first of the relations (5.185) implies

$$\lim_{\substack{\epsilon\to 0\\ \epsilon u\to\rho}} (e^{\theta G_1})_{n,m} = (e^{\theta K_1})_{n,m}, \qquad G_1 = \epsilon J_1, \qquad (5.186)$$

or

$$\lim_{\epsilon\to 0} (i)^{n-m}\left[\frac{(\rho/\epsilon+m)!\,(\rho/\epsilon-n)!}{(\rho/\epsilon-m)!\,(\rho/\epsilon+n)!}\right]^{1/2} P_{\rho/\epsilon}^{-n,m}(\cos\epsilon\theta) = i^{m-n} J_{n-m}(\rho\theta).$$

Simplifying this expression we obtain

$$\lim_{k\to+\infty} (-k)^{m-n} P_k^{-n,m}(\cos x/k) = J_{n-m}(x), \qquad n, m = 0, \pm 1, \pm 2, \ldots, \quad x \text{ real}. \qquad (5.187)$$

This demonstration of the limit relation (5.187) is not rigorous since the validity of (5.186) has not been explicitly verified. However, it is now easy to establish (5.187) directly from the power series expansions for the functions involved.

Using similar arguments we could derive formulas expressing Bessel functions as limits of matrix elements of irreducible representations of G_3. Also we could use the fact that $\mathscr{G}(0, 1)$ is a contraction of $\mathscr{G}(1, 0)$ to obtain the associated Laguerre polynomials as limits of Jacobi polynomials. Both of these constructions are left to the reader.