CHAPTER 9

Some Generalizations

Up to this point all identities for special functions have been derived from a study of the representation theory of the Lie algebras $\mathscr{G}(a, b)$ and $\mathscr{T}_6$. However, the Lie theory approach to special functions is of much wider applicability than these examples may indicate. To bolster this contention we use the results of Chapter 8 to discover some new Lie algebras which have realizations by generalized Lie derivatives in one and two complex variables. Corresponding to each of the new Lie algebras we will derive identities for special functions by relating these functions to the representation theory of the Lie algebra.

The first five sections of this chapter are devoted to a brief examination of the Lie algebra $\mathscr{K}_5$. A study of $\mathscr{K}_5$ leads to new identities for the Hermite functions, the best known of which is Mehler's theorem, (9.36). The special function theory of $\mathscr{K}_5$ presented here is by no means exhaustive, and the interested reader is invited to derive additional results. (Weisner [2] has obtained identities for the Hermite functions even more general than those derived here, by studying a 6-dimensional Lie algebra which contains $\mathscr{K}_5$ as a subalgebra. However, his 6-dimensional algebra has no realizations by generalized Lie derivatives in one complex variable.)

In the last half of the chapter we introduce a family of 3-dimensional Lie algebras $\mathscr{G}_{p,q}$ which forms a natural generalization of $\mathscr{T}_3 \simeq \mathscr{G}_{1,1}$. The special functions associated with this family form a natural generalization of Bessel functions, and the identities obeyed by these functions are analogous to those derived for Bessel functions in Chapter 3.

The above examples are indicative of the use of Lie theory for the study of special functions. This theory can be employed both to obtain properties of known special functions and to derive new functions which are interesting enough to deserve the label "special." Additional examples of the applications of Lie theory to special functions are found in Vilenkin [2], [3], notably the special function theory of unitary representations of the orthogonal and Euclidean groups in n-space.

9-1 The Lie Algebra $\mathcal{K}_5$

$\mathcal{K}_5$ is the 5-dimensional complex Lie algebra with basis $\mathcal{J}^\pm$, $\mathcal{J}^3$, $\mathcal{E}$, $\mathcal{Q}$ and commutation relations

$$\begin{gathered}[\mathcal{J}^3, \mathcal{J}^\pm] = \pm\mathcal{J}^\pm, \qquad [\mathcal{J}^3, \mathcal{Q}] = 2\mathcal{Q} \\ [\mathcal{J}^-, \mathcal{J}^+] = \mathcal{E}, \qquad [\mathcal{J}^-, \mathcal{Q}] = 2\mathcal{J}^+, \qquad [\mathcal{J}^+, \mathcal{Q}] = \mathcal{O} \\ [\mathcal{J}^\pm, \mathcal{E}] = [\mathcal{J}^3, \mathcal{E}] = [\mathcal{Q}, \mathcal{E}] = \mathcal{O}.\end{gathered} \tag{9.1}$$

Clearly, the 4-dimensional subalgebra of $\mathcal{K}_5$ generated by $\mathcal{J}^\pm$, $\mathcal{J}^3$, $\mathcal{E}$ is isomorphic to $\mathcal{G}(0, 1)$. The Lie algebra $\mathcal{K}_5$ is of interest to us because it has realizations by differential operators in one complex variable. In fact, for $q = 2$ in expression (8.28) one obtains the operators

$$\alpha_1\colon \quad \frac{d}{dz}, \quad z\frac{d}{dz}, \quad 1, \quad z, \quad z^2; \qquad r = 5, \quad k = 4, \quad s = 2,$$

which define an effective realization of $\mathcal{K}_5$. Every transitive effective realization of $\mathcal{K}_5$ by gd's in one complex variable is an element of $\mathcal{O}\!\ell(\tilde{\alpha}_1)$.

$\mathcal{K}_5$ also has realizations in two complex variables. Using Lie's tables (Lie [1], Vol. III, pp. 71–73), and the methods of Chapter 8 one can show that every transitive effective realization of $\mathcal{K}_5$ by gd's in two complex variables is an element of $\mathcal{O}\!\ell(\tilde{\beta}_j)$, $j = 1,\ldots, 5$, where

$$\begin{aligned}\beta_1\colon \quad & \frac{\partial}{\partial z_1}, \quad e^{z_1}\left(\frac{\partial}{\partial z_2} - \frac{z_2}{2}\right), \quad e^{-z_1}\left(-\frac{\partial}{\partial z_2} - \frac{z_2}{2}\right), \quad 1, \\ & e^{2z_1}\left(-z_2\frac{\partial}{\partial z_2} - \frac{\partial}{\partial z_1} + \frac{z_2^2}{2}\right); \qquad r = 5, \quad k = 3, \quad s = 1;\end{aligned}$$

$$\begin{aligned}\beta_2\colon \quad & z_2\frac{\partial}{\partial z_2} - z_1\frac{\partial}{\partial z_1}, \quad \frac{\partial}{\partial z_1} + z_2, \quad \frac{\partial}{\partial z_2}, \quad 1, \quad 2z_2\frac{\partial}{\partial z_1} + z_2^2; \\ & \qquad r = 5, \quad k = 3, \quad s = 1;\end{aligned}$$

$$\begin{aligned}\beta_3\colon \quad & z_1\frac{\partial}{\partial z_1}, \quad z_1\frac{\partial}{\partial z_2}, \quad \frac{\partial}{\partial z_1}, \quad \frac{\partial}{\partial z_2}, \quad z_1^2\frac{\partial}{\partial z_2}; \\ & \qquad r = 5, \quad k = 3, \quad s = 2;\end{aligned}$$

$$\beta_4:\quad \frac{\partial}{\partial z_1},\quad e^{z_1}z_2,\quad e^{-z_1}\frac{\partial}{\partial z_2},\quad 1,\quad e^{2z_1}\left(\frac{\partial}{\partial z_1} - z_2\frac{\partial}{\partial z_2} + z_2^2\right);$$
$$r = 5,\quad k = 3,\quad s = 1;$$
$$\beta_5:\quad \frac{\partial}{\partial z_1},\quad e^{z_1}z_2,\quad e^{-z_1}\frac{\partial}{\partial z_2},\quad 1,\quad e^{2z_1}z_2^2;$$
$$r = 5,\quad k = 3,\quad s = 0. \tag{9.2}$$

9-2 The Lie Group K_5

K_5 is the 5-dimensional complex Lie group with elements

$$g(q, a, b, c, \tau), \qquad q, a, b, c, \tau \in \mathcal{C},$$

and multiplication law

$$g(q, a, b, c, \tau)\, g(q', a', b', c', \tau')$$
$$= g(q + e^{2\tau}q', a + a' + e^{\tau}cb', b + e^{\tau}b' + 2e^{2\tau}cq', c + e^{-\tau}c', \tau + \tau'). \tag{9.3}$$

In particular the identity element of K_5 is $g(0, 0, 0, 0, 0)$ and the inverse of $g(q, a, b, c, \tau)$ is

$$g(-qe^{-2\tau}, -a + bc - 2c^2q, -be^{-\tau} + 2cqe^{-\tau}, -ce^{\tau}, -\tau).$$

The associative law can be verified directly. This group has the 5×5 matrix realization

$$g(q, a, b, c, \tau) \equiv \begin{pmatrix} 1 & ce^{\tau} & be^{-\tau} & 2a - bc & \tau \\ 0 & e^{\tau} & 2qe^{-\tau} & b - 2qc & 0 \\ 0 & 0 & e^{-\tau} & -c & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{9.4}$$

where now the group operation is matrix multiplication. It is clear from (9.3) that the set of all group elements with $q = 0$ forms a subgroup of K_5 isomorphic to $G(0, 1)$ (compare with (4.11)).

A simple computation using (9.3) or (9.4) shows that the Lie algebra of K_5 is isomorphic to $\mathscr{K}_5$. In fact we can make the identification

$$g(q, a, b, c, \tau) = \exp(q\mathscr{Q}) \exp(a\mathscr{E}) \exp(b\mathscr{J}^+) \exp(c\mathscr{J}^-) \exp(\tau\mathscr{J}^3) \tag{9.5}$$

where the elements $\mathscr{J}^{\pm}$, $\mathscr{J}^3$, $\mathscr{E}$, $\mathscr{Q}$ generate $\mathscr{K}_5$ and satisfy the commutation relations (9.1). Equation (9.5) uniquely determines K_5 as a local Lie group. Moreover, as a global group K_5 is simply connected.

9-3 Representations of $\mathscr{K}_5$

Rather than try to give a complete analysis of the representation theory of $\mathscr{K}_5$ we restrict ourselves to irreducible representations which can be realized by elements of $\mathscr{A}(\tilde{\alpha}_1)$ acting on spaces of analytic functions. For such representations it will be easy to compute the matrix elements. In particular, we study the irreducible representations ρ of $\mathscr{K}_5$ satisfying property A: $\rho/\mathscr{G}(0, 1)$ (ρ restricted to the subalgebra $\mathscr{G}(0, 1)$) is isomorphic to one of the irreducible representations $R(\omega, m_o, \mu)$ or $\uparrow_{\omega,\mu}$ of $\mathscr{G}(0,1)$. Such representations will yield additional information about special functions associated with $\mathscr{G}(0,1)$.

Let ρ be an abstract irreducible representation of $\mathscr{K}_5$ on a vector space V which satisfies the above requirement and let

$$\rho(\mathscr{J}^{\pm}) = \mathrm{J}^{\pm}, \qquad \rho(\mathscr{J}^{3}) = \mathrm{J}^{3}, \qquad \rho(\mathscr{E}) = \mathrm{E}, \qquad \rho(\mathscr{Q}) = \mathrm{Q}$$

be operators on V. These operators clearly satisfy the commutation relations

$$[\mathrm{J}^3, \mathrm{J}^{\pm}] = \pm\mathrm{J}^{\pm}, \qquad [\mathrm{J}^3, \mathrm{Q}] = 2\mathrm{Q}, \qquad [\mathrm{J}^-, \mathrm{J}^+] = \mathrm{E},$$
$$[\mathrm{J}^-, \mathrm{Q}] = 2\mathrm{J}^+, \qquad [\mathrm{J}^+, \mathrm{Q}] = 0, \qquad [\mathrm{J}^{\pm}, \mathrm{E}] = [\mathrm{J}^3, \mathrm{E}] = [\mathrm{Q}, \mathrm{E}] = 0.$$

Theorem 9.1 Every irreducible representation ρ of $\mathscr{K}_5$ satisfying property A is isomorphic to a representation in the following list:

(1) $R'(\omega, m_o, \mu)$, $\quad \omega, m_o, \mu \in \mathscr{C}$, $\quad \mu \neq 0$,

$0 \leqslant \operatorname{Re} m_o < 1$, $\quad \omega + m_o$ not an integer.

The spectrum of J^3 is the set $S = \{m_o + n\colon n \text{ an integer}\}$.

(2) $\uparrow'_{\omega,\mu}$ $\quad \omega, \mu \in \mathscr{C}$, $\quad \mu \neq 0$.

The spectrum of J^3 is the set $S = \{-\omega + n\colon n \text{ a nonnegative integer}\}$.

For each of the above classes the representation space V has a basis $\{f_m\}$, $m \in S$, such that

$$\begin{aligned} \mathrm{J}^3 f_m &= m f_m, \qquad \mathrm{E} f_m = \mu f_m, \qquad \mathrm{Q} f_m = \mu f_{m+2}, \\ \mathrm{J}^+ f_m &= \mu f_{m+1}, \qquad \mathrm{J}^- f_m = (m + \omega) f_{m-1} \end{aligned} \tag{9.6}$$

for all $m \in S$ on the left-hand sides of these equations. All of the irreducible representations in classes (1) and (2) satisfy property A and no two of them are isomorphic.

The proof of this theorem is elementary and is left to the reader. We could of course find many other representations of $\mathscr{K}_5$, but for purposes of illustration, the representations listed in Theorem 9.1 will suffice.

9-4 The Representation $R'(\omega, m_o, \mu)$

We will find a realization of $R'(\omega, m_o, \mu)$ given by some $\tau' \in \mathcal{O}\!\ell(\tilde{\alpha}_1)$ acting on a vector space $\mathscr{V}_1'$ of analytic functions of z. According to the theory of Section 8-2, the space of cohomology classes of $\mathcal{O}\!\ell(\tilde{\alpha}_1)$ is 2-dimensional. Moreover, every element of $\mathcal{O}\!\ell(\tilde{\alpha}_1)$ is cohomologous to a realization τ' of the form

$$\begin{gathered} \mathrm{J}^3 = z\frac{d}{dz} + c_1, \qquad \mathrm{J}^+ = c_2 z, \qquad \mathrm{J}^- = \frac{d}{dz}, \\ \mathrm{E} = c_2, \qquad \mathrm{Q} = c_2 z^2; \qquad c_1, c_2 \in \mathcal{C}. \end{gathered} \tag{9.7}$$

Two realizations of the form (9.7) are cohomologous if and only if they are identical.

To construct a realization of $R'(\omega, m_o, \mu)$ in terms of the operators (9.7) let $\mathscr{V}_1'$ be the space of all finite linear combinations of the functions $f_m'(z) = z^{m+\omega}$, defined for all $m \in S$ where S is the spectrum of the representation, and let $c_1 = -\omega$, $c_2 = \mu$. Then

$$\begin{gathered} \mathrm{J}^3 f'_m = \left(z\frac{d}{dz} - \omega\right) z^{m+\omega} = mz^{m+\omega} = mf'_m, \\ \mathrm{J}^+ f'_m = \mu z^{m+1+\omega} = \mu f'_{m+1}, \qquad \mathrm{J}^- f'_m = \frac{d}{dz} z^{m+\omega} = (m+\omega) f'_{m-1}, \\ \mathrm{E} f'_m = \mu z^{m+\omega} = \mu f'_m, \qquad \mathrm{Q} f'_m = \mu z^{m+\omega+2} = \mu f'_{m+2} \end{gathered} \tag{9.8}$$

for all $m \in S$. Comparison of these expressions with (9.6) yields a realization of $R'(\omega, m_o, \mu)$ in terms of linear differential operators.

Since $m + \omega$ is not an integer the functions $f_m'(z) = z^{m+\omega}$ are not analytic and single-valued in a neighborhood of $z = 0$. This is very annoying for computational purposes. To remedy the defect we map the space $\mathscr{V}_1'$ onto the space $\mathscr{V}_1 = \boldsymbol{\varphi}^{-1}(\mathscr{V}_1')$ where $\boldsymbol{\varphi}^{-1}[f'](z) = (\varphi(z))^{-1} f'(z) \in \mathscr{V}_1$, $f' \in \mathscr{V}_1'$, and $\varphi(z) = z^{m_o+\omega}$. Then $\mathscr{V}_1$ has a basis of the form $f_m(z) = \boldsymbol{\varphi}^{-1}[f_m'](z) = z^k$ where $k = m - m_o$ is an integer. $\boldsymbol{\varphi}$ induces a transformation $\tau = \boldsymbol{\varphi}^{-1}\tau'\boldsymbol{\varphi}$ mapping a realization τ' of $\mathscr{K}_5$ by gd's on $\mathscr{V}_1'$ into a realization τ by gd's on $\mathscr{V}_1$; see (8.3). Under this transformation the operators (9.7) with $c_1 = -\omega$, $c_2 = \mu$ are mapped into the operators

$$J^3 = z\frac{d}{dz} + m_o, \qquad J^+ = \mu z, \qquad J^- = \frac{d}{dz} + m_o + \omega,$$
$$E = \mu, \qquad Q = \mu z^2 \tag{9.9}$$

on $\mathscr{V}_1$. The operators (9.9) and the basis functions $f_m(z) = z^k$, $m = m_o + k$, provide the desired realization of $R'(\omega, m_o, \mu)$. (Note that with the exception of $Q = \mu z^2$, these operators and basis functions are identical with those derived for a realization of the representation $R(\omega, m_o, \mu)$ of $\mathscr{G}(0, 1)$ in Section 4-1.)

Following our usual procedure we can extend this realization of $R'(\omega, m_o, \mu)$ defined on $\mathscr{V}_1$ to a local multiplier representation of K_5 on $\mathscr{A}_1$, where $\mathscr{A}_1$ is the complex vector space of all functions of z analytic in some neighborhood of the point $z = 1$. Here $\mathscr{A}_1 \supset \mathscr{V}_1$ and $\mathscr{A}_1$ is invariant under the operators (9.9).

The operators $\mathbf{A}(g)$, $g = g(q, a, b, c, \tau) \in K_5$, defining the multiplier representation can be computed in a straightforward manner from expressions (9.5) and (9.9). The result is

$$[\mathbf{A}(g)f](z) = \exp[\mu(qz^2 + bz + a) + m_o\tau](1 + c/z)^{m_o+\omega} f(e^\tau z + e^\tau c),$$
$$f \in \mathscr{A}_1, \quad |c/z| < 1. \tag{9.10}$$

Just as in Section 4-1 we can show that every function $h \in \mathscr{V}_1 \subset \mathscr{A}_1$ has a unique Laurent expansion

$$h(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

which converges absolutely for all $|z| > d_n$ where $1 > d_h \geqslant 0$. ($\mathscr{V}_1$ is invariant under $\mathbf{A}(g)$, see Section 2-2.) Thus, the functions $f_m(z) = h_k(z) = z^k$, $m = m_o + k$, in $\mathscr{V}_1$ form an analytic basis for $\mathscr{V}_1$. The matrix elements $A_{lk}(g)$ of the operators $\mathbf{A}(g)$ with respect to this basis are defined by

$$[\mathbf{A}(g)h_k](z) = \sum_{l=-\infty}^{\infty} A_{lk}(g)h_l(z), \qquad k = 0, \pm 1, \pm 2, \ldots,$$

or

$$\exp[\mu(qz^2 + bz + a) + (m_o + k)\tau](1 + c/z)^{m_o+\omega+k} z^k$$
$$= \sum_{l=-\infty}^{\infty} A_{lk}(g)z^l, \qquad |c/z| < 1. \tag{9.11}$$

Since the operators $\mathbf{A}(g)$ define a local multiplier representation of K_5 we obtain the addition theorem

$$A_{lk}(g_1g_2) = \sum_{j=-\infty}^{\infty} A_{lj}(g_1)A_{jk}(g_2), \qquad l, k = 0, \pm 1, \pm 2, \ldots, \tag{9.12}$$

valid for g_1, g_2 in a sufficiently small neighborhood of the identity element. To be precise we should determine the exact domain of validity of (9.12), just as in Section 4-1. However, it will soon be evident that both sides of this equation are entire analytic functions of the group parameters of g_1 and g_2. Thus, by analytic continuation equation (9.12) holds without restriction for all $g_1, g_2 \in K_5$.

The matrix elements $A_{lk}(g)$ can be evaluated directly from the generating function (9.11). We examine some special cases:

(1) $g(0, a, b, c, \tau)$ $(q = 0)$. In this case (9.11) is identical with the generating function (4.8). Thus the matrix elements are proportional to the Laguerre functions,

$$A_{lk}(g) = e^{\mu a + (m_o + k)\tau} c^{k-l} L_{\rho+l}^{(k-l)}(-\mu bc) \tag{9.13}$$

where $\rho = m_o + \omega$ is not an integer.

(2) $g(q, 0, b, 0, 0)$. The generating function becomes

$$\exp[\mu(qz^2 + bz)]\, z^k = \sum_{l=-\infty}^{\infty} A_{lk}(g) z^l.$$

Comparing this expression with the generating function (4.77) for the Hermite polynomials we find

$$A_{lk}(g) = \begin{cases} 0 & \text{if } \ l < k, \\ \dfrac{(-\mu q)^{(l-k)/2}}{(l-k)!} H_{l-k}\left(\dfrac{\mu b}{2(-\mu q)^{1/2}}\right) & \text{if } \ l \geqslant k. \end{cases} \tag{9.14}$$

Here, the matrix elements are entire functions of μ and q.

(3) $g(q, 0, 0, c, 0)$. The generating function is

$$\exp(\mu q z^2)(1 + c/z)^\rho = \sum_{n=-\infty}^{\infty} R_n^\rho(\mu q, c) z^n$$

where $R_{l-k}^{\rho+k}(\mu q, c) = A_{lk}(g)$. Thus,

$$R_{l-k}^{\rho+k}(\mu q, c) = c^{k-l}\Gamma(\rho + k + 1) \sum_j \frac{(\mu q c^2)^j}{j!(2j + k - l)!\, \Gamma(\rho + l - 2j + 1)}, \tag{9.15}$$

where j ranges over all integral values such that the summand makes sense. There is no simple expression for the functions R_n^ρ in terms of functions of hypergeometric type.

Various identities relating these matrix elements can be obtained from the addition theorem (9.12). (We have already derived identities for the matrix elements (9.13) in Section 4-1.) Thus, the relation

$$g(q, 0, b, 0, 0)\, g(q', 0, b', 0, 0) = g(q + q', 0, b + b', 0, 0)$$

implies (after some simplification) the identity

$$\frac{(q+q')^{n/2}}{n!} H_n\left(\frac{b+b'}{2(q+q')^{1/2}}\right) = \sum_{m=0}^{n} \frac{q^{(n-m)/2}}{(n-m)!} H_{n-m}\left(\frac{b}{2\sqrt{q}}\right) \cdot \frac{q'^m}{m!} H_m\left(\frac{b'}{2\sqrt{q'}}\right). \qquad (9.16)$$

The relation

$$g(0, 0, 0, c, 0)\, g(-b/2c, 0, b, 0, 0) = g(-b/2c, bc, 0, c, 0)$$

implies

$$\exp(2bc^2)\, R_{l-k}^{\rho+k}(-b, c) = \sum_{j=\max(l,k)}^{\infty} c^{j-l} \binom{\rho+j}{j-l} \frac{b^{(j-k)/2}}{(j-k)!} H_{j-k}\left(\frac{bc}{\sqrt{b}}\right). \qquad (9.17)$$

There are many other identities contained in (9.12), but the reader can derive them for himself.

In analogy with Section 4-6 we can construct a realization of $R'(\omega, m_o, \mu)$ by gd's on a space of two complex variables where the action of $\mathscr{K}_5$ on this space is given by a member of $\mathcal{O}(\tilde{\beta}_1)$, (9.2). (The elements of $\mathcal{O}(\tilde{\beta}_1)$ when restricted to $\mathscr{G}(0, 1)$ are identical with the *type D'* operators.) From the theory of Section 8-2 it is easy to show that the space of cohomology classes of $\mathcal{O}(\tilde{\beta}_1)$ is 2-dimensional and that every element of this space is cohomologous to a realization of the form

$$J^3 = \frac{\partial}{\partial y}, \quad J^+ = e^y\left(\frac{\partial}{\partial x} - \mu x\right), \quad J^- = -e^{-y}\frac{\partial}{\partial x},$$
$$E = \mu, \quad Q = e^{2y}\left(-x\frac{\partial}{\partial x} - \frac{\partial}{\partial y} + \mu x^2 + c\right). \qquad (9.18)$$

Two realizations of this form are cohomologous if and only if they are identical. Note: The selection of a representative in each cohomology class given by (9.18) differs on the subalgebra $\mathscr{G}(0, 1)$ from that chosen in Section 4-6. The representative selected here has been chosen for convenience in the computations to follow.

To construct a realization of $R'(\omega, m_o, \mu)$ using the operators (9.18) we must find nonzero functions $f_m(x, y) = Z_m(x)\, e^{my}$ such that Eqs. (9.6) are valid for all $m \in S = \{m_o + n\colon n \text{ an integer}\}$. In terms of the functions $Z_m(x)$ these relations become

$$\begin{aligned}
&\text{(i)} \quad \left(\frac{d}{dx} - \mu x\right) Z_m(x) = \mu Z_{m+1}(x),\\
&\text{(ii)} \quad -\frac{d}{dx} Z_m(x) = (m + \omega)\, Z_{m-1}(x), \qquad (9.19)\\
&\text{(iii)} \quad \left(-x\frac{d}{dx} + \mu x^2 + c - m\right) Z_m(x) = \mu Z_{m+2}(x).
\end{aligned}$$

Just as in Section 4-6 we can assume $\omega = 0$, $\mu = 1$ without any loss of generality for special function theory. Furthermore, a simple computation shows (i), (ii), and (iii) are compatible if and only if $c = -1$. Thus our problem reduces to finding a realization of $R'(0, m_o, 1)$ in terms of the operators (9.18) where $c = -1$. In this case Eqs. (9.19) have the following linearly independent solutions for all $m \in S$:

$$\begin{aligned}
&(1) \quad Z_m(x) = (-1)^{m-m_o}\, 2^{-m/2} H_m(x/\sqrt{2}) = (-1)^{m-m_o} \exp(x^2/4)\, D_m(x),\\
&(2) \quad Z_m(x) = \exp(x^2/2)\, e^{-i\pi(m+1)/2}\Gamma(m+1)\, 2^{(m+1)/2} H_{-m-1}(ix/\sqrt{2})\\
&\qquad\qquad = \exp(x^2/4)\, e^{-i\pi(m+1)/2}\Gamma(m+1)\, D_{-m-1}(ix). \qquad (9.20)
\end{aligned}$$

The $H_m(x)$ are Hermite functions defined in terms of parabolic cylinder functions by $H_m(x) = 2^{m/2} \exp x^2/2 D_m(\sqrt{2}x)$. When m is a positive integer, which is not the case here, $H_m(x)$ is a Hermite polynomial. The fact that expressions (9.20) satisfy the recursion relations (i) and (ii) follows easily from (4.66) and (4.67). Relation (iii) can be derived from (i) and (ii). These solutions are entire functions of x.

Clearly, if the functions Z_m, $m \in S$, are given by either (1) or (2), then the functions $f_m(x, y) = Z_m(x)\, e^{my}$ form an analytic basis for a realization of the representation $R'(0, m_o, 1)$ of $\mathscr{K}_5$. As usual this representation of $\mathscr{K}_5$ can be extended to a local multiplier representation of K_5 by operators $\mathbf{T}(g)$, $g \in K_5$, on the space $\mathscr{F}$ of all functions analytic in a neighborhood of the point $(x^o, y^o) = (0, 0)$. A straightforward computation using (9.5) and (9.18) yields

$$\begin{aligned}
[\mathbf{T}(g)f](x, t) = (1 + 2t^2q)^{-1/2} \exp\left[\frac{x^2t^2q - txb - b^2t^2/2}{1 + 2t^2q} + a\right]\\
\cdot f\left(\frac{x + bt - ct^{-1} - 2ctq}{(1 + 2t^2q)^{1/2}}, \frac{te^{\tau}}{(1 + 2t^2q)^{1/2}}\right),\\
f \in \mathscr{F}, \qquad (9.21)
\end{aligned}$$

where $t = e^y$.

The matrix elements of the operators $\mathbf{T}(g)$ with respect to a basis $f_m(x, t) = Z_m(x)\, t^m$ satisfying (9.19) are given by the functions $A_{lk}(g)$, (9.11), where $\omega = 0$, $\mu = 1$. Thus we obtain the relations

$$[\mathbf{T}(g) f_{m_0+k}](x, t) = \sum_{l=-\infty}^{\infty} A_{lk}(g) f_{m_0+l}(x, t).$$

Corresponding to the basis

$$f_{m_0+k}(x, t) = (-1)^k\, 2^{-(m_0+k)/2}\, H_{m_0+k}(x/\sqrt{2})\, t^{m_0+k}$$

for all integers k, this equation can be written in the form

$$\exp\left[\frac{x^2t^2q - txb - b^2t^2/2}{1 + 2t^2q} + a\right] H_{m_0+k}\left(\frac{x + bt - ct^{-1} - 2ct\,q}{(2(1 + 2t^2q))^{1/2}}\right)$$
$$\cdot \frac{(-t/\sqrt{2})^k}{(1 + 2t^2q)^{(m_0+k+1)/2}}\, e^{(m_0+k)\tau} = \sum_{l=-\infty}^{\infty} A_{lk}(g)\, H_{m_0+l}\left(\frac{x}{\sqrt{2}}\right)\left(-\frac{t}{\sqrt{2}}\right)^l. \tag{9.22}$$

For $q = 0$, (9.22) is identical with (4.70) so we will omit this case.

If $a = c = \tau = 0$, $q = -\frac{1}{2}$, (9.22) reduces (after some simplification) to the identity

$$(1 - t^2)^{-(m+1)/2} \exp\left[\frac{2xtb - (x^2 + b^2)\,t^2}{1 - t^2}\right] H_m\left(\frac{x - bt}{(1 - t^2)^{1/2}}\right)$$
$$= \sum_{n=0}^{\infty} \frac{t^n 2^{-n}}{n!} H_n(b)\, H_{m+n}(x), \qquad |\, t\,| < 1, \tag{9.23}$$

valid for all $m \in \mathscr{C}$ not an integer. (In the next section we will show that (9.23) is also valid for m a nonnegative integer.)

If $a = b = \tau = 0$, $\quad q = -\frac{1}{2}$, (9.22) simplifies to

$$(1 - t^2)^{-(m+1)/2} \exp\left[\frac{-x^2t^2}{1 - t^2}\right] H_m\left(\frac{x + c(t + t^{-1})}{(1 - t^2)^{1/2}}\right)$$
$$= \sum_{n=-\infty}^{\infty} h_n^m(c)\, H_{m+n}(x)\, t^n, \qquad |\, t\,| < 1,$$

where $h_n^m(c) = (-\sqrt{2})^{-n}\, R_n^m(-\frac{1}{2}, \sqrt{2}c)$ has the generating function

$$\exp(-z^2/4)(1 - 2c/z)^m = \sum_{n=-\infty}^{\infty} h_n^m(c)\, z^n, \qquad |\, 2c/z\,| < 1.$$

Here m is not an integer. However, in the next section we will show that this equation remains true for m a nonnegative integer.

Similarly, the second set of basis vectors (9.20) can be used to derive identities for the Hermite functions. For example the case $a = c = \tau = 0$, $q = -\frac{1}{2}$, yields the identity

$$(1 + t^2)^{m/2} H_m\left(\frac{x + bt}{(1 + t^2)^{1/2}}\right) = \sum_{n=0}^{\infty} \binom{m}{n} H_n(b)\, H_{m-n}(x)\, t^n, \qquad |t| < 1,$$

a generalization of (9.16).

By substituting appropriate choices for the group parameter into (9.22) the reader can derive additional identities obeyed by the Hermite functions.

9-5 The Representation $\uparrow'_{\omega,\mu}$

In analogy with the procedure of the last section we look for a realization of the representation $\uparrow'_{\omega,\mu}$ given by some $\tau \in \mathcal{O}(\tilde{\alpha}_1)$ acting on a vector space of analytic functions of z. A comparison of expressions (4.17) and (9.7) shows how to proceed. Namely, in (9.7) we set $c_1 = -\omega$, $c_2 = \mu$ to obtain

$$\mathrm{J}^3 = z\frac{d}{dz} - \omega, \qquad \mathrm{J}^+ = \mu z, \qquad \mathrm{J}^- = \frac{d}{dz}, \qquad \mathrm{E} = \mu, \qquad \mathrm{Q} = \mu z^2. \tag{9.24}$$

Designate by $\mathscr{V}_2$ the space of all finite linear combinations of the functions $h_k(z) = z^k$, $k = 0, 1, 2, \ldots$, and define the basis vectors f_m of $\mathscr{V}_2$ by $f_m(z) = h_k(z) = z^k$ where $m = -\omega + k$, $k \geqslant 0$. Clearly,

$$\begin{gathered}
\mathrm{J}^3 f_m = \left(-\omega + z\frac{d}{dz}\right) z^k = m f_m, \qquad \mathrm{J}^+ f_m = (\mu z)\, z^k = \mu f_{m+1}, \\
\mathrm{J}^- f_m = \frac{d}{dz} z^k = (m + \omega) f_{m-1}, \qquad \mathrm{E} f_m = \mu f_m, \\
\mathrm{Q} f_m = (\mu z^2)\, z^k = \mu f_{m+2}.
\end{gathered} \tag{9.25}$$

These relations define a realization of $\uparrow'_{\omega,\mu}$ on $\mathscr{V}_2$.

Moreover, this realization can easily be extended to a local multiplier representation B of K_5 on the space $\mathcal{O}_2$ consisting of entire functions of z. The operators $\mathbf{B}(g)$, $g = g(q, a, b, c, \tau) \in K_5$, defining this representation are easily computed:

$$[\mathbf{B}(g) f](z) = \exp[\mu(qz^2 + bz + a) - \omega\tau]\, f(e^\tau z + e^\tau c), \qquad f \in \mathcal{O}_2. \tag{9.26}$$

Furthermore it is straightforward to show that the operators $\mathbf{B}(g)$ form a global representation of K_5. The matrix elements $B_{lk}(g)$ of $\mathbf{B}(g)$ with respect to the basis $\{f_m = h_k\}$ are given by

$$[\mathbf{B}(g)\,h_k](z) = \sum_{l=0}^{\infty} B_{lk}(g)\,h_l(z), \qquad k = 0, 1, 2,...,$$

or

$$\exp[\mu(qz^2 + bz + a) + (k - \omega)\,\tau](z + c)^k = \sum_{l=0}^{\infty} B_{lk}(g)\,z^l. \tag{9.27}$$

The addition theorem for the matrix elements is

$$B_{lk}(g_1 g_2) = \sum_{j=0}^{\infty} B_{lj}(g_1)\,B_{jk}(g_2), \qquad l, k \geqslant 0, \tag{9.28}$$

valid for all $g_1, g_2 \in K_5$. Corresponding to some special choices of the group parameters the matrix elements have the following explicit expressions:

(1) $g(0, a\, b, c, \tau)$. For $q = 0$, (9.27) becomes identical with the generating function (4.22). The matrix elements are thus proportional to associated Laguerre polynomials.

$$B_{lk}(g) = e^{\mu a + (k-\omega)\tau} c^{k-l} L_l^{(k-l)}(-\mu bc). \tag{9.29}$$

(2) $g(q, 0, b, 0, 0)$. Exactly as in (9.14) we find

$$B_{lk}(g) = \begin{cases} 0 & \text{if} \quad 0 \leqslant l \leqslant k, \\ \dfrac{(-\mu q)^{(l-k)/2}}{(l-k)!}\, H_{l-k}\left(\dfrac{\mu b}{2(-\mu q)^{1/2}}\right) & \text{if} \quad l \geqslant k \geqslant 0. \end{cases}$$

(3) $g(q, 0, 0, c, 0)$. In this case (9.27) reduces to

$$\exp(\mu q z^2)(z + c)^k = \sum_{l=0}^{\infty} R^k_{l-k}(\mu q, c)\,z^l \tag{9.30}$$

where $R^k_{l-k}(\mu q, c) = B_{lk}(g)$. Thus,

$$R^k_{l-k}(\mu q, c) = c^{k-l} k! \sum_j \frac{(\mu q c^2)^j}{j!(2j + k - l)!\,(l - 2j)!} \tag{9.31}$$

where j ranges over all integral values such that the summand is defined. Comparing (9.30) with the generating function (4.77) for the Hermite polynomials we find

$$e^{-bc}\,\frac{(q)^{l/2}}{l!}\,H_l\left(\frac{-b}{2\sqrt{q}}\right) = \sum_{k=0}^{\infty} \frac{(\mu b)^k}{k!}\,R^k_{l-k}(\mu q, c). \tag{9.32}$$

Just as in the last section, the addition theorem (9.28) can be used to derive identities obeyed by the matrix elements. This task will be left to the reader.

Armed with a knowledge of the matrix elements of $\uparrow'_{\omega,\mu}$ we now procede to construct a realization of this representation by gd's on a space of two complex variables, where the action of $\mathscr{K}_5$ is given by a member of $\mathcal{O}(\tilde{\beta}_1)$, (9.2). In particular we will use the operators (9.18) with $c = -\mu$. To construct a realization of $\uparrow'_{\omega,\mu}$ using these operators it is necessary to find nonzero functions $f_m(x, y) = Z_m(x)\, e^{my}$, $m = -\omega, -\omega + 1, \ldots$, such that Eqs. (9.6) are valid with

$$\mathrm{J}^3 = \frac{\partial}{\partial y}, \qquad \mathrm{J}^+ = e^y\left(\frac{\partial}{\partial x} - \mu x\right), \qquad \mathrm{J}^- = -e^{-y}\,\frac{\partial}{\partial x},$$

$$\mathrm{E} = \mu, \qquad \mathrm{Q} = e^{2y}\left(-x\,\frac{\partial}{\partial x} - y\,\frac{\partial}{\partial y} + \mu x^2 - \mu\right).$$

Just as in the derivation of Eqs. (9.19), it is easy to show that without loss of generality for special function theory we can assume $\omega = 0$, $\mu = 1$. Then, expressed in terms of the functions $Z_m(x)$, Eqs. (9.6) become the recursion relations

$$\begin{aligned}
&\text{(i)}\quad \left(\frac{d}{dx} - x\right) Z_m(x) = Z_{m+1}(x),\\
&\text{(ii)}\quad -\frac{d}{dx}\, Z_m(x) = mZ_{m-1}(x),\\
&\text{(iii)}\quad \left(-x\,\frac{d}{dx} + x^2 - 1 - m\right) Z_m(x) = Z_{m+2}(x),
\end{aligned} \tag{9.33}$$

where m takes the values $0, 1, 2, \ldots$. The functions $Z_m(x)$ are determined to within an arbitrary constant by (9.33). In fact relation (ii) implies $Z_0(x) = c$ for some constant c. If we set $c = 1$, the remaining solutions $Z_m(x)$ are uniquely determined:

$$Z_m(x) = (-1)^m\, 2^{-m/2} H_m(x/\sqrt{2}) = (-1)^m \exp(x^2/4)\, D_m(x), \tag{9.34}$$

where the $H_m(x)$ are Hermite polynomials. Thus the vectors $f_m(x, y) = Z_m(x)\, e^{my}$, $m = 0, 1, 2,\ldots$, where $Z_m(x)$ is given by (9.34), form a basis for a realization of the representation $\uparrow'_{0,1}$ of $\mathscr{K}_5$.

This infinitesimal representation of $\mathscr{K}_5$ can be extended to a local multiplier representation of K_5 by operators $\mathbf{T}(g)$, $g \in K_5$, on the space $\mathscr{F}$ of all functions analytic in a neighborhood of the point $(x^o, y^o) = (0, 0)$. Indeed we have already computed the operators $\mathbf{T}(g)$ on $\mathscr{F}$, Eq. (9.21).

The matrix elements of $\mathbf{T}(g)$ with respect to an analytic basis $\{f_m(x, y)\}$ satisfying (9.33) are the functions $B_{lk}(g)$ given by (9.27) ($\omega = 0$, $\mu = 1$). Consequently,

$$[\mathbf{T}(g) f_k](x, y) = \sum_{l=0}^{\infty} B_{lk}(g) f_l(x, y), \qquad k = 0, 1, 2,\ldots,$$

or

$$\exp\left[\frac{x^2t^2q - txb - b^2t^2/2}{1 + 2t^2q} + a\right] H_k\left(\frac{x + bt + ct^{-1} - 2ctq}{(2(1 + 2t^2q))^{1/2}}\right)$$

$$\cdot \frac{(-t/\sqrt{2})^k\, e^{k\tau}}{(1 + 2t^2q)^{(k+1)/2}} = \sum_{l=0}^{\infty} B_{lk}(g)\, H_l\left(\frac{x}{\sqrt{2}}\right)\left(-\frac{t}{\sqrt{2}}\right)^l. \tag{9.35}$$

For $q = 0$ this expression is identical with (4.76). For $a = c = \tau = 0$, $q = -\frac{1}{2}$, it reduces to (9.23) where now m is a nonnegative integer. In the special case $n = 0$, (9.23) becomes

$$(1 - t^2)^{-1/2} \exp\left[\frac{2xtb - (x^2 + b^2)\, t^2}{1 - t^2}\right] = \sum_{n=0}^{\infty} \frac{t^n 2^{-n}}{n!} H_n(b)\, H_n(x), \qquad |\, t\,| < 1, \tag{9.36}$$

which is Mehler's theorem, (4.158).

Finally, if $a = b = \tau = 0$, $q = -\frac{1}{2}$, (9.35) reduces to the identity

$$(1 - t^2)^{-(k+1)/2} \exp\left[\frac{-x^2t^2}{1 - t^2}\right] H_k\left(\frac{x + c(t + t^{-1})}{(1 - t^2)^{1/2}}\right)$$

$$= \sum_{l=0}^{\infty} h^k_{l-k}(c)\, H_l(x)\, t^{l-k}, \tag{9.37}$$

where $h^k_{l-k}(c) = (-\sqrt{2})^{k-l} R^k_{l-k}(-\frac{1}{2}, \sqrt{2}c)$ is given by the generating function

$$\exp(-z^2/4)(z - 2c)^k = \sum_{l=0}^{\infty} h^k_{l-k}(c)\, z^l.$$

This completes our analysis of the special function theory of $\mathscr{K}_5$. Note that the identities for Hermite functions derived here have been obtained from a study of the operator realizations α_1 and β_1, (8.28) and (9.2). We could also have used β_5, but this would not have led to any new results. The operators β_2, β_3, β_4, on the other hand, do not yield realizations of the representations $R'(\omega, m_o, \mu)$ or $\uparrow'_{\omega,\mu}$, but rather lead to realizations of representations of $\mathscr{K}_5$ which we have not classified here. In particular, β_2 can be used to realize certain **reducible** representations in such a way that the basis vectors turn out to be proportional to generalized Laguerre functions. Thus a study of β_2 leads to new identities for Laguerre functions.

9-6 The Lie Algebra $\mathscr{G}_{p,q}$

Corresponding to a pair of positive integers (p, q) let $\mathscr{G}_{p,q}$ be the complex 3-dimensional Lie algebra with basis $\mathscr{J}^3$, $\mathscr{J}^+$, $\mathscr{J}^-$ and commutation relations

$$[\mathscr{J}^3, \mathscr{J}^+] = p\mathscr{J}^+, \quad [\mathscr{J}^3, \mathscr{J}^-] = -q\mathscr{J}^-, \quad [\mathscr{J}^+, \mathscr{J}^-] = \mathscr{O}. \tag{9.38}$$

Clearly, $\mathscr{G}_{1,1}$ is isomorphic to $\mathscr{T}_3$. In addition the following isomorphisms are easily established:

Lemma 9.1 $\mathscr{G}_{p,q} \cong \mathscr{G}_{q,p}$; $\mathscr{G}_{np,nq} \cong \mathscr{G}_{p,q}$; for all positive integers p, q, n.

According to this lemma every Lie algebra $\mathscr{G}_{p',q'}$ is isomorphic to a Lie algebra $\mathscr{G}_{p,q}$ such that (1) p and q are relatively prime positive integers; (2) p is odd; and (3) if q is odd then $p > q$. Consequently, from now on the pair (p, q) will be assumed to satisfy properties (1)–(3). It is obvious that two Lie algebras $\mathscr{G}_{p,q}$, $\mathscr{G}_{p',q'}$ with subscripts satisfying these properties, are isomorphic if and only if $p = p'$ and $q = q'$.

Applying the techniques developed in Chapter 8 we can determine all of the transitive effective realizations of $\mathscr{G}_{p,q}$ by gd's in one or two complex variables. Thus, from (8.26) there follows the realization

$$\gamma_1: \ \frac{\partial}{\partial y}, \ e^{py}, \ e^{-qy}; \qquad r = 3, \ k = 2, \ s = 0. \tag{9.39}$$

Every transitive effective realization of $\mathscr{G}_{p,q}$ by gd's in one complex variable is an element of $\mathscr{O}\!\!\iota(\tilde{\gamma}_1)$.

Similarly, by making use of Lie's tables (Lie [1], Vol. III, pp. 71–73), it can be shown that every transitive effective realization of $\mathscr{G}_{p,q}$ by gd's in two complex variables is an element of $\mathscr{O}(\delta_j)$, $j = 1, 2, 3$, where

$$\begin{aligned}
&\delta_1: \quad \frac{\partial}{\partial z_1}, \quad e^{pz_1}\frac{\partial}{\partial z_2}, \quad e^{-qz_1}\frac{\partial}{\partial z_2}; && r = 3, \quad k = 1, \quad s = 0;\\
&\delta_2: \quad -pz_1\frac{\partial}{\partial z_1} + qz_2\frac{\partial}{\partial z_2}, \quad \frac{\partial}{\partial z_1}, \quad \frac{\partial}{\partial z_2}; && r = 3, \quad k = 1, \quad s = 0; \qquad (9.40)\\
&\delta_3: \quad \frac{\partial}{\partial z_1}, \quad e^{pz_1}\frac{\partial}{\partial z_2}, \quad e^{-qz_1}; && r = 3, \quad k = 1, \quad s = 0.
\end{aligned}$$

9-7 The Lie Group $G_{p,q}$

Corresponding to a pair of relatively prime positive integers (p, q) such that p is odd, denote by $G_{p,q}$ the 3-dimensional complex Lie group with elements

$$g(b, c, \tau), \qquad b, c, \tau \in \mathscr{C},$$

and group multiplication

$$g(b_1, c_1, \tau_1)\, g(b_2, c_2, \tau_2) = g(b_1 + e^{p\tau_1}b_2, c_1 + e^{-q\tau_1}c_2, \tau_1 + \tau_2). \qquad (9.41)$$

It is easy to check that the multiplication is associative. Furthermore, $\mathbf{e} = g(0, 0, 0)$ is the identity element and $g(-e^{-p\tau}b, -e^{q\tau}c, -\tau)$ is the unique inverse of the group element $g(b, c, \tau)$.

$G_{p,q}$ has a 4×4 matrix realization

$$g(b, c, \tau) \equiv \begin{pmatrix} 1 & 0 & 0 & \tau \\ 0 & e^{-q\tau} & 0 & c \\ 0 & 0 & e^{p\tau} & b \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (9.42)$$

where matrix multiplication corresponds to the group operation. Comparing (9.42) with (1.35) we see that $G_{1,1}$ is isomorphic to T_3.

The Lie algebra of $G_{p,q}$ can be computed from either of the expressions (9.41) or (9.42), and is easily recognized to be isomorphic to $\mathscr{G}_{p,q}$. Indeed we can make the identification

$$g(b, c, \tau) = \exp(b\mathscr{J}^+) \exp(c\mathscr{J}^-) \exp(\tau\mathscr{J}^3) \qquad (9.43)$$

where $\mathscr{J}^\pm$, $\mathscr{J}^3$ generate $\mathscr{G}_{p,q}$ and satisfy the commutation relations (9.38).

9-8 Representations of $\mathscr{G}_{p,q}$

Let ρ be a representation of $\mathscr{G}_{p,q}$ on an abstract vector space V and let $\rho(\mathscr{J}^{\pm}) = \mathrm{J}^{\pm}$, $\rho(\mathscr{J}^3) = \mathrm{J}^3$, be operators on V. Clearly,

$$[\mathrm{J}^3, \mathrm{J}^+] = p\mathrm{J}^+, \qquad [\mathrm{J}^3, \mathrm{J}^-] = -q\mathrm{J}^-, \qquad [\mathrm{J}^+, \mathrm{J}^-] = 0. \tag{9.44}$$

Note that the operator $\mathrm{C}^{p,q} = (\mathrm{J}^+)^q(\mathrm{J}^-)^p = (\mathrm{J}^-)^p(\mathrm{J}^+)^q$ commutes with all operators $\rho(\alpha)$, $\alpha \in \mathscr{G}_{p,q}$, on V. For ρ irreducible we would expect $\mathrm{C}^{p,q}$ to be a multiple of I.

In strict analogy with our study of the representation theory of $\mathscr{T}_3$ we will classify all representations ρ satisfying the properties:

(i) ρ is irreducible

(ii) Each eigenvalue of J^3 has multiplicity equal to one. There is a countable basis for V consisting of eigenvectors of J^3. (9.45)

We give without proof the results of the straightforward classification.

Theorem 9.2 Every representation ρ of $\mathscr{G}_{p,q}$ satisfying (9.45) and for which $\mathrm{C}^{p,q} \not\equiv 0$, on V, is isomorphic to a representation of the form $Q_{p,q}(\omega, m_o)$ defined for $\omega, m_o \in \mathcal{C}$ such that $\omega \neq 0$ and $0 \leqslant \mathrm{Re}\, m_o < 1$. The spectrum of J^3 corresponding to $Q_{p,q}(\omega, m_o)$ is given by $S = \{m_o + n\colon n \text{ an integer}\}$ and there is a basis $\{f_m\}$, $m \in S$, for V such that

$$\begin{aligned} \mathrm{J}^3 f_m &= m f_m, \qquad \mathrm{J}^+ f_m = \omega f_{m+p}, \\ \mathrm{J}^- f_m &= \omega f_{m-q}, \qquad \mathrm{C}^{p,q} f_m = (\mathrm{J}^+)^q(\mathrm{J}^-)^p f_m = \omega^{p+q} f_m. \end{aligned} \tag{9.46}$$

There exist isomorphisms $Q_{p,q}(\omega, m_o) \cong Q_{p,q}(\omega', m_o')$ if and only if $m_o = m_o'$, $\omega = \omega' d$, where d is a $(p+q)$th root of unity.

In accordance with the usual procedure, we can use the operators $\mathcal{O}(\tilde{\gamma}_1)$ to construct a realization of the representation $Q_{p,q}(\omega, m_o)$. This can be done by introducing the change of variable $z = e^y$ in (9.39) and considering the operators

$$\mathrm{J}^3 = z\frac{d}{dz} + c_1, \qquad \mathrm{J}^+ = c_2 z^p, \qquad \mathrm{J}^- = c_2 z^{-q} \tag{9.47}$$

in $\mathcal{O}(\tilde{\gamma}_1)$. Thus, let $\mathscr{V}_1$ be the complex vector space of all finite linear combinations of the functions $h_n(z) = z^n$, $n = 0, \pm 1, \pm 2, \ldots$, and define operators $\mathrm{J}^{\pm}$, J^3 on $\mathscr{V}_1$ by (9.47) where $c_1 = m_o$, $c_2 = \omega$. Define the basis vectors f_m of $\mathscr{V}_1$ by $f_m(z) = h_n(z)$ where $m = m_o + n$ and n runs over the integers. Then

$$J^3 f_m = \left(z \frac{d}{dz} + m_o\right) z^n = m z^n = m f_m ,$$

$$J^+ f_m = (\omega z^p)\, z^n = \omega f_{m+p} , \qquad J^- f_m = (\omega z^{-q})\, z^n = \omega f_{m-q} , \tag{9.48}$$

$$C^{p,q} f_m = \omega^{p+q} f_m .$$

These equations agree with (9.46) and yield a realization of $Q_{p,q}(\omega, m_o)$.

The differential operators (9.47) ($c_1 = m_o$, $c_2 = \omega$) induce a multiplier representation A of $G_{p,q}$ on the space $\mathscr{A}_1$ consisting of those functions $f(z)$ which are analytic and single-valued for all $z \neq 0$. This multiplier representation is defined by operators $\mathbf{A}(g)$, $g = g(b, c, \tau) \in G_{p,q}$:

$$[\mathbf{A}(g) f](z) = \exp[\omega(b z^p + c z^{-q}) + m_o \tau]\, f(e^\tau z), \qquad f \in \mathscr{A}_1 . \tag{9.49}$$

Clearly, $\mathscr{A}_1$ is invariant under the operators $\mathbf{A}(g)$ and the group property $\mathbf{A}(g_1 g_2) = \mathbf{A}(g_1)\, \mathbf{A}(g_2)$ is valid for all $g_1, g_2 \in G_{p,q}$.

The matrix elements $A_{lk}(g)$ of $\mathbf{A}(g)$ with respect to the analytic basis $\{f_m = h_n\}$ of $\mathscr{A}_1$ are defined by

$$[\mathbf{A}(g)\, h_k](z) = \sum_{l=-\infty}^{\infty} A_{lk}(g)\, h_l(z), \qquad g \in G_{p,q} , \qquad k = 0, \pm 1, \pm 2, \ldots, \tag{9.50}$$

or

$$\exp[\omega(b z^p + c z^{-q}) + (m_o + k)\, \tau]\, z^k = \sum_{l=-\infty}^{\infty} A_{lk}(g)\, z^l \tag{9.51}$$

where $g = g(b, c, \tau)$. Explicitly,

$$A_{lk}(g) = e^{(m_o + k)\tau} F^{p,q}_{l-k}(b, c) \tag{9.52}$$

where

$$F^{p,q}_l(b, c) = \omega^{s_l + n_l} b^{n_l} c^{s_l} \sum_{j=0}^{\infty} \frac{(\omega^{p+q} b^q c^p)^j}{(n_l + jq)!\, (s_l + jp)!} . \tag{9.53}$$

The nonnegative integers s_l, n_l are uniquely determined by the properties:

(1) $l = n_l p - s_l q$,

(2) If $l = n_l' p - s_l' q$ where n_l', s_l' are nonnegative integers, then $n_l' + s_l' \geqslant n_l + s_l$.

Since p and q are relatively prime, the integers n_l, s_l can easily be shown to exist for all l. For example, if $p = q = 1$ then $s_l = 0$, $n_l = l$ for $l \geqslant 0$ and $s_l = -l$, $n_l = 0$ for $l < 0$.

If $bc \neq 0$ we can introduce new group parameters r, v defined by $b = rv^p/(p+q)$, $c = rv^{-q}/(p+q)$. In terms of these new parameters the matrix elements are

$$A_{lk}(g) = e^{(m_0+k)\tau} v^{l-k} I^{p,q}_{l-k}(\omega r) \tag{9.54}$$

where

$$I^{p,q}_l(r) = \left(\frac{r}{p+q}\right)^{n_l+s_l} \sum_{j=0}^{\infty} \frac{(r/(p+q))^{j(p+q)}}{(n_l+jq)!\,(s_l+jp)!}. \tag{9.55}$$

It follows from the ratio test that $I^{p,q}_l(r)$ is an entire function of r for all integers l. Here $I^{1,1}_l(r) = (-i)^l J_l(ir)$, $l \geqslant 0$, is the ordinary "modified Bessel function" (see Erdélyi *et al.* [1], Vol. II). Thus we can consider the functions $I^{p,q}_l(r)$ to be a group-theoretic generalization of Bessel functions. Substitute (9.54) into (9.51) to obtain the simple generating function

$$\exp\left[\frac{r}{p+q}(z^p + z^{-q})\right] = \sum_{l=-\infty}^{\infty} I^{p,q}_l(r)\, z^l. \tag{9.56}$$

The group property of the operators $\mathbf{A}(g)$ implies the addition theorem

$$A_{lk}(g_1 g_2) = \sum_{j=-\infty}^{\infty} A_{lj}(g_1)\, A_{jk}(g_2), \tag{9.57}$$

valid for all $g_1, g_2 \in G_{p,q}$. This leads immediately to the identity

$$F^{p,q}_l(b_1+b_2, c_1+c_2) = \sum_{j=-\infty}^{\infty} F^{p,q}_{l-j}(b_1, c_1) F^{p,q}_j(b_2, c_2). \tag{9.58}$$

We could also use the addition theorem to derive identities involving the functions $I^{p,q}_l(r)$. However, this will not be done here as most of the results so obtained are special cases of identities which will be derived in Section 9-10.

9-9 Recursion Relations for the Matrix Elements

Denote by $\mathcal{A}(G_{p,q})$ the space consisting of all entire functions on the group $G_{p,q}$, i.e., of all entire functions of the parameters b, c, τ. Clearly, the matrix elements $A_{lk}(g)$ are elements of $\mathcal{A}(G_{p,q})$. The representation P of $G_{p,q}$ on $\mathcal{A}(G_{p,q})$ defined by

$$[\mathbf{P}(g')f](g) = f(gg'), \qquad f \in \mathcal{A}(G_{p,q}), \tag{9.59}$$

for all $g, g' \in G_{p,q}$ satisfies the relation

$$\mathbf{P}(g_1g_2)f = \mathbf{P}(g_1)[\mathbf{P}(g_2)f]$$

and determines $G_{p,q}$ as a transformation group. According to (9.57) the action of P on a matrix element A_{jk} is given by

$$[\mathbf{P}(g')\, A_{jk}](g) = A_{jk}(gg') = \sum_{l=-\infty}^{\infty} A_{lk}(g')\, A_{jl}(g). \tag{9.60}$$

Comparing this expression with (9.50) we find that for fixed j the functions $\{A_{jk}\}$, $k = 0, \pm 1, \pm 2, \ldots$, form a basis for a realization of the representation $Q_{p,q}(\omega, m_o)$ of $G_{p,q}$ under the action of the operators $\mathbf{P}(g)$. Therefore, the Lie derivatives $\mathrm{J}^{\pm}$, J^3 defined on $\mathcal{O}(G_{p,q})$ by

$$\begin{aligned}
\mathrm{J}^+f(g) &= \frac{d}{db}[\mathbf{P}(\exp b\mathcal{J}^+)f](g)\Big|_{b=0} \\
\mathrm{J}^-f(g) &= \frac{d}{dc}[\mathbf{P}(\exp c\mathcal{J}^-)f](g)\Big|_{c=0} \\
\mathrm{J}^3f(g) &= \frac{d}{d\tau}[\mathbf{P}(\exp \tau\mathcal{J}^3)f](g)\Big|_{\tau=0}, \qquad f \in \mathcal{O}(G_{p,q}),
\end{aligned} \tag{9.61}$$

satisfy the relations

$$\begin{aligned}
\mathrm{J}^3A_{jk}(g) &= (m_o + k)\, A_{jk}(g), \qquad \mathrm{J}^+A_{jk}(g) = \omega A_{jk+p}(g), \\
&\qquad \mathrm{J}^-A_{jk}(g) = \omega A_{jk-q}(g), \\
\mathrm{C}^{p,q}A_{jk}(g) &= (\mathrm{J}^+)^q\, (\mathrm{J}^-)^p\, A_{jk}(g) = \omega^{p+q}A_{jk}(g), \qquad j, k = 0, \pm 1, \pm 2, \ldots .
\end{aligned} \tag{9.62}$$

Equations (9.62) yield recursion relations and differential equations for the matrix elements A_{jk}. We will use these equations to derive recursion relations for the generalized Bessel functions $I_l^{p,q}(r)$. Thus, we assign to a group element $g = g(b, c, \tau)$, $bc \neq 0$, the local coordinates $[r, v, \tau]$, where $b = rv^p/(p+q)$, $c = rv^{-q}/(p+q)$. Then if $g' = g(b', c', \tau')$, a straightforward computation shows the local coordinates of gg' are

$$\begin{aligned}
&[r(1 + (p+q)\, r^{-1}v^{-p}e^{p\tau}b')^{q/(p+q)}\, (1 + (p+q)\, r^{-1}v^{q}e^{-q\tau}c')^{p/(p+q)}, \\
&v(1 + (p+q)\, r^{-1}v^{-p}e^{p\tau}b')^{1/(p+q)}\, (1 + (p+q)\, r^{-1}v^{q}e^{-q\tau}c')^{1/(p+q)}, \tau + \tau']
\end{aligned}$$

for $|\, b'\,|, |\, c'\,|$ sufficiently small. It follows from the definition (9.61) that the Lie derivatives $\mathrm{J}^{\pm}$, J^3 are

$$\mathrm{J}^+ = e^{p\tau}v^{-p}\left(q\frac{\partial}{\partial r} + v\frac{\partial}{\partial v}\right), \qquad \mathrm{J}^- = e^{-q\tau}v^{q}\left(p\frac{\partial}{\partial r} + v\frac{\partial}{\partial v}\right), \qquad \mathrm{J}^3 = \frac{\partial}{\partial \tau}. \tag{9.63}$$

Substituting (9.54) and (9.63) into (9.62) and factoring out the dependence on v and τ, we obtain the equations

$$\left(q\frac{\partial}{\partial r}+\frac{m}{r}\right)I_m^{p,q}(r)=I_{m-p}^{p,q}(r),\qquad \left(p\frac{\partial}{\partial r}-\frac{m}{r}\right)I_m^{p,q}(r)=I_{m+q}^{p,q}(r),$$

$$\left(q\frac{\partial}{\partial r}+\frac{m+p}{r}\right)\left(q\frac{\partial}{\partial r}+\frac{m+2p}{r}\right)\cdots\left(q\frac{\partial}{\partial r}+\frac{m+qp}{r}\right)$$

$$\cdot\left(p\frac{\partial}{\partial r}-\frac{m+(p-1)q}{r}\right)\left(p\frac{\partial}{\partial r}-\frac{m+(p-2)q}{r}\right)\cdots\left(p\frac{\partial}{\partial r}-\frac{m}{r}\right)I_m^{p,q}(r)$$

$$=I_m^{p,q}(r),\qquad m=0,\pm1,\pm2,\ldots. \tag{9.64}$$

In particular, the functions $I_m^{p,q}(r)$ are solutions of an ordinary differential equation of order $p+q$.

9-10 Realizations in Two Variables

The operators δ_1 and δ_3, (9.40) can be used to construct realizations of $Q_{p,q}(\omega, m_o)$ on spaces of functions of two complex variables. However, the basis functions so obtained are elementary and of little interest. On the other hand a realization of $Q_{p,q}(\omega, m_o)$ by δ_2 does lead to new results. To see this we introduce new variables x, y, where $z_1 = xe^{py}/(p+q)$, $z_2 = xe^{-qy}/(p+q)$, in the expression for δ_2:

$$\mathrm{J}^3=-\frac{\partial}{\partial y},\quad \mathrm{J}^+=e^{-py}\left(q\frac{\partial}{\partial x}+\frac{1}{x}\frac{\partial}{\partial y}\right),\quad \mathrm{J}^-=e^{qy}\left(p\frac{\partial}{\partial x}-\frac{1}{x}\frac{\partial}{\partial y}\right). \tag{9.65}$$

To construct a realization of $Q_{p,q}(\omega, m_o)$ using the operators (9.65) we must find nonzero functions $f_m(x, y) = Z_m(x)\,e^{-my}$; $m = m_o + n$, $n = 0, \pm1, \pm2, \ldots$, such that expressions (9.46) are valid. These expressions are equivalent to the equations

$$\left(q\frac{d}{dx}-\frac{m}{x}\right)Z_m(x)=Z_{m+p}(x),\qquad \left(p\frac{d}{dx}+\frac{m}{x}\right)Z_m(x)=Z_{m-q}(x), \tag{9.66}$$

$$\left(q\frac{d}{dx}-\frac{m-p}{x}\right)\left(q\frac{d}{dx}-\frac{m-2p}{x}\right)\cdots\left(q\frac{d}{dx}-\frac{m-qp}{x}\right)\left(p\frac{d}{dx}+\frac{m-(p-1)q}{x}\right)$$

$$\cdot\left(p\frac{d}{dx}+\frac{m-(p-2)q}{x}\right)\cdots\left(p\frac{d}{dx}+\frac{m}{x}\right)Z_m(x)=Z_m(x), \tag{9.67}$$

where without loss of generality for special function theory, we have set $\omega = 1$. Thus, the functions $Z_m(x)$ are solutions of differential equations of order $p+q$.

In general it is possible to find $p + q$ linearly independent solutions of Eqs. (9.66) and (9.67). However, we will be content with a single solution. It is a consequence of (9.64) that if $m_o = 0$, the functions $Z_m(x) = I^{p,q}_{-m}(x)$ are solutions of these equations. Following a standard procedure in the theory of special functions we shall use analytic continuation to define $I^{p,q}_m(x)$ for m an arbitrary complex number and show that the functions $Z_m(x) = I^{p,q}_{-m}(x)$ satisfy the required relations (9.66) and (9.67) even if $m_o \neq 0$.

An application of the Cauchy residue theorem to (9.56) yields

$$2\pi i I^{p,q}_m(r) = \int_C z^{-m-1} \exp\left[\frac{r}{p+q}(z^p + z^{-q})\right] dz,$$
$$m = 0, \pm 1, \pm 2, \ldots,$$

where C is a simple closed contour surrounding the origin in the z-plane. Without changing the value of the integral, the contour C can be deformed into a loop D which starts at infinity on the negative real z-axis, encircles the origin counterclockwise and returns to its starting point. (The proof of this statement is almost word for word the same as the well-known proof of the analogous statement in the special case $p = q = 1$, see Whittaker and Watson [1], Chapter 17). Since p is odd, the integrand is bounded on D for $\operatorname{Re} r > 0$. Thus, we have

$$2\pi i I^{p,q}_m(r) = \int_{-\infty}^{(0+)} z^{-m-1} \exp\left[\frac{r}{p+q}(z^p + z^{-q})\right] dz \tag{9.68}$$

where $\int_{-\infty}^{(0+)}$ denotes integration along D. Since the integrand is now single valued on D even for m complex, we can use (9.68) to define the function $I^{p,q}_{-m}(r)$ for arbitrary values of m and $\operatorname{Re} r > 0$. Moreover, by differentiating under the integral sign in (9.68) we can verify that Eqs. (9.64) remain valid for arbitrary m. Thus the functions $Z_m(x) = I^{p,q}_{-m}(x)$, $m = m_o + n$, define a realization of the representation $Q_{p,q}(m_o, 1)$ of $\mathcal{G}_{p,q}$.

We can obtain more information about the generalized Bessel functions $I^{p,q}_m$ by mimicking the standard treatment of ordinary Bessel functions in terms of contour integrals as given, for example, by Whittaker and Watson [1], Chapter 17. If $r > 0$, introduction of the new variable $w = (r/(p+q))^{1/p} z$ in (9.68) leads to the expression

$$2\pi i I^{p,q}_m(r) = \left(\frac{r}{p+q}\right)^{m/p} \int_{-\infty}^{(0+)} w^{-m-1} \exp\left[w^p + \left(\frac{r}{p+q}\right)^{(p+q)/p} w^{-q}\right] dw. \tag{9.69}$$

The right-hand side of this equation can be analytically continued to define $I^{p,q}_m(r)$ as an analytic, but not single valued, function of r for all

$r \neq 0$. In particular $r^{-m/p} I_m^{p,q}(r)$ is an entire function of $r^{(p+q)/p}$. To obtain a series expansion for $I_m^{p,q}(r)$ we use

$$\exp\left[\left(\frac{r}{p+q}\right)^{(p+q)/p} w^{-q}\right] = \sum_{k=0}^{\infty} \left(\frac{r}{p+q}\right)^{k(p+q)/p} \left(\frac{w}{k!}\right)^{-kq}$$

in (9.69) and integrate term by term. From the theory of the gamma function (Whittaker and Watson [1], Chapter 12),

$$\int_{-\infty}^{(0+)} \exp(w^p)\, w^{-kq-m-1}\, dw = \frac{-2i \sin[\pi(kq+m+1)]}{kq+m}\, \Gamma\left(\frac{p-kq-m}{p}\right).$$

Thus,

$$I_m^{p,q}(r) = \frac{1}{\pi}\left(\frac{r}{p+q}\right)^{m/p} \sum_{k=0}^{\infty} \left(\frac{r}{p+q}\right)^{k(p+q)/p} \frac{\sin[\pi(kq+m+1)]}{k!\,p} \cdot \Gamma\left(\frac{-kq-m}{p}\right). \tag{9.70}$$

For $p = 1$ this expression simplifies to

$$I_m^{1,q}(r) = \left(\frac{r}{1+q}\right)^m \sum_{k=0}^{\infty} \left(\frac{r}{1+q}\right)^{k(1+q)} \frac{1}{k!\,\Gamma(m+kq+1)}. \tag{9.71}$$

We could construct additional solutions of equations (9.64), linearly independent of the functions $I_m^{p,q}(r)$, by choosing a different path of integration in (9.68) (see Khriptun [1]). However, for purposes of illustration we will be content with the solutions $I_m^{p,q}(r)$.

9-11 Generating Functions for Generalized Bessel Functions

As shown above, the functions $f_m(x, y) = I_{-m}^{p,q}(x)\, e^{-my}$, $m = m_o + n$, n an integer, form a basis for a realization of the representation $Q_{p,q}(1, m_o)$ of $\mathcal{G}_{p,q}$. Following our usual procedure we will extend this representation to a local multiplier representation of $G_{p,q}$ in the space $\mathcal{F}$ of functions analytic in a neighborhood of $(x^o, y^o) = (1, 0)$. The operators $\mathbf{T}(g)$, $g = g(b, c, \tau) \in G_{p,q}$ defining this multiplier representation can easily be computed from (9.65) and (9.43). The result is

$$[\mathbf{T}(g)f](x, t) = f\left[x\left(1 + \frac{b(p+q)}{xt^p}\right)^{q/(p+q)} \left(1 + \frac{c(p+q)\,t^q}{x}\right)^{p/(p+q)},\right.$$
$$\left. te^{-\tau}\left(1 + \frac{b(p+q)}{xt^p}\right)^{1/(p+q)} \left(1 + \frac{c(p+q)\,t^q}{x}\right)^{-1/(p+q)}\right],$$
$$f \in \mathcal{F}, \tag{9.72}$$

valid for $|b(p+q)/xt^p| < 1$, $|c(p+q)\,t^q/x| < 1$, where $t = e^y$. The matrix elements of $\mathbf{T}(g)$ with respect to the analytic basis $\{f_m(x,t) = I^{p,q}_{-m}(x)\,t^{-m}\}$, are the functions $A_{lk}(g)$ given by (9.52) and (9.54). Thus,

$$[\mathbf{T}(g) f_{m_0+k}](x,t) = \sum_{l=-\infty}^{\infty} A_{lk}(g) f_{m_0+l}(x,t), \qquad k = 0, \pm 1, \pm 2, \ldots,$$

or

$$\begin{aligned} I^{p,q}_m &\left[x\left(1+\frac{b(p+q)}{xt^p}\right)^{q/(p+q)}\left(1+\frac{c(p+q)\,t^q}{x}\right)^{p/(p+q)}\right] \\ &\cdot\left(\frac{x+b(p+q)\,t^{-p}}{x+c(p+q)\,t^q}\right)^{m/(p+q)} \\ &= \sum_{l=-\infty}^{\infty} F^{p,q}_{-l}(b,c)\, I^{p,q}_{m+l}(x)\, t^l, \\ &\qquad |b(p+q)/xt^p| < 1, \qquad |c(p+q)\,t^q/x| < 1, \end{aligned} \tag{9.73}$$

for all $m \in \mathcal{C}$. If $c = 0$, $t = 1$, this expression simplifies to

$$\begin{aligned} I^{p,q}_m &\left[x\left(1+\frac{b(p+q)}{x}\right)^{q/(p+q)}\right]\left(1+\frac{b(p+q)}{x}\right)^{m/(p+q)} \\ &= \sum_{l=0}^{\infty} \frac{b^l}{l!} I^{p,q}_{m-lp}(x), \qquad |b(p+q)/x| < 1, \end{aligned}$$

while for $b = 0$, $t = 1$ it becomes

$$\begin{aligned} I^{p,q}_m &\left[x\left(1+\frac{c(p+q)}{x}\right)^{p/(p+q)}\right]\left(1+\frac{c(p+q)}{x}\right)^{-m/(p+q)} \\ &= \sum_{l=0}^{\infty} \frac{c^l}{l!} I^{p,q}_{m+lq}(x), \qquad |c(p+q)/x| < 1. \end{aligned}$$

If $b = c = r/(p+q) \neq 0$ the matrix elements can be expressed in terms of generalized Bessel functions and (9.73) becomes

$$\begin{aligned} I^{p,q}_m &\left[x\left(1+\frac{r}{xt^p}\right)^{q/(p+q)}\left(1+\frac{rt^q}{x}\right)^{p/(p+q)}\right]\left(\frac{x+rt^{-p}}{x+rt^q}\right)^{m/(p+q)} \\ &= \sum_{l=-\infty}^{\infty} I^{p,q}_{-l}(r)\, I^{p,q}_{m+l}(x)\, t^l, \qquad |r/xt^p| < 1, \qquad |rt^q/x| < 1. \end{aligned}$$

For $p = q = 1$, Eq. (9.73) is equivalent to the Bessel function identity (3.29).

Similarly, in analogy with Weisner's treatment of Bessel functions as given in Section 3-4, we can use the operators

$$\mathrm{J}^3 = -t\frac{\partial}{\partial t}, \qquad \mathrm{J}^+ = t^{-p}\left(q\frac{\partial}{\partial x} + \frac{t}{x}\frac{\partial}{\partial t}\right), \qquad \mathrm{J}^- = t^q\left(p\frac{\partial}{\partial x} - \frac{t}{x}\frac{\partial}{\partial t}\right),$$

expressions (9.65), to derive generating functions which are not related to the matrix elements $A_{lk}(g)$. The basic observation to be made is: If $f(x, t)$ is a solution of the equation $C^{p,q}f = (\mathrm{J}^+)^q(\mathrm{J}^-)^p f = \omega f$ then the function $\mathbf{T}(g)f$ given by the formal expression (9.72) is also a solution of this equation. Moreover, if f is a solution of the equation

$$(x_1\mathrm{J}^+ + x_2\mathrm{J}^- + x_3\mathrm{J}^3)f(x, t) = \lambda f(x, t)$$

for constants x_1, x_2, x_3, λ, then $\mathbf{T}(g)f$ is a solution of

$$[\mathbf{T}(g)(x_1\mathrm{J}^+ + x_2\mathrm{J}^- + x_3\mathrm{J}^3)\,\mathbf{T}(g^{-1})][\mathbf{T}(g)f] = \lambda[\mathbf{T}(g)f]$$

where

$$\begin{aligned}\mathbf{T}(g)&(x_1\mathrm{J}^+ + x_2\mathrm{J}^- + x_3\mathrm{J}^3)\,\mathbf{T}(g^{-1})\\ &= (x_1e^{p\tau} - bpx_3)\,\mathrm{J}^+ + (x_2e^{-q\tau} + cqx_3)\,\mathrm{J}^- + x_3\mathrm{J}^3.\end{aligned}$$

As an illustration of these remarks, consider the function $f(x, t) = I_m^{p,q}(x)\,t^m$, $m \in \mathcal{C}$. In this case $C^{p,q}f = f$, $\mathrm{J}^3 f = -mf$, so the function $\mathbf{T}(g)f$ satisfies

$$C^{p,q}[\mathbf{T}(g)f] = \mathbf{T}(g)f, \qquad (-bp\mathrm{J}^+ + cq\mathrm{J}^- + \mathrm{J}^3)[\mathbf{T}(g)f] = -m[\mathbf{T}(g)f] \tag{9.74}$$

where $g = g(b, c, \tau) \in G_{p,q}$. If $c = \tau = 0$, $b = 1$, $\mathbf{T}(g)f$ can be written in the form

$$\begin{aligned}h(x, t) = [\mathbf{T}(g)f](x, t) &= \left[x^p\left(x + \frac{p+q}{t^p}\right)^q\right]^{-m/(p(p+q))}\\ &\cdot I_m^{p,q}\left\{\left[x^p\left(x + \frac{p+q}{t^p}\right)^q\right]^{1/(p+q)}\right\}(xt^p + p + q)^{m/p}.\end{aligned} \tag{9.75}$$

Since $x^{-m/p}I_m^{p,q}(x)$ is an entire function of $x^{(p+q)/p}$, h has a Laurent expansion in t^p about $t = 0$:

$$h(x, t) = \sum_{n=-\infty}^{\infty} h_n(x)\,t^{np}, \qquad |xt^p| < p + q.$$

Substituting this expansion into the first equation (9.74), we find $Z_{np}(x) = h_{-n}(x)$ is a solution of the generalized Bessel equation (9.67) for $n = 0, \pm 1, \pm 2, \ldots$. At this point we use the fact: For m an integer the only solutions of (9.67) which are regular at $x = 0$ are $cI_{-n}^{p,q}(x)$, c a constant. This statement can be verified by computing the indicial equation of (9.67) (see Ince [1], Chapter 7).

Since $h(x, t)$ is bounded for $x = 0$, we have $h_n(x) = c_n I_{np}^{p,q}(x)$, $c_n \in \mathcal{C}$. Hence,

$$h(x, t) = \sum_{n=-\infty}^{\infty} c_n I_{np}^{p,q}(x)\, t^n.$$

The equation $(-p\mathrm{J}^+ + \mathrm{J}^3)\, h(x, t) = -mh(x, t)$ implies $pc_{n+1} = (m - np)\, c_n$. The constant c_0 can be determined by setting $x = 0$ in $h(x, t)$ and using (9.70). Consequently,

$$c_0 = \frac{\Gamma(-m/p)\sin[\pi(m+1)]}{p\pi}, \qquad c_n = \frac{(-1)^n\,\Gamma[(np-m)/p]\sin[\pi(m+1)]}{p\pi},$$

and we obtain the identity

$$\begin{aligned} &\left[x^p\left(x + \frac{p+q}{t^p}\right)^q\right]^{-m/(p(p+q))} I_m^{p,q}\left\{\left[x^p\left(x + \frac{p+q}{t^p}\right)^q\right]^{1/(p+q)}\right\}(xt^p + p + q)^{m/p} \\ &\qquad = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{p\pi}\,\Gamma\left(\frac{np-m}{p}\right)\sin[\pi(m+1)]\, I_{np}^{p,q}(x)\, t^{np}, \\ &\qquad |xt^p| < p + q. \end{aligned} \tag{9.76}$$

When $p = q = 1$ this expression is equivalent to (3.38).