Math 1281
Fall 2002
Solutions to Second Midterm Exam

1. What is $\frac{d^2}{dx^2}(\cos 2x)$?

(a) $-2\sin 2x$
(b) $-4\sin 2x$
(c) $-4\cos 2x$
(d) $4\cos 2x$
(e) None of the above.

Solution. The answer is (c).

$$\frac{d}{dx}(\cos 2x) = -2\sin 2x$$
$$\frac{d^2}{dx^2}(\cos 2x) = \frac{d}{dx}(-2\sin 2x) = -4\cos 2x$$

2. Suppose that the function $f(x)$ is differentiable and that $f(1)=2$, $f(2)=3$, $f'(2)=4$, $f'(1)=5$. Let $g(x)=f^{-1}(x)$. Which of the following statements is true?

(a) $g'(1)=\frac{1}{2}$
(b) $g'(2)=\frac{1}{3}$
(c) $g'(3)=\frac{1}{4}$
(d) $g'(4)=\frac{1}{5}$
(e) None of the above.

Solution. The answer is (c). First note that $f^{-1}(2)=1$ and that $f^{-1}(3)=2$. Therefore,

$$g'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)} = \frac{1}{5}, \text{ and}$$
$$g'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(2)} = \frac{1}{4}$$

3. Compute the following limits.

a. $\lim\limits_{x\to 0}\dfrac{2^x-1}{2x}$

Answer: $\dfrac{\ln 2}{2}$

$$\lim_{x\to 0}\frac{2^x-1}{2x}=\frac{1}{2}\lim_{x\to 0}\frac{2^x-1}{x}=\frac{1}{2}\ln 2$$

b. $\lim\limits_{x\to\infty}\dfrac{e^x}{1-e^x}$

Answer: -1

$$\lim_{x\to\infty}\frac{e^x}{1-e^x}=\lim_{x\to\infty}\frac{1}{e^{-x}-1}=-1$$

c. $\lim\limits_{x\to 0}\dfrac{\sin 2x}{x}$

Answer: 2

$$\lim_{x\to 0}\frac{\sin 2x}{x}=2\cdot\lim_{2x\to 0}\frac{\sin 2x}{2x}=2$$

4. A population of bacteria is growing exponentially. Let $M(t)$ be the mass of the population measured in grams, where t is the elapsed time since the beginning of the experiment, measured in hours. At the beginning of the experiment, the mass of the population, $M(0)$, is 10^{-3} grams. Three hours later, the mass of the population, $M(3)$, is 1.5×10^{-3} grams. Assume that the same exponential growth continues indefinitely.

a. Compute $\dfrac{dM}{dt}$ exactly five days after the beginning of the experiment.

Answer: $\frac{1}{3}\ln(\frac{3}{2})\cdot 10^{-3}(\frac{3}{2})^{40}\approx 1494$ grams/hr

Model of exponential growth: $M(t)=M(0)e^{\lambda t}$.

$$\begin{aligned} M(3)&=M(0)e^{\lambda\cdot 3}\\ e^{\lambda\cdot 3}&=\frac{M(3)}{M(0)}=\frac{1.5\times 10^{-3}}{10^{-3}}=1.5=\frac{3}{2}\\ 3\lambda&=\ln(\tfrac{3}{2})\\ \lambda&=\tfrac{1}{3}\ln(\tfrac{3}{2}) \end{aligned}$$

Five days is 120 hours.

$$M(120) = M(0)e^{\lambda \cdot 120} = 10^{-3} e^{40\ln(\frac{3}{2})} = 10^{-3}(\tfrac{3}{2})^{40}$$
$$\frac{dM}{dt}(120) = \lambda M(120) = \tfrac{1}{3}\ln(\tfrac{3}{2}) \cdot 10^{-3}(\tfrac{3}{2})^{40}$$

b. Compute $\frac{dM}{dt}$ when the mass of the population reaches 6×10^{21} metric tons (1 metric ton = 10^6 grams).

Answer: $2\ln(\tfrac{3}{2})\cdot 10^{27} \approx 8.1\times 10^{26}$ grams/hr

$$\frac{dM}{dt}(t) = \lambda M(t) = \tfrac{1}{3}\ln(\tfrac{3}{2}) \cdot 6\times 10^{27}$$

5. Consider the curve $y + \ln y + x = 2x^2$. Compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $(x, y) = (1,1)$.

Answer: $\frac{dy}{dx} = \frac{3}{2}$ and $\frac{d^2y}{dx^2} = \frac{25}{8}$.

$$\frac{d}{dx}(y + \ln y + x) = \frac{d}{dx}(2x^2)$$
$$\frac{dy}{dx} + \frac{1}{y}\frac{dy}{dx} + 1 = 4x$$

When $(x, y) = (1,1)$,

$$\frac{dy}{dx} + \frac{1}{1}\cdot\frac{dy}{dx} + 1 = 4\cdot 1$$
$$\frac{dy}{dx} = \frac{3}{2}$$

Now for the second derivative.

$$\left(1+\frac{1}{y}\right)\frac{dy}{dx}+1=4x$$

$$\frac{d}{dx}\left[\left(1+\frac{1}{y}\right)\frac{dy}{dx}+1\right]=\frac{d}{dx}(4x)$$

$$\frac{d}{dx}\left(1+\frac{1}{y}\right)\frac{dy}{dx}+\left(1+\frac{1}{y}\right)\frac{d}{dx}\frac{dy}{dx}=4x$$

$$-\frac{1}{y^2}\frac{dy}{dx}\frac{dy}{dx}+\left(1+\frac{1}{y}\right)\frac{d^2y}{dx^2}=4x$$

When $(x,y)=(1,1)$,

$$-\frac{1}{1^2}\cdot\frac{3}{2}\cdot\frac{3}{2}+\left(1+\frac{1}{1}\right)\frac{d^2y}{dx^2}=4\cdot 1$$

$$\frac{d^2y}{dx^2}=\frac{25}{8}$$

6. Use the method of linear approximation to estimate $\sin(31^\circ)$. Hint: $\sin(30^\circ)=\frac{1}{2}$.

Answer: $\frac{1}{2}+(\frac{\sqrt{3}}{2})(\frac{\pi}{180})\approx 0.515$.

Convert to radians: $31^\circ=\frac{\pi}{6}+\frac{\pi}{180}$ radians.

$$\sin(\frac{\pi}{6}+\frac{\pi}{180})\approx\sin\frac{\pi}{6}+(\cos\frac{\pi}{6})(\frac{\pi}{180})$$
$$=\frac{1}{2}+(\frac{\sqrt{3}}{2})(\frac{\pi}{180})$$

7. Compute the following derivatives. Simplify your answer.

a. $\frac{d}{d\theta}\left(\frac{1}{1+\cos\theta}\right)$

$$\frac{d}{d\theta}\left(\frac{1}{1+\cos\theta}\right)=\frac{d}{d\theta}(1+\cos\theta)^{-1}=-(1+\cos\theta)^{-2}(-\sin\theta)$$
$$=\frac{\sin\theta}{(1+\cos\theta)^2}$$

b. $\dfrac{d}{dx}\left(\dfrac{e^x - e^{-x}}{x-1}\right)$

$$\begin{aligned}\frac{d}{dx}\left(\frac{e^x - e^{-x}}{x-1}\right) &= \frac{\left(\frac{d}{dx}(e^x - e^{-x})\right)(x-1) - (e^x - e^{-x})\left(\frac{d}{dx}(x-1)\right)}{(x-1)^2} \\ &= \frac{(e^x + e^{-x})(x-1) - (e^x - e^{-x})}{(x-1)^2} \\ &= \frac{(x-2)e^x + xe^{-x}}{(x-1)^2}\end{aligned}$$

c. $\dfrac{d}{d\theta}\big((\sin\theta)(\ln(\sin\theta))\big)$

$$\begin{aligned}\frac{d}{d\theta}\big((\sin\theta)(\ln(\sin\theta))\big) &= \left(\frac{d}{d\theta}(\sin\theta)\right)(\ln(\sin\theta)) + (\sin\theta)\left(\frac{d}{d\theta}(\ln(\sin\theta))\right) \\ &= \cos\theta \ln(\sin\theta) + \sin\theta \frac{1}{\sin\theta}\cos\theta \\ &= (1 + \ln(\sin\theta))\cos\theta\end{aligned}$$