CHAPTER 4

The Wave Equation

4.1 The Equation $\Psi_{tt} - \Delta_2 \Psi = 0$

Here we are concerned with the real wave equation

$$(\partial_{00} - \partial_{11} - \partial_{22})\Psi(x) = 0, \qquad x = (x_0, x_1, x_2). \tag{1.1}$$

It is well known that the symmetry algebra of (1.1) is ten dimensional with basis the momentum and energy operators

$$P_\alpha = \partial_\alpha, \qquad \alpha = 0, 1, 2, \tag{1.2}$$

the generators of homogeneous Lorentz transformations

$$M_{12} = x_1 \partial_2 - x_2 \partial_1, \qquad M_{01} = x_0 \partial_1 + x_1 \partial_0, \qquad M_{02} = x_0 \partial_2 + x_2 \partial_0, \tag{1.3}$$

the generator of dilatations

$$D = -\left(\tfrac{1}{2} + x_0 \partial_0 + x_1 \partial_1 + x_2 \partial_2\right), \tag{1.4}$$

and the generators of special conformal transformations

$$\begin{aligned} K_0 &= -x_0 + (x \cdot x - 2x_0^2)\partial_0 - 2x_0 x_1 \partial_1 - 2x_0 x_2 \partial_2, \\ K_1 &= x_1 + (x \cdot x + 2x_1^2)\partial_1 + 2x_1 x_0 \partial_0 + 2x_1 x_2 \partial_2, \\ K_2 &= x_2 + (x \cdot x + 2x_2^2)\partial_2 + 2x_2 x_0 \partial_0 + 2x_2 x_1 \partial_1 \end{aligned} \tag{1.5}$$

where

$$x \cdot y = x_0 y_0 - x_1 y_1 - x_2 y_2 = x_0 y_0 - \mathbf{x} \cdot \mathbf{y}.$$

(We are ignoring the trivial symmetry E.)

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It is convenient to introduce another basis for the symmetry algebra which clearly displays the isomorphism between this algebra and $so(3,2)$. We define $so(3,2)$ as the ten-dimensional Lie algebra of 5×5 real matrices $\mathcal{Q}$ such that $\mathcal{Q}G^{3,2}+G^{3,2}\mathcal{Q}'=0$ where

$$G^{3,2}=\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & -1 \end{pmatrix}=\sum_{j=1}^{3}\mathcal{E}_{jj}-\sum_{k=4}^{5}\mathcal{E}_{kk}$$

and $\mathcal{E}_{ij}$ is defined by (6.4), Section 3.6. It is straightforward to check that the matrices

$$\begin{aligned} &\Gamma_{ab}=\mathcal{E}_{ab}-\mathcal{E}_{ba}=-\Gamma_{ba}, && a\neq b,\\ &\Gamma_{aB}=\mathcal{E}_{aB}+\mathcal{E}_{Ba}=\Gamma_{Ba}, && 1\leqslant a,b\leqslant 3,\ \ 4\leqslant A,B\leqslant 5,\\ &\Gamma_{AB}=-\mathcal{E}_{AB}+\mathcal{E}_{BA}=-\Gamma_{BA}, \end{aligned} \tag{1.6}$$

form a basis for $so(3,2)$ with commutation relations

$$\begin{aligned} &[\Gamma_{ab},\Gamma_{cd}]=\delta_{bc}\Gamma_{ad}+\delta_{ad}\Gamma_{bc}+\delta_{ca}\Gamma_{db}+\delta_{db}\Gamma_{ca},\\ &[\Gamma_{aB},\Gamma_{cd}]=-\delta_{ad}\Gamma_{cB}+\delta_{ac}\Gamma_{dB}, \quad [\Gamma_{Ab},\Gamma_{45}]=\delta_{A5}\Gamma_{4b}-\delta_{A4}\Gamma_{5b},\\ &[\Gamma_{aB},\Gamma_{cD}]=\delta_{BD}\Gamma_{ac}-\delta_{ac}\Gamma_{BD}, \quad [\Gamma_{ab},\Gamma_{45}]=0. \end{aligned} \tag{1.7}$$

This Γ basis is related to our other basis via the identifications

$$\begin{aligned} &P_0=\Gamma_{14}+\Gamma_{45}, && P_1=\Gamma_{12}+\Gamma_{25}, && P_2=\Gamma_{13}+\Gamma_{35},\\ &K_0=\Gamma_{14}-\Gamma_{45}, && K_1=\Gamma_{12}-\Gamma_{25}, && K_2=\Gamma_{13}-\Gamma_{35},\\ &M_{12}=\Gamma_{23}, && M_{01}=\Gamma_{42}, && M_{02}=\Gamma_{43}, \quad D=\Gamma_{15}. \end{aligned} \tag{1.8}$$

The symmetry operators can be exponentiated to obtain a local Lie transformation group of symmetries of (1.1). In particular, the momentum and Lorentz operators generate the Poincaré group of symmetries

$$\Psi(x)\to\Psi(x\Lambda+a), \qquad a=(a_0,a_1,a_2),\ \Lambda\in SO(1,2); \tag{1.9}$$

the dilatation operator generates

$$\exp(\lambda D)\Psi(x)=\exp(-\lambda/2)\Psi\big[\exp(-\lambda)x\big]; \tag{1.10}$$

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and the K_α generate the special conformal transformations

$$\exp(a\cdot K)\Psi(x)=\left[1+2x\cdot a+(a\cdot a)(x\cdot x)\right]^{-1/2}\Psi\left(\frac{x+a(x\cdot x)}{1+2x\cdot a+(a\cdot a)(x\cdot x)}\right). \tag{1.11}$$

In addition, we shall consider the inversion, space reflection, and time reflection symmetries,

$$\begin{aligned} &I\Psi(x)=\left[-x\cdot x\right]^{-1/2}\Psi(-x/(x\cdot x)), \qquad S\Psi(x)=\Psi(x_0,-x_1,x_2),\\ &T\Psi(x)=\Psi(-x_0,x_1,x_2), \qquad I=I^{-1}, \qquad S=S^{-1}, \qquad T=T^{-1}, \end{aligned} \tag{1.12}$$

which are not generated by the local symmetry operators. It follows from the expression for the inversion I that

$$\begin{aligned} &IK_\alpha I^{-1}=-P_\alpha, \qquad IDI^{-1}=-D,\\ &IM_{\alpha\beta}I^{-1}=M_{\alpha\beta}. \end{aligned} \tag{1.13}$$

In analogy with the treatment of the Laplace equation in Section 3.6, we can verify that the wave equation is class I. Furthermore, although the space of symmetric second-order elements in the enveloping algebra of $so(3,2)$ is 55 dimensional, there are 20 linearly independent relations between these operators on the solution space of (1.1). For example, we have

$$\begin{aligned} &\text{(i)} \quad P_0^2-P_1^2-P_2^2=K_0^2-K_1^2-K_2^2=0,\\ &\text{(ii)} \quad \Gamma_{12}^2+\Gamma_{13}^2+\Gamma_{23}^2=\tfrac{1}{4}+\Gamma_{45}^2,\\ &\text{(iii)} \quad M_{12}^2-M_{01}^2-M_{02}^2=\tfrac{1}{4}-D^2,\\ &\text{(iv)} \quad \Gamma_{45}^2-\Gamma_{41}^2-\Gamma_{51}^2=\tfrac{1}{4}+\Gamma_{23}^2, \end{aligned} \tag{1.14}$$

valid when applied to solutions of (1.1).

As is well known [44, 66, 118], by formally taking the Fourier transform in the variables x_α we can express a solution $\Psi(x)$ of (1.1) in the form

$$\Psi(x)=(4\pi)^{-1}\iint_{-\infty}^{\infty}\left[\exp(ik\cdot x)f(\mathbf{k})+\exp(i\tilde{k}\cdot x)\tilde{f}(\mathbf{k})\right]d\mu(\mathbf{k}) \tag{1.15}$$

where $k_0=(k_1^2+k_2^2)^{1/2}$, $\tilde{k}=(-k_0,k_1,k_2)$ and $d\mu(\mathbf{k})=dk_1dk_2/k_0$. Let $\mathcal{H}=\mathcal{H}_+\oplus\mathcal{H}_-$ be the space of all ordered pairs of complex-valued functions

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$\mathbf{F}(\mathbf{k})=\{f(\mathbf{k}),\tilde{f}(\mathbf{k})\}$ defined on R^2 such that

$$\int\int(|f|^2+|\tilde{f}|^2)\,d\mu(\mathbf{k})<\infty$$

(Lebesgue integral), and consider the indefinite inner product on $\mathcal{H}$ given by

$$\langle\mathbf{F},\mathbf{G}\rangle=\int\int\left(f\bar{g}-\tilde{f}\bar{\tilde{g}}\right)d\mu(\mathbf{k}). \tag{1.16}$$

Then [44, 66] the functions Ψ, Φ related to $\mathbf{F}$, $\mathbf{G}$ by (1.15) satisfy

$$\langle\Psi,\Phi\rangle\equiv\langle\mathbf{F},\mathbf{G}\rangle=2i\int\int_{x_0=t}\left(\Psi(x)\partial_0\bar{\Phi}(x)-\left[\partial_0\Psi(x)\right]\bar{\Phi}(x)\right)dx_1\,dx_2, \tag{1.17}$$

independent of t. (More precisely, (1.17) can be derived from (1.16) by first considering the dense subspace of $\mathcal{H}$ consisting of C^∞ functions with compact support bounded away from $(0,0)$ and then passing to the limit. For $\mathbf{F}\in\mathcal{H}$ the corresponding $\Psi(x)$ is a solution of (1.1) in the weak sense of distribution theory; it may not be true that Ψ is two times continuously differentiable in each variable.)

The operators (1.2)–(1.5) acting on solutions of (1.1) induce corresponding operators on $\mathcal{H}$ under which $\mathcal{H}_+$ and $\mathcal{H}_-$ are separately invariant. Indeed, with repeated integrations by parts we can establish that the action of these operators on $\mathcal{H}_+$ is

$$\begin{aligned}
&P_0=ik_0, && P_j=-ik_j,\ \ j=1,2, && M_{12}=k_1\partial_2-k_2\partial_1,\\
&M_{01}=k_0\partial_1, && M_{02}=k_0\partial_2, && D=\tfrac{1}{2}+k_1\partial_1+k_2\partial_2,\\
&K_0=ik_0(\partial_{11}+\partial_{22}), && && K_1=i(k_1\partial_{11}-k_1\partial_{22}+2k_2\partial_{12}+\partial_1),\\
&K_2=i(-k_2\partial_{11}+k_2\partial_{22}+2k_1\partial_{12}+\partial_2). && &&
\end{aligned} \tag{1.18}$$

The action on $\mathcal{H}_-$ is the same except that k_0 is replaced by $-k_0$ in each of expressions (1.18). Moreover, it is straightforward to verify that these operators are skew-Hermitian on $\mathcal{H}_+$ and $\mathcal{H}_-$ separately.

The induced operators S,T on $\mathcal{H}$ are

$$S\mathbf{F}(k_1,k_2)=\mathbf{F}(-k_1,k_2),\qquad T\mathbf{F}(\mathbf{k})=\left(\tilde{f}\,(\mathbf{k}),f(\mathbf{k})\right). \tag{1.19}$$

Thus, $\mathcal{H}_+$ and $\mathcal{H}_-$ are invariant under S but are interchanged by T. In view of this interchange property of T, we will henceforth limit ourselves to consideration of elements in the Hilbert space $\mathcal{H}_+$, that is, the positive

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energy solutions

$$\Psi(x)=(4\pi)^{-1}\iint_{-\infty}^{\infty}\exp(ik\cdot x)f(\mathbf{k})\,d\mu(\mathbf{k}). \tag{1.20}$$

The inner product on $\mathcal{H}_+$ is

$$\langle f,g\rangle=\iint_{-\infty}^{\infty}f(\mathbf{k})\,\bar{g}(\mathbf{k})\,d\mu(\mathbf{k}) \tag{1.21}$$

and

$$\begin{aligned}\langle\Psi,\Phi\rangle\equiv\langle f,g\rangle&=4i\iint_{x_0=t}\Psi(x)\,\partial_0\bar{\Phi}(x)\,dx_1\,dx_2\\&=-4i\iint_{x_0=t}\bar{\Phi}(x)\,\partial_0\Psi(x)\,dx_1\,dx_2.\end{aligned} \tag{1.22}$$

Furthermore, if Ψ is given by (1.20), we have

$$f(\mathbf{k})=k_0\pi^{-1}\iint_{-\infty}^{\infty}\Psi(x)\exp(-ik\cdot x)\,dx_1\,dx_2. \tag{1.23}$$

By employing arguments analogous to those in [66], we can show that $\mathcal{H}_+$ is invariant under I and

$$If(\mathbf{k})=(2\pi)^{-1}\iint_{-\infty}^{\infty}\cos\left[(2l\cdot k)^{1/2}\right]f(\mathbf{l})\,d\mu(\mathbf{l}),\qquad f\in\mathcal{H}_+,\ I^2=E, \tag{1.24}$$

where E is the identity operator on $\mathcal{H}_+$. Clearly, I extends to a unitary self-adjoint operator on $\mathcal{H}_+$ with eigenvalues ± 1.

If $\{\Psi_\alpha(x)\}$ is an ON basis for $\mathcal{H}_+$, then (in the sense of distributions)

$$\sum_\alpha\bar{\Psi}_\alpha(x)\Psi_\alpha(x')=\Delta_+(x-x')=(16\pi^2)^{-1}\iint\exp\left[ik(x'-x)\right]d\mu(\mathbf{k}) \tag{1.25}$$

where the distribution Δ_+ has the explicit expression

$$\Delta_+(x)=\begin{cases}2\pi i(t^2-r^2)^{-1/2}, & t>r\\ -2\pi i(t^2-r^2)^{-1/2}, & t<-r,\\ 2\pi(r^2-t^2)^{-1/2}, & -r<t<r,\end{cases}\qquad r=(x_1^2+x_2^2)^{1/2},\quad t=x_0. \tag{1.26}$$

The computation of (1.26) is carried out in analogy with the corresponding

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result for four-dimensional space-time [32]. It follows immediately that

$$\Psi(x)=\langle\Psi,\Delta_+(x'-x)\rangle \tag{1.27}$$

where the integration is carried out over x'.

It is well known that the representation of $so(3,2)$ on $\mathcal{H}_+$ induced by the operators (1.18) exponentiates to a global irreducible unitary representation of a covering group $\widetilde{SO}(3,2)$ of the identity component of $SO(3,2)$ [44]. The maximal connected compact subgroup of $\widetilde{SO}(3,2)$ is $SO(3)\times SO(2)$ where $SO(3)$ is generated by Γ_{12}, Γ_{13}, Γ_{23} and $SO(2)$ by Γ_{45}. We will determine the explicit action of this subgroup on $\mathcal{H}_+$ as well as the action of several other interesting subgroups of $\widetilde{SO}(3,2)$.

The operators M_{01}, M_{02}, M_{12} generate a subgroup of $\widetilde{SO}(3,2)$ isomorphic to $SO(2,1)$ (see section 4.3). The action of this subgroup on $\mathcal{H}_+$ is determined by

$$\begin{aligned}\exp(\theta M_{12})f(\mathbf{k})&=f(k_1\cos\theta-k_2\sin\theta,\,k_1\sin\theta+k_2\cos\theta),\\ \exp(aM_{01})f(\mathbf{k})&=f(k_1(a),k_2),\\ k_1(a)&=\left[e^a(k_1+k_0)^2-e^{-a}k_2^2\right]/2(k_1+k_0),\qquad f\in\mathcal{H}_+.\end{aligned} \tag{1.28}$$

(The result for M_{02} follows easily from that for M_{01}.)

The P_α generate a translation subgroup of $\widetilde{SO}(3,2)$:

$$\exp\left(\sum a_\alpha P_\alpha\right)f(\mathbf{k})=\exp(ia\cdot k)f(\mathbf{k}). \tag{1.29}$$

Unitary operators of the form $\exp(\sum a_\alpha K_\alpha)$ are more difficult to compute. In [60] it is shown that

$$\begin{aligned}\exp(aK_0)f(\mathbf{s})=&-i(2\pi a)^{-1}\iint_{-\infty}^{\infty}\exp\left[i(s_0+l_0)/a\right]\\ &\times\cos\left\{a^{-1}\left[2(s_0l_0+s_1l_1+s_2l_2)\right]^{1/2}\right\}f(\mathbf{l})\,d\mu(\mathbf{l}),\end{aligned} \tag{1.30}$$

$$\begin{aligned}\exp(aK_1)f(\mathbf{s})=&(8\pi|a|)^{-1}\iint_{-\infty}^{\infty}\exp\left[i\frac{(s_1+l_1)}{a}\right]\\ &\times\cos\left[\frac{s_1(l_2+l_0)-l_1(s_2+s_0)}{a(s_2+s_0)^{1/2}(l_2+l_0)^{1/2}}\right]f(\mathbf{l})\,d\mu(\mathbf{l})\end{aligned} \tag{1.31}$$

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for $f\in\mathcal{H}_+$, and

$$\exp[a(K_0+K_1)]f(\mathbf{s})=[4\pi ia(s_0-s_1)]^{-1/2}\int_{-\infty}^{\infty}$$
$$\times\exp\left[\frac{-(s_2-w)^2}{4ia(s_0-s_1)}\right]f\left(\frac{w^2-(s_0-s_1)^2}{2(s_0-s_1)},w\right)dw. \tag{1.32}$$

The dilatation operator D generates the subgroup

$$\exp(aD)f(\mathbf{k})=\exp(a/2)f(e^a\mathbf{k}). \tag{1.33}$$

We can now easily exponentiate the compact generator $\Gamma_{45}=(P_0-K_0)/2$. Indeed, the operators P_0, D, and K_0 generate a $SL(2,R)$ subgroup of $\widetilde{SO}(3,2)$. From (1.17) in Chapter 2 it is easy to verify the relation

$$\exp(2\theta\Gamma_{45})=\exp(\tan(\theta)P_0)\exp(-K_0\sin\theta\cos\theta)\exp(-2D\ln\cos\theta),$$

and evaluating the right-hand side we find

$$\exp(2\theta\Gamma_{45})f(\mathbf{k})=i(2\pi)^{-1}\csc(\theta)\int\int\exp[-i(k_0+l_0)\cot\theta]$$
$$\times\cos\{\csc(\theta)[2(k_0l_0+k_1l_1+k_2l_2)]^{1/2}\}f(\mathbf{l})\,d\mu(\mathbf{l}), \tag{1.34}$$
$$\theta\neq n\pi.$$

Similarly, the operators P_1, D, K_1 generate an $SL(2,R)$ subgroup of $\widetilde{SO}(3,2)$ and we can verify the relation

$$\exp(2\theta\Gamma_{12})=\exp(\tan(\theta)P_1)\exp(K_1\sin\theta\cos\theta)\exp(-2D\ln\cos\theta),$$
$$2\Gamma_{12}=K_1+P_1,$$

or

$$\exp(2\theta\Gamma_{12})f(\mathbf{k})=(8\pi|\sin\theta|)^{-1}\exp(ik_1\cot\theta)\int\int\exp(il_1\cot\theta)$$
$$\times\cos\left[\frac{k_1(l_2+l_0)-l_1(k_2+k_0)}{\sin\theta(k_2+k_0)^{1/2}(l_2+l_0)^{1/2}}\right]f(\mathbf{l})\,d\mu(\mathbf{l}), \tag{1.35}$$
$$\theta\neq n\pi.$$

The operators (1.35) together with the operators $\exp(\theta M_{12})$, (1.28), determine the action of the $SO(3)$ subgroup.

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The known R-separable coordinate systems for (1.1) each correspond to a two-dimensional (commuting) subspace of the space of second-order symmetric elements in the enveloping algebra of $so(3,2)$. If the commuting operators form a basis for such a subspace, then the corresponding separated solutions of (1.1) are characterized by the eigenvalue equations

$$S_j\Psi=\lambda_j\Psi, \qquad j=1,2,$$

see [60–62]. Coordinate systems are considered equivalent if they can be mapped into one another by transformations generated by $\widetilde{SO}(3,2)$, S, T, and I. If a separable system corresponds to a subspace with basis $S_j=L_j^2$, $j=1,2$, such that $[L_1,L_2]=0$ and $L_j\in so(3,2)$, we call these coordinates *split*. In this case one can diagonalize the first-order operators L_j. Such systems are the best known and most tractable. More generally, if a system corresponds to a subspace in which there exists a basis $S_1=L^2$, S_2, $L\in so(3,2)$, we call these coordinates *semisplit*. Here, we can diagonalize the first-order operator L. If there exists no basis S_1,S_2 such that S_1 is the square of some $L\in so(3,2)$, we call the coordinates *nonsplit*. Nonsplit coordinates are the most intractable of all separable coordinates and appear the least frequently in applications.

A detailed (but still incomplete) study of R-separable solutions of (1.1) was carried out in [60–62]. Here we will be content with an examination of some of the most important semisplit systems. A given $L\in so(3,2)$ may correspond to several (or to no) semisplit systems. Indeed, if Ψ satisfies (1.1) and $L\Psi=i\lambda\Psi$, then since L is a symmetry of (1.1) we can introduce new variables y_0, y_1, y_2 such that $L=\partial_{y_0}+f(y)$ (where f may be zero) and $\Psi(y)=r(y)\exp(i\lambda y_0)\Phi_\lambda(y_1,y_2)$ where r is a fixed function satisfying $\partial_{y_0}r+fr=0$. Then (1.1) reduces to a second-order partial differential equation for Φ_λ in the two variables y_1, y_2. The semisplit systems we will study each correspond to systems such that the reduced equation separates. In particular, $S_1=L^2$ and S_2 corresponds to a second-order symmetry of the reduced equation.

In the next few sections we shall examine various possibilities for L that lead to semisplit systems.

4.2 The Laplace Operator on the Sphere

The first systems we shall study correspond to diagonalization of the operator Γ_{45}, (1.8). On restriction of our unitary irreducible representation of $\widetilde{SO}(3,2)$ on $\mathcal{H}_+$ to the compact subgroup $SO(3)$, this representation decomposes into a direct sum of irreducible representations D_l of $SO(3)$, $\dim D_l=2l+1$. We will determine a convenient basis for $\mathcal{H}_+$ which exhibits the decomposition. This is a basis of eigenfunctions of the com-

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muting operators Γ_{45} and $\Gamma_{23}=M_{12}$:

$$\Gamma_{45}f=i\lambda f, \qquad \Gamma_{23}f=imf, \qquad -i\Gamma_{45}=(k_0/2)(-\partial_{11}-\partial_{22}+1). \quad (2.1)$$

With $k_1=k\cos\theta$, $k_2=k\sin\theta$, $k_0=k$ it is easy to show that the ON basis of eigenvectors is

$$\begin{aligned} f_m^{(l)}(\mathbf{k})&=[(l-m)!/\pi(l+m)!]^{1/2}(2k)^m e^{-k}L_{l-m}^{(2m)}(2k)e^{im\theta}, \\ \lambda&=l+\tfrac{1}{2},\ l=0,1,\ldots,m=l,l-1,\ldots,-l. \end{aligned} \quad (2.2)$$

From this result and (1.14ii), we see that the $\{f_m^{(l)}\}$ for fixed l form an ON basis for the representation D_l of $SO(3)$. Furthermore, the restriction of our representation of $\widetilde{SO}(3,2)$ to $SO(3)$ decomposes as $\sum_{l=0}^{\infty}\oplus D_l$. The known recurrence formulas for Laguerre polynomials imply

$$\begin{aligned} \Gamma_{15}f_m^{(l)}&=\tfrac{1}{2}\left[(l-m+1)(l+m+1)\right]^{1/2}f_m^{(l+1)}-\tfrac{1}{2}\left[(l-m)(l+m)\right]^{1/2}f_m^{(l-1)}, \\ \Gamma_{42}f_m^{(l)}&=-\tfrac{1}{4}\left[(l+m+2)(l+m+1)\right]^{1/2}f_{m+1}^{(l+1)}+\tfrac{1}{4}\left[(l-m)(l-m-1)\right]^{1/2}f_{m+1}^{(l-1)} \\ &\quad+\tfrac{1}{4}\left[(l+m)(l+m-1)\right]^{1/2}f_{m-1}^{(l-1)}-\tfrac{1}{4}\left[(l-m+1)(l-m+2)\right]^{1/2}f_{m-1}^{(l+1)}. \end{aligned} \quad (2.3)$$

Using (2.1), (2.3) and taking commutators, we can compute the action of $\Gamma_{\alpha\beta}$ on this basis.

Note the close connection between the eigenvalue equation $\Gamma_{45}f=i\lambda f$ and the quantum Kepler problem in two-dimensional space:

$$\begin{aligned} &Hg=\mu g, \qquad H=-\partial_{xx}-\partial_{yy}+e/r, \\ &r=(x^2+y^2)^{1/2}, \iint_{R^2}|g|^2\,dx\,dy<\infty. \end{aligned} \quad (2.4)$$

The two eigenvalue equations can be identified provided $k_1=x(-\mu)^{1/2}$, $k_2=y(-\mu)^{1/2}$, $\mu=-e^2/4\lambda^2$. The eigenvalue problems are defined on Hilbert spaces with different inner products, but from the Virial theorem [31, p. 51] we see that if the energy eigenvalue μ belongs to the point spectrum of H and g is a corresponding eigenvector, then g also has finite norm in $\mathcal{H}_+$. Conversely, if f is an eigenfunction of Γ_{45}, then $\iint|f|^2\,dx\,dy<\infty$ and f corresponds to an energy eigenvalue μ in the point spectrum of H. Since the eigenvalues λ of Γ_{45} are $\lambda=l+\frac{1}{2}$, $l=0,1,\ldots$, it follows that the point eigenvalues of H are $\mu_l=-e^2/4(l+\frac{1}{2})^2$. Although this is a satisfying explanation of the point spectrum of H, it sheds no light on the continuous spectrum of H, since Γ_{45} has only a point spectrum.

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Using the mapping (1.20) we can compute the corresponding ON basis of positive energy solutions of (1.1):

$$\Psi_m^{(l)}(x)=\left[\frac{(l-m)!}{4\pi(l+m)!}\right]^{1/2}\exp\left[im\left(\alpha-\frac{\pi}{2}\right)\right]$$

$$\times\int_0^\infty \exp[(ix_0-1)k](2k)^m J_m(kr)L_{l-m}^{(2m)}(2k)dk, \qquad (2.5)$$

$$x_1=r\cos\alpha,\ x_2=r\sin\alpha.$$

In terms of the coordinates

$$x_0=\sin\psi/(\cos\sigma-\cos\psi), \qquad x_1=\sin\sigma\cos\alpha/(\cos\sigma-\cos\psi),$$
$$x_2=\sin\sigma\sin\alpha/(\cos\sigma-\cos\psi), \qquad (2.6)$$

variables R-separate in (1.1), (2.5) to give

$$\Psi_m^{(l)}(x)=(-i)^{m-1}\left[(\cos\sigma-\cos\psi)/(4l+2)\right]^{1/2}\exp\left[-i\psi\left(l+\tfrac{1}{2}\right)\right]Y_l^m(\sigma,\alpha), \qquad (2.7)$$

where Y_l^m is a spherical harmonic. (We can always parametrize so $\cos\sigma-\cos\psi>0$, see [63].) Indeed, on the solution space of (1.1) we find

$$\Gamma_{45}=-\partial_\psi+\tfrac{1}{2}\frac{\sin\psi}{\cos\sigma-\cos\psi}, \qquad \Gamma_{23}=\partial_\alpha, \qquad (2.8)$$

so

$$\Psi_m^{(l)}=(\cos\sigma-\cos\psi)^{1/2}\exp\left[-i\psi\left(l+\tfrac{1}{2}\right)\right]\exp(im\alpha)\,g(\sigma),$$

and substituting into (1.1), we see that variables R-separate and $g(\sigma)$ is a linear combination of $P_l^m(\cos\sigma)$, $Q_l^m(\cos\sigma)$. Evaluating the integral (2.5) for special values of the parameters (e.g., $\sigma=0,\pi$), we establish (2.7).

There is another model of our irreducible representation of $\widetilde{SO}(3,2)$ in which the eigenfunctions of Γ_{45} and Γ_{23} take an especially simple form. The representation space is the Bargmann–Segal Hilbert space $\mathfrak{F}_2$ consisting of all entire functions $h(z_1,z_2)$ such that [11]

$$\int_{\mathbb{C}\times\mathbb{C}}|h|^2 d\xi(\mathbf{z})<\infty, \quad d\xi(\mathbf{z})=\pi^{-2}\exp[-(|z_1|^2+|z_2|^2)]dx_1dx_2dy_1dy_2, \qquad (2.9)$$

$$z_j=x_j+iy_j.$$

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The inner product is

$$\langle f,h\rangle=\int_{\mathbb{C}\times\mathbb{C}} f\bar{h}\,d\xi(\mathbf{z}).$$

The carrier space for our representation is not $\mathfrak{F}_2$ but the subspace $\mathfrak{F}_2^+$ consisting of all $h\in\mathfrak{F}_2$ such that $h(-z_1,-z_2)=h(z_1,z_2)$. The functions

$$f_m^{(l)}(z)=z_1^{l+m}z_2^{l-m}/\left[(l+m)!(l-m)!\right]^{1/2},\qquad l=0,1,2,\ldots,m=l,\ldots,-l, \tag{2.10}$$

form an ON basis for $\mathfrak{F}_2^+$. Setting

$$\begin{aligned}\Gamma_{45}&=\frac{i}{2}(z_1\partial_{z_1}+z_2\partial_{z_2}+1), & \Gamma_{15}&=\frac{1}{2}(z_1z_2-\partial_{z_1z_2}),\\ \Gamma_{23}&=\frac{i}{2}(z_1\partial_{z_1}-z_2\partial_{z_2}), & \Gamma_{42}&=\frac{1}{4}\left(\partial_{z_1z_1}+\partial_{z_2z_2}-z_1^2-z_2^2\right),\end{aligned} \tag{2.11}$$

and comparing with expressions (2.3), we see that there is a new model of our representation of $\widetilde{SO}(3,2)$ in which the functions $f_m^{(l)}(\mathbf{k})$ can be identified with the functions (2.10). The explicit unitary mapping U from $\mathcal{H}^+$ to $\mathfrak{F}_2^+$ that commutes with the group action is

$$Uf(\mathbf{z})=\iint_{R^2} U(\mathbf{k},\mathbf{z})f(\mathbf{k})\,d\mu(\mathbf{k}),\qquad f\in\mathcal{H}^+, \tag{2.12}$$

where

$$\begin{aligned}U(\mathbf{k},\mathbf{z})=\sum_{l,m}\bar{f}_m^{(l)}(\mathbf{k})f_m^{(l)}(\mathbf{z})\ &=\pi^{-1/2}\exp(-k+z_1z_2)\\ &\times\cosh\left\{\sqrt{2k}\left[z_1\exp(i\theta/2)-z_2\exp(i\theta/2)\right]\right\},\\ &k_1=k\cos\theta,\ k_2=k\sin\theta.\end{aligned} \tag{2.13}$$

(Note that $f_m^{(l)}(\mathbf{k})\in\mathcal{H}^+$ and $f_m^{(l)}(\mathbf{z})\in\mathfrak{F}_2^+$.)

To understand more clearly the significance of the coordinates (2.6), note that if Ψ is a solution of (1.1) such that $\Gamma_{45}\Psi=i(l+\frac{1}{2})\Psi$, then $\Psi(\sigma,\alpha,\psi)=(\cos\alpha-\cos\psi)^{1/2}\exp[-i\psi(l+\frac{1}{2})]\Phi(\sigma,\alpha)$ where Φ is an eigenfunction of the Laplace operator on the sphere ((2.20), Section 3.2) ($\sigma=\theta,\alpha=\varphi$). Equation (2.20) separates in two coordinate systems, as we saw in Section 3.3. The first system (spherical coordinates $\{\sigma,\alpha\}$) leads to the R-separable solutions (2.7) of (1.1) that are characterized by diagonalization of the operators

1. Γ_{45}^2, Γ_{23}^2.

However, there is also a Lamé-type system which leads to R-separable

ISBN-0-201-13503-5

solutions of (1.1) characterized by diagonalization of

2. $\Gamma_{45}^2, \Gamma_{12}^2 + a^2\Gamma_{13}^2, \qquad a>0.$

The overlaps between these bases are just those computed in Section 3.3.

4.3 Diagonalization of P_0, P_2, and D

We next look for those coordinate systems permitting separation of variables in (1.1) such that the corresponding basis functions Ψ are eigenfunctions of $P_0: P_0\Psi = i\omega\Psi$. For such systems we have $\Psi(x) = \exp(i\omega x_0)\Phi(x_1, x_2)$ where

$$(\partial_{11} + \partial_{22} + \omega^2)\Phi = 0. \tag{3.1}$$

Thus the equation for the eigenfunctions reduces to the Helmholtz equation. Now P_0 commutes with every element in the Euclidean Lie algebra $\mathcal{E}(2)$ generated by P_1, P_2, M_{12} and, as we know from Chapter 1, $\mathcal{E}(2)$ is the symmetry algebra of (3.1). Furthermore, (3.1) separates in four coordinate systems, each system corresponding to a symmetric second-order element in the enveloping algebra of $\mathcal{E}(2)$ (see Table 1). The four associated separable systems for (1.1) are characterized by diagonalization of the operators in Table 18.

On $\mathcal{H}_+$ the requirement $P_0 f = i\omega f$ implies $f(\mathbf{k}) = \delta(k-\omega)g_\omega(\theta)$ where $\omega > 0$, $k_1 = k\cos\theta$, $k_2 = k\sin\theta$. The search for the functions g_ω reduces to a study of the Hilbert space $L_2(S_2)$ on which $E(2)$ acts via

$$P_1 = -i\omega\cos\theta, \qquad P_2 = -i\omega\sin\theta, \qquad M_{12} = \partial_\theta.$$

These operators determine a unitary irreducible representation of $E(2)$ on $L_2(S_2)$. Once the eigenfunctions $g_{\omega\mu}(\theta)$ of the second operator in 3–6 in Table 18 have been determined, the corresponding separated solutions $\Psi_{\omega\mu}$ of (1.1) can be obtained from the relation

$$\Psi_{\omega\mu}(x) = (4\pi)^{-1}\exp(i\omega x_0)\int_{-\pi}^{\pi}\exp[-i\omega(x_1\cos\theta + x_2\sin\theta)]\,g_{\omega\mu}(\theta)\,d\theta. \tag{3.2}$$

Table 18

3	P_0^2, P_1^2	Cartesian
4	P_0^2, M_{12}^2	Polar
5	$P_0^2, \{M_{12}, P_2\}$	Parabolic cylinder
6	$P_0^2, M_{12}^2 + d^2P_2^2$	Elliptic

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Note that this model is essentially identical to the circle model studied in Chapter 1. Thus, the spectral resolutions and overlaps computed there can be immediately carried over to the wave equation.

Now we search for coordinate systems allowing separation of variables in (1.1) such that the basis functions Ψ are eigenfunctions of $P_2: P_2\Psi = -i\omega\Psi$. Here we have $\Psi(x)=\exp(-i\omega x_2)\Phi(x_0,x_1)$ where

$$(\partial_{00}-\partial_{11}+\omega^2)\Phi=0. \tag{3.3}$$

The operator P_2 commutes with the subalgebra $\mathcal{E}(1,1)$ generated by P_0, P_1, M_{01} and, indeed, $\mathcal{E}(1,1)$ is the symmetry algebra of (3.3). This equation separates in ten coordinate systems associated with ten symmetric second-order operators in the enveloping algebra of $\mathcal{E}(1,1)$ (see Table 2). The pairs of commuting operators associated with the corresponding separable solutions of (1.1) are listed in Table 19. The case 3′ is equivalent to 3 in Table 18.

On $\mathcal{H}_+$ the requirement $P_2 f = -i\omega f$ implies $f(\mathbf{k})=\delta(k_2-\omega)g_\omega(\zeta)$ where $-\infty<\omega<\infty, k_1=|k_2|\sinh\zeta, k_0=|k_2|\cosh\zeta$. The search for eigenfunctions reduces to a study of the Hilbert space $L_2(R)$ on which $E(1,1)$ acts via

$$P_0=i|\omega|\cosh\xi, \qquad P_1=-i|\omega|\sinh\xi, \qquad M_{01}=\partial_\xi. \tag{3.4}$$

These operators define a unitary irreducible representation of $E(1,1)$ on $L_2(R)$. After the eigenfunctions $g_{\omega\mu}(\xi)$ of the second operator in 7–15 (Table 19) have been determined, the corresponding separable solutions $\Psi_{\omega\mu}$ of (1.1) follow from

$$\Psi_{\omega\mu}(x)=(4\pi)^{-1}\exp(-i\omega x_2)\int_{-\infty}^{\infty}\exp\left[i|\omega|(x_0\cosh\xi-x_1\sinh\xi)\right]g_{\omega\mu}|(\xi)\,d\xi. \tag{3.5}$$

This is virtually identical to the $L_2(R)$ model discussed in Chapter 1, and the spectral resolutions and overlaps derived there can be carried over to the wave equation.

Next we look for coordinate systems yielding separation of variables in (1.1) such that the basis functions Ψ are eigenfunctions of $D: D\Psi=-i\nu\Psi$.

Table 19

3′	P_2^2, P_0, P_1	11	$P_2^2, M_{01}^2-P_0P_1$
7	P_2^2, M_{01}^2	12	$P_2^2, M_{01}^2+(P_0+P_1)^2$
8	$P_2^2, \{M_{01}, P_1\}$	13	$P_2^2, M_{01}^2-(P_0+P_1)^2$
9	$P_2^2, \{M_{01}, P_0\}$	14	$P_2^2, M_{01}^2+P_1^2$
10	$P_2^2, \{M_{01}, P_0-P_1\}+(P_0+P_1)^2$	15	$P_2^2, M_{01}^2-P_1^2$

ISBN-0-201-13503-5

In this case we have $\Psi(x)=\rho^{i\nu-\frac{1}{2}}\Phi(s)$ where

$$x_\alpha=\rho s_\alpha \qquad (\rho\geqslant 0), \qquad s_0^2-s_1^2-s_2^2=\varepsilon,$$

and $\varepsilon=\pm 1$ or 0 depending on whether $x\cdot x>0$, <0, or $=0$. It follows from (1.14iii) that

$$\left(M_{12}^2-M_{01}^2-M_{02}^2\right)\Phi(s)=\left(\nu^2+\tfrac{1}{4}\right)\Phi(s). \tag{3.6}$$

The operators $M_{\alpha\beta}$, (1.3), satisfy the commutation relations

$$[M_{12},M_{01}]=-M_{02}, \qquad [M_{12},M_{02}]=M_{01}, \qquad [M_{01},M_{02}]=M_{12}, \tag{3.7}$$

so they form a basis for the subalgebra $sl(2,R)\cong so(2,1)$ (see Section 2.1). Now D commutes with this subalgebra and in fact $SO(2,1)$ is the symmetry group of (3.6). The Casimir operator $M_{12}^2-M_{01}^2-M_{02}^2$ commutes with all elements of $so(2,1)$. As shown in [139], the space of second-order symmetry operators in the enveloping algebra of $so(2,1)$, modulo the Casimir operator, is decomposed into nine orbit types under the adjoint action of $SO(2,1)$. (The groups $SO(2,1)$ and $SL(2,R)$ are locally isomorphic.) Moreover, the reduced equation (3.6) separates in nine coordinate systems, each system associated with a unique operator orbit. The coordinate systems for $\varepsilon=1$ can be found in [58, 104], in which case (3.6) is the eigenvalue equation for the *Laplace operator on the hyperboloid*. Coordinates for all cases $\varepsilon=\pm 1,0$ are derived in [61]. Referring to the papers just cited for details, we give here (Table 20) only the functional forms of the separated solutions of (3.6), the names of the coordinate systems, and the pairs of commuting operators associated with the corresponding separable solutions of (1.1). System 7′ is equivalent to 7.

On $\mathcal{H}_+$ the requirement $Df=-i\nu f$ implies $f(\mathbf{k})=k^{-i\nu-\frac{1}{2}}h_\nu(\theta)$ where $-\infty<\nu<\infty$, $k_1=k\cos\theta$, $k_2=k\sin\theta$. The eigenfunction problem thus reduces to a study of the Hilbert space $L_2(S_2)$ on which $SO(2,1)$ acts via

$$\begin{aligned} M_{12}&=\partial_\theta, \qquad M_{01}=-\sin\theta\,\partial_\theta-\left(i\nu+\tfrac{1}{2}\right)\cos\theta,\\ M_{02}&=\cos\theta\,\partial_\theta-\left(i\nu+\tfrac{1}{2}\right)\sin\theta. \end{aligned} \tag{3.8}$$

These operators define a unitary irreducible representation of $SO(2,1)$ that is single valued and belongs to the principal series: $l=-\frac{1}{2}+i|\nu|$ (see [10, 115]). Once the eigenfunctions $h_{\nu\alpha}(\theta)$ of the second operator in 16–23 (Table 20) have been determined, the corresponding separable solutions

ISBN-0-201-13503-5

Table 20

Operators	Coordinates	Separated functions
16 D^2, M_{12}^2	Spherical	Exponential Associated Legendre
17 D^2, M_{01}^2	Equidistant	Exponential Associated Legendre
7′ $D^2, (M_{12}-M_{02})^2$	Horocyclic	Exponential Macdonald
18 $D^2, M_{12}^2+a^2M_{01}^2$	Elliptic	Periodic Lamé Periodic Lamé
19 $D^2, M_{01}^2-a^2M_{12}^2$, $0<a<1$	Hyperbolic	Lamé–Wangerin Lamé–Wangerin
20 $D^2, aM_{01}^2-\{M_{12}, M_{02}\}$, $0<a$	Semihyperbolic	Lamé–Wangerin Lamé–Wangerin
21 $D^2, aM_{01}^2+M_{02}^2+M_{12}^2-\{M_{12}, M_{02}\}$, $0<a$	Elliptic-parabolic	Associated Legendre Associated Legendre
22 $D^2, -aM_{01}^2+M_{02}^2+M_{12}^2-\{M_{12}, M_{02}\}$, $0<a$	Hyperbolic-parabolic	Associated Legendre Associated Legendre
23 $D^2, \{M_{01}, M_{02}\}-\{M_{12}, M_{01}\}$	Semicircular-parabolic	Bessel Macdonald

$\Psi_{\nu\alpha}$ of (1.1) can be obtained from

$$\Psi_{\nu\alpha}(x)=\rho^{i\nu-\frac{1}{2}}(4\pi)^{-1}\Gamma\left(\tfrac{1}{2}-i\nu\right)\int_0^{2\pi}\exp\left[\pm i\pi\left(\tfrac{1}{2}-i\nu\right)/2\right]$$
$$\times|s_0-s_1\cos\theta-s_2\sin\theta|^{i\nu-\frac{1}{2}}h_{\nu\alpha}(\theta)\,d\theta \tag{3.9}$$

where the plus sign occurs when $s_0-s_1\cos\theta-s_2\sin\theta>0$ and the minus sign occurs when this expression is <0. The spectral resolutions of the operators 16–23 and various overlaps computed in the $L_2(S_2)$ model can be found in [58]. (See also [54] for mixed-basis matrix elements corresponding to subgroup systems.)

4.4 The Schrödinger and EPD Equations

Of special interest are the coordinate systems permitting separation of variables in (1.1) such that the basis functions Ψ are eigenfunctions of P_0+P_1:$(P_0+P_1)\Psi=i\beta\Psi$. For this case we have $\Psi(x)=e^{is\beta}\Phi(t,x_2)$ where $2s=x_0+x_1, 2t=x_1-x_0$. The reduced equation for Φ is the free-particle Schrödinger equation

$$(i\beta\partial_t+\partial_{x_2x_2})\Phi(t,x_2)=0. \tag{4.1}$$

ISBN-0-201-13503-5

This equation admits as symmetries the operators

$$\mathcal{K}_{-1}=P_2,\qquad \mathcal{K}_{-2}=P_1-P_0,\qquad \mathcal{K}_0=P_0+P_1,\qquad \mathcal{K}_1=\tfrac{1}{2}(M_{02}-M_{12}),$$
$$\mathcal{K}^0=-D-M_{01},\qquad \mathcal{K}_2=-\tfrac{1}{2}(K_0+K_1), \tag{4.2}$$

which all commute with $P_0+P_1=\mathcal{K}_0$. As we showed in Section 2.1, these operators form a basis for the six-dimensional Schrödinger algebra $\mathcal{G}_2$, the symmetry algebra of (4.1). (Note that the constant β can be set equal to 1 in (4.1) by a renormalization of t and x_2.) The pairs of commuting operators associated with separable systems for (4.1) are listed in Table 21. (Coordinates 3″ are essentially equivalent to 3 in Table 18.) These results follow from Table 6. Note that here the defining operators are first order, rather than second order, in the enveloping algebra. This is because they appear as first order in the explicit separation equations. All of the earlier listed semisubgroup coordinate systems have been orthogonal with respect to the Minkowski metric. However, the four systems in Table 21 are nonorthogonal.

On $\mathcal{K}_+$ the requirement $(P_0+P_1)f=i\beta f$ implies $f(\mathbf{k})=u\delta(u-\beta)l_\beta(v)$ where $\beta>0, u=k_0-k_1, v=k_2$. Thus, the search for the l_β reduces to a study of the Hilbert space $L_2(R)$ on which the Schrödinger group acts via

$$\mathcal{K}_0=i\beta,\qquad \mathcal{K}_{-1}=-iv,\qquad \mathcal{K}_1=\frac{\beta}{2}\partial_v,\qquad \mathcal{K}^0=-\tfrac{1}{2}-v\partial_v,$$
$$\mathcal{K}_{-2}=-\frac{iv^2}{\beta},\quad \mathcal{K}_2=-i\frac{\beta}{2}\partial_{vv}. \tag{4.3}$$

As shown in Section 2.1, these operators determine an irreducible unitary representation of the Schrödinger group on $L_2(R)$. (Indeed, for $\beta=1$ the operators (1.24), Section 2.1, are unitary equivalent via the Fourier transform to the operators given here, and for $\beta\neq 1$ our earlier results can easily be modified to yield the global group action.) Once the eigenfunctions $l_{\beta\mu}(v)$ of the second operators in 3″,24–26 (Table 21) have been determined, the corresponding separable solutions $\Psi_{\beta\mu}$ of (1.1) can be computed from

$$\Psi_{\beta\mu}(x)=(4\pi)^{-1}\exp(i\beta s)\int_{-\infty}^{\infty}\exp\left[-i\left(v^2t/\beta+vx_2\right)\right]l_{\beta\mu}(v)\,dv. \tag{4.4}$$

Table 21

3″	P_0+P_1, P_2	Free particle
24	$P_0+P_1, P_0-P_1-\frac{1}{4}K_0-\frac{1}{4}K_1$	Oscillator
25	$P_0+P_1, P_0-P_1+aM_{12}-aM_{02}$, $a\neq 0$	Linear potential
26	$P_0+P_1, D+M_{01}$	Repulsive oscillator

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Next we look for coordinate systems yielding separation of variables for (1.1) such that the basis functions Ψ are eigenfunctions of $M_{12}: M_{12}\Psi = im\Psi$. We have $\Psi(x) = e^{im\varphi}\Phi(x_0, r)$ where $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$, and Φ satisfies the Euler–Poisson–Darboux (EPD) equation

$$(\partial_{00} - \partial_{rr} - r^{-1}\partial_r + m^2 r^{-2})\Phi = 0 \tag{4.5}$$

or

$$(\Gamma_{45}^2 - \Gamma_{41}^2 - \Gamma_{51}^2)\Phi = (\Gamma_{23}^2 + \tfrac{1}{4})\Phi = -(m + \tfrac{1}{2})(m - \tfrac{1}{2})\Phi \tag{4.6}$$

from (1.14*iv*). The symmetry algebra of (4.5) is $sl(2, R)$, generated by the operators $\Gamma_{45}, \Gamma_{41}, \Gamma_{51}$, and the symmetry group (for m integer) is $SL(2, R)$:

$$[\Gamma_{41}, \Gamma_{51}] = -\Gamma_{45}, \qquad [\Gamma_{41}, \Gamma_{45}] = -\Gamma_{51}, \qquad [\Gamma_{51}, \Gamma_{45}] = \Gamma_{41}. \tag{4.7}$$

In [63] it is shown that the EPD equation R-separates in exactly nine coordinate systems corresponding to the nine $SL(2, R)$ orbit types of symmetric second-order operators in the enveloping algebra of $sl(2, R)$, modulo the Casimir operator $\Gamma_{45}^2 - \Gamma_{41}^2 - \Gamma_{51}^2$. We list in Table 22 only the operator characterizations of the R-separable solutions of (1.1) together with the functional forms of the associated solutions of (4.5) ($\Gamma_{23} = M_{12}, \Gamma_{51} = D$, $\Gamma_{45} = (P_0 - K_0)/2$, $\Gamma_{41} = (P_0 + K_0)/2$). The truly R-separable systems are 1′ and 29–31.

Table 22

Operators		Separated functions
1′	$\Gamma_{23}^2, \Gamma_{45}^2$	Exponential
		Gegenbauer
4′	$\Gamma_{23}^2, (\Gamma_{45} + \Gamma_{41})^2$	Exponential
		Bessel
16′	$\Gamma_{23}^2, \Gamma_{51}^2$	Exponential
		Associated Legendre
27	$\Gamma_{23}^2, 2\Gamma_{41}^2 + \{\Gamma_{45}, \Gamma_{41}\}$	Associated Legendre
		Associated Legendre
28	$\Gamma_{23}^2, 2\Gamma_{45}^2 + \{\Gamma_{45}, \Gamma_{41}\}$	Associated Legendre
		Associated Legendre
29	$\Gamma_{23}^2, \Gamma_{41}^2 + a\{\Gamma_{45}, \Gamma_{51}\}$	Lamé–Wangerin
		Lamé–Wangerin
30	$\Gamma_{23}^2, \Gamma_{45}^2 + a\Gamma_{51}^2$,	Lamé–Wangerin
	$a > 0$	Lamé–Wangerin
31	$\Gamma_{23}^2, a\Gamma_{41}^2 + \Gamma_{51}^2$,	Lamé–Wangerin
	$a > 1$	Lamé–Wangerin
32	$\Gamma_{23}^2, \{\Gamma_{51}, \Gamma_{41} + \Gamma_{45}\}$	Bessel
		Bessel

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On $\mathcal{H}_+$ the requirement $M_{12}f = imf$ implies $f(\mathbf{k}) = e^{im\theta}j_m(k)$ where $m = 0, \pm 1, \ldots, k_1 = k\cos\theta, k_2 = k\sin\theta$. The eigenfunction problem reduces to a study of the Hilbert space $L_2[0,\infty]$ on which $SL(2,R)$ acts via

$$\begin{aligned} \Gamma_{45} &= \frac{ik}{2}\left(-\partial_{kk} - k^{-1}\partial_k + m^2k^{-2} + 1\right), \\ \Gamma_{41} &= \frac{ik}{2}\left(\partial_{kk} + k^{-1}\partial_k - m^2k^{-2} + 1\right), \qquad \Gamma_{51} = k\,\partial_k + \tfrac{1}{2}. \end{aligned} \tag{4.8}$$

This action is irreducible and unitary equivalent to a single-valued representation of $SL(2,R)$, not $SO(2,1)$, from the negative discrete series $D^-_{|m|-\frac{1}{2}}$, as can be seen from (4.6) and (2.2). (Compare with Section 2.3.) Indeed, the eigenvalues of Γ_{45} in this model are $i(n+\frac{1}{2}), n = |m|, |m|+1, |m|+2, \ldots$. This model of D_l^- has been studied by a number of authors (e.g., [24, 96].)

Once the eigenfunctions $j_{m\mu}(k)$ of the second operators in Table 22 have been determined, the corresponding separable solutions $\Psi_{m\mu}$ of (1.1) can be computed from

$$\Psi_{m\mu}(x) = \exp\left[im(\theta - \pi/2)\right]\int_0^\infty \exp(ix_0k)J_m(kr)j_{m\mu}(k)\,dk. \tag{4.9}$$

More generally one can study the EPD equation (4.5) for any real $m > 0$. The separable coordinate systems and model (4.8) are unchanged but the symmetry group becomes the universal covering group $\widetilde{SL}(2,R)$ of $SL(2,R)$, as in Section 2.3. The mapping from $L_2[0,\infty]$ to the solution space of (4.5) is

$$\Phi(x_0,r) = \exp(-im\pi/2)\int_0^\infty \exp(ix_0k)J_m(kr)f(k)\,dk = U[f] \tag{4.10}$$

and the associated inner product is

$$\begin{aligned} (\Phi_1,\Phi_2) \equiv \langle f_1, f_2\rangle &= i\int_0^\infty \Phi_1(x_0,r)\,\partial_0\overline{\Phi}_2(x_0,r)r\,dr \\ &= -i\int_0^\infty \overline{\Phi}_2(x_0,r)\,\partial_0\Phi_1(x_0,r)r\,dr, \end{aligned} \tag{4.11}$$

independent of x_0. Details concerning the spectral resolutions of the operators that determine the separated solutions can be found in [63].

We have characterized the solutions Φ_m of the EPD equation (4.5) as solutions of the wave equation (1.1) that are eigenfunctions of $L = -iM_{12}: L\Psi_m = m\Psi_m, \Psi_m = e^{im\varphi}\Phi_m(x_0,r)$. One can choose a basis $\{L_j\}$ for the complexification $so(3,2)^c \cong so(5,\mathbb{C})$ of the conformal symmetry algebra

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such that $[L, L_j] = \alpha_j L_j$ where $\alpha_j = 0, \pm 1$. Indeed, the commutation relations

$$[L, P_1 \pm iP_2] = \pm(P_1 \pm iP_2), \qquad [L, M_{01} \pm iM_{02}] = \pm(M_{01} \pm iM_{02}),$$
$$[L, K_1 \pm iK_2] = \pm(K_1 \pm iK_2),$$

together with the fact that $[L, L'] = 0$ for $L' = D, P_0, K_0$ provides such a basis. It follows from these relations that $L_j \Psi_m$ is an eigenfunction of L with eigenvalue $m + \alpha_j = m, m \pm 1$; that is, $L_j(e^{im\varphi}\Phi_m) = \exp[i(m + \alpha_j)\varphi]\Phi_{m+\alpha_j}$. Factoring out the φ dependence, we see that each symmetry operator maps a solution of (4.5) for m to a solution for $m + \alpha_j$. Similarly, the operators (1.12) induce mappings from one EPD equation to another, as do certain of the group symmetry operators.

We see that this series of recurrence formulas relating distinct EPD equations to one another is a direct consequence of the conformal symmetry of the wave equation, from which the EPD equation arises by partial separation of variables. Weinstein [131, 132] has made use of two of these recurrence relations in his study of boundary value problems for the EPD equation. A complete group-theoretic discussion appears in [93], where it is also shown that quadratic transformation formulas for the ${}_2F_1$ [36] are related to the conformal symmetry of the wave equation.

We have mentioned all the semisplit systems for the wave equation with the exception of some curious nonorthogonal systems which correspond to diagonalization of the operator $\frac{1}{2}M_{12} + \frac{1}{4}K_0 - \frac{1}{4}P_0$ and are discussed in [60, 62], as well as some highly singular solutions, discussed in [62], that arise because diagonalization of a given first-order operator does not uniquely determine the corresponding coordinate. Orthogonal nonsplit coordinates are treated in [61].

4.5 The Wave Equation $(\partial_{tt} - \Delta_3)\Psi(x) = 0$

In many respects the real wave equation in four-dimensional space-time

$$(\partial_{00} - \partial_{11} - \partial_{22} - \partial_{33})\Psi(x) = 0 \tag{5.1}$$

is the most important equation in this book. In addition to the well-known physical importance of (5.1), [12, 107], it is a fact that virtually every equation examined in the earlier chapters is either a special case of (5.1) or is obtained from (5.1) by a partial separation of variables. Moreover, whereas the three-space wave equation and its complexification are associated with the generating functions for Gegenbauer functions and polynomials, (5.1) is associated with generating functions for the general Gaussian hypergeometric function and Jacobi polynomials.

Although (5.1) is presently undergoing intensive study from a group-theoretic viewpoint, the results at this writing are still fragmentary. We shall

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limit ourselves here to the indication of some general features of the separation of variables problem for (5.1) and a brief discussion of relevant published papers.

The 15-dimensional symmetry algebra $so(4,2)$ of (5.1) was computed in [14] and can be obtained in obvious analogy to that of (1.1). The symmetry group, locally isomorphic to $SO(4,2)$, is called the conformal group. It contains the homogeneous Lorentz group $SO(3,1)$, the Poincaré group $E(3,1)$, and the compact orthogonal group $SO(4,R)$ as proper subgroups. There is also an inversion symmetry analogous to I, (1.12). By utilizing the Fourier transform one can construct a Hilbert space $\mathcal{H}_+$ of positive energy solutions on which there is defined a unitary irreducible representation of the conformal group. This is carried out in analogy with (1.20) and details are presented in [44, 66, 118].

One expects the R-separable solutions of (5.1) to be characterized as simultaneous eigenfunctions of triplets of independent commuting operators that are at most second order in the enveloping algebra of $so(4,2)$. We will discuss a few of the special cases in which the details have been worked out.

By restricting the symmetry algebra of (1.1) to the compact subalgebra $so(3)$ we were led to the Laplace operator on the sphere S_2 and obtained two separable systems. Similarly, by restricting $so(4,2)$ to the compact subalgebra $so(4)$, we obtain the Laplace operator on the unit sphere S_3 in four-dimensional space. This operator is studied in [65], where it is shown that the eigenvalue equation separates in exactly six coordinate systems associated with six commuting pairs of second-order symmetry operators in the enveloping algebra of $so(4)$. The relationship between $so(4)$ and the Schrödinger equation for the Kepler problem in three space variables is also discussed.

Diagonalization of the symmetry operator $P_0 = \partial_0$ reduces (5.1) to the Helmholtz equation, which separates in eleven coordinate systems. Diagonalization of $P_3 = \partial_3$ reduces (5.1) to the Klein–Gordon equation

$$(\partial_{00} - \partial_{11} - \partial_{22} + \omega^2)\Phi = 0. \tag{5.2}$$

In [61], 53 Minkowski-orthogonal separable systems for (5.2) were classified. Diagonalization of the dilatation symmetry $\Sigma_{\alpha=0}^{3} x_\alpha \partial_\alpha$ reduces (5.1) to the eigenvalue equation for the Laplace operator on a hyperboloid in four-dimensional space. The reduced equation admits the homogeneous Lorentz group $SO(3,1)$ as its symmetry group and separates in 34 coordinate systems, each corresponding to a pair of second-order symmetric operators in the enveloping algebra of $so(3,1)$ [104, 64]. Diagonalization of $P_0 + P_1 = \partial_0 + \partial_1$ reduces (5.1) to the free-particle Schrödinger equation

$$(i\beta\partial_t + \partial_{22} + \partial_{33})\Phi = 0, \tag{5.3}$$

which separates in 17 corrdinate systems. Similarly, diagonalization of the symmetry $M_{23}=x_2\partial_3-x_3\partial_2$ leads to a reduced EPD-like equation. Bateman has used the complexification of the reduced equation obtained by diagonalizing both M_{23} and $M_{01}=x_0\partial_1+x_1\partial_0$ to derive generating functions for Jacobi polynomials [13, p. 392], and Koornwinder [68,68a], has used it in connection with his study of the addition theorem for Jacobi polynomials. Henrici employed the same equation to derive generating functions for products of Gegenbauer polynomials [48].

Although the systems above were obtained in complete analogy with our treatment of (1.1), there are some novel types of nonsplit coordinates that appear for (5.1). For example, diagonalization of $P_2^2+P_3^2$ reduces (5.1) to the two equations

$$(\partial_{00}-\partial_{11}+\omega^2)\Phi=0, \qquad (\partial_{22}+\partial_{33}+\omega^2)\Theta=0, \tag{5.4}$$

where $\Psi=\Phi\Theta$. The possible separable systems for the reduced equations can be read off from Tables 1 and 2.

The explicit connection between the functions ${}_2F_1$ and the wave equation will be discussed in the following chapter.

Exercises

1. Compute the symmetry algebra of the wave equation (1.1).

2. Let $y_0=\cos\sigma, y_1=\sin\sigma\cos\alpha, y_2=\sin\sigma\sin\alpha$ where (ψ,σ,α) are the R-separable coordinates (2.6) for the wave equation (1.1). Show that substitution of $\Psi=[\cos\sigma-\cos\psi]^{1/2}\exp[-i\psi(l+\frac{1}{2})]\Phi(y_1,y_2,y_3)$ into the wave equation leads to the reduced equation $(\Gamma_{12}^2+\Gamma_{13}^2+\Gamma_{23}^2)\Phi=-l(l+1)\Phi$, the eigenvalue equation for the Laplace operator on the sphere $y_0^2+y_1^2+y_2^2=1$. Here $\Gamma_{12},\Gamma_{13},\Gamma_{23}$ are the usual angular momentum operators on the sphere.

3. Show that the space of second-order symmetry operators in the enveloping algebra of $so(2,1)$, modulo the Casimir operator, is decomposed into nine orbit types under the adjoint action of $SO(2,1)$. (Hint: This problem is equivalent to classifying the equivalence classes of 3×3 real symmetric matrices Q under the conjugacy transformations $Q\to A^tQA, A\in SO(2,1)$. For more details see [139].)

4. Show that the EPD equation (4.5) separates in the variables

$$x=\frac{1}{2}\left[(t+r)^{1/2}+(t-r)^{1/2}\right], \qquad y=\frac{1}{2}\left[(t+r)^{1/2}-(t-r)^{1/2}\right],$$
$$t\pm r>0,$$

corresponding to the operators $\Gamma_{23}^2, \{\Gamma_{51},\Gamma_{41}+\Gamma_{45}\}$. The separated solutions are products of Bessel functions [63].

5. As shown in the text, a function $\Phi(x_0,r)$ is a solution of the EPD equation

$$(\partial_{00}-\partial_{rr}-r^{-1}\partial_r+m^2r^{-2})\Phi=0$$

if and only if $\Psi_m = e^{im\varphi}\Phi$ is a solution of the wave equation (1.1), where $x_1 = r\cos\varphi, x_2 = r\sin\varphi$. Thus the solutions of the wave equation that are eigenfunctions of $M_{12} = \partial_\varphi$ correspond to solutions of the EPD equation. Use the expressions $[iM_{12}, \pm iM_{01} + M_{02}] = \mp(\pm iM_{01} + M_{02})$ to derive differential recurrence relations mapping solutions of the EPD equation for $m = m_0$ to those for $m = m_0 \mp 1$, respectively. Similarly, the other Lie symmetries of the wave equation yield mappings between EPD equations (see [93]).

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