Studying an Exotic Option
Highlighting Market Priced Parameters

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1 Abstract

In this report, we study an exotic option whose price is dependent on the correlation between different underlyings. We begin our analysis with a rather simple initial model with static pairwise correlation and common volatility. Then we propose a model in which correlation has a term structure. Afterwards we develop a single factor model for the underlyings, with idiosyncratic noise. We also suggest a Markovian correlation model and a stochastic correlation model in this setting. We prove some analytic results, calibrate desired parameters from historical data, simulate the payoff function and compute the expected payoff under different models.

We find that the option price is always higher with dynamic correlations than under the assumption of static correlation.

2 Introduction

Since the groundbreaking work of Black, Scholes, and Merton (1973), mathematical modeling of financial phenomena has become both more coherent and more important. The field is not without appropriate criticisms, however. The interested reader should look at Mandelbrot and Taleb for some unique, if not constructive, perspectives on the failings of current models.

The purpose of the current work is to study an exotic option in which the correlation between individual underlyings plays an important role. We proceed as follows. First, we define the payoff of the specific option under inspection, and then proceed to derive results using Black-Scholes dynamics for all underlyings. Our initial model is rather simple, and uses a static pairwise correlation and constant volatility for all assets.

Like most priced parameters in finance, correlation in financial data is time varying. In light of this fact, we suggest a model for stochastic correlation that captures some stylized facts observed in markets. Our first dynamic correlation model is determined by a stochastic differential equation with mean reversion and a term ensuring that we obtain correlations in the appropriate bounds. We obtain an asymptotic density result for our proposed correlation dynamics, and use this density to calibrate model parameters to historical data using conditional maximum likelihood.

We proceed to construct a single factor Black-Scholes-like stochastic differential equation with systematic and idiosyncratic noise, and produce analytical results analagous to those found in our initial simplistic model. This model is then expanded to include a Markovian model of correlation with states calibrated to historical data. We also incorporate the stochastic correlation model into this single factor setting.

The models suggested in this paper are then used in simulations. The results show that dispersion is significantly impacted by introducing dynamic correlation.

2.1 Option definition and Initial Study

The option under study has payoff given by

$$\max\left(\frac{1}{N}\sum_{i=1}^{N}|R_i-\mu|-K,0\right)$$

where N is the number of underlying stocks, R_i is the compounded return of stock i, and μ is the average of $R_i's$, i.e. $\mu=\frac{1}{N}\sum_{i=1}^{N}R_i$.

Intuition leads us to believe that this option should be long volatility, σ, and short correlation, ρ. This is a model-free statement. That is, this should be evidenced by any model we propose to price this option.

3 The Initial Model

We start by building a stochastic model of returns with easily identifiable parameters. We suppose all pairwise correlations between stocks are the same; namely, ρ. Further, we assume all of the stocks have the same volatility σ. In addition, we assume both ρ and σ are constants. As stated above, We will add dynamics to ρ in later analysis. The current model under consideration is

$$\frac{dS_t^i}{S_t^i} = \mu dt + \sigma dW_t^i,$$

$$E(dW_t^i dW_t^j) = \rho dt.$$

Note that R_t^i, the compound return, is defined as $R_t^i = \frac{S_t^i}{S_0^i} - 1$, but that the payoff function is unchanged if we define R_t^i to be $R_t^i = \frac{S_t^i}{S_0^i}$. We will use this latter definition in this report.

We clearly have that R_t^i has the same dynamics as S_t^i:

$$\frac{dR_t^i}{R_t^i} = \mu dt + \sigma dW_t^i.$$

R_t^i therefore has a lognormal distribution, i.e.,

$$\ln R_t^i \sim \mathbf{N}(\mu - \frac{1}{2}\sigma^2, \sigma).$$

We define D to be the dispersion portion of the option payoff,

$$D := \frac{1}{N}\sum_i \left| R_t^i - \frac{1}{N}\sum_k R_t^k \right|.$$

Our focus is primarily on D in the current study. It is difficult to find the distribution, or even the expectation, of the absolute value of a linear combination of a sequence of lognormal distributions. However if we use the first two terms of a Taylor expansion of natural log function around $x = 1$, i.e., $\ln x \sim x - 1$, we get the following:

$$R_t^i \sim \ln R_t^i + 1,$$

and can estimate the expectation of the payoff as

$$E\left(\left| R_t^i - \frac{1}{N}\sum_k R_t^k \right|\right) \sim E\left(\left| \ln R_t^i + 1 - \frac{1}{N}\sum_k (\ln R_t^k + 1) \right|\right) = E\left(\left| \ln R_t^i - \frac{1}{N}\sum_k \ln R_t^k \right|\right).$$

While $\{\ln R_t^i\}_{i=1}^N$ is multi-variate normally distributed, with the expectation $\mu - \sigma^2/2$ for each i, and covariance matrix

$$\sigma^2 t \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \vdots & \vdots \\ \rho & \rho & \cdots & 1 \end{pmatrix},$$

$\ln R_t^i - \frac{1}{N}\sum_k(\ln R_t^k)$ is a one dimensional normal distribution, i.e.,

$$\ln R_t^i - \frac{1}{N}\sum_k(\ln R_t^k) \sim (0, \sqrt{(1-1/N)t}\ \sigma\sqrt{1-\rho})$$

the expectation of the absolute value of the above is $\sqrt{\frac{2}{\pi}}\sqrt{(1-1/N)t}\ \sigma\sqrt{1-\rho}$.

Therefore the expectation of the payoff can be calculated

$$E(D) \sim \frac{1}{N}\sum_i E(|\ln R_t^i - \frac{1}{N}\sum_k(\ln R_t^k)|) = \sqrt{\frac{2}{\pi}}\sqrt{(1-1/N)t}\ \sigma\sqrt{1-\rho}$$

This agrees with the intuition that the payoff is long volatility and short correlation. We also see that under the presumed dynamics, we are long both the number of names and time.

3.1 Calibration of σ and ρ

Given the $m \times m$ historical covariance matrix Σ, we seek a single σ and ρ for calibration. We choose these variables as the solution to the minimization problem,

$$\min_{\sigma,\ \rho} \left\| \Sigma - \sigma^2 \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \cdots & 1 \end{pmatrix} \right\|$$

subject to

$$-\frac{1}{m-1} < \rho < 1$$

where the norm taken is the Frobenius norm. The bounds on ρ guarantee that the calibrated covariance matrix is positive definite.

3.2 Stochastic Correlation

By examining historical financial data, it is evident that it is necessary to incorporate stochastic correlation. In light of these results, we suggest the following SDE for correlation:

$$d\rho_t = k(\bar{\rho}_t - \rho_t)dt + b\sqrt{(1-\rho_t)(1+(N-1)\rho_t)}dZ_t.$$

The $k(\bar{\rho}_t - \rho_t)dt$ term makes the dynamic mean-reverting, with mean $\bar{\rho}$, and mean reversion rate, k. To ensure that ρ always stays within the appropriate bounds, we introduce the factor $\sqrt{(1-\rho_t)(1+(N-1)\rho_t)}$. Additionally, we assume that the systematic noise factor, dZ is the same as that affecting returns. This is not only a matter of convenience. Research shows that correlations in a market upswing tend to be lower than those found in a bear market.

From these dynamics, we may obtain an asymptotic transition density for ρ. This allows us to formulate and solve a maximum likelihood estimation problem with the appropriate density.

3.2.1 Transition Density for Stochastic Correlation

Although the above SDE for stochastic correlation cannot be solved explicitly, it is possible to find the asymptotic transition distribution of ρ_t. That is, we may find the conditional distribution of ρ_t given ρ_{t-1} as $t \to \infty$. The transition density of ρ_t, $p(t, \rho)$, is defined as

$$p(t,\rho)d\rho = Pr(\rho < \rho_t < \rho + d\rho | \rho_s),\ t > s$$

It can be shown that $p(t, \rho)$ satisfies the following Kolmogorov forward (Fokker-Planck) equation:

$$\frac{\partial p}{\partial t} = \frac{1}{2}b^2(1-\rho^2)\frac{\partial^2 p}{\partial \rho^2} - [k\bar{\rho} - (k-2b^2)]\frac{\partial p}{\partial \rho} + (k-b^2)p,\ t > s$$
$$p(s,\rho) = \delta(\rho - \rho_s)$$

Two structural conditions are imposed. First, p is required to be a density

$$\int_{-1}^{1} p(t,\rho)d\rho = 1$$

Moreover, we postulate that p preserves the expectation

$$\lim_{t\to\infty}\int_{-1}^{1} \rho\ p(t,\rho)d\rho = \bar{\rho}$$

We assume the correlation time series is stationary, and hence it is sufficient to derive the stationary solution $p(\rho) = \lim_{t\to\infty} p(t,\rho)$ to the steady state equation which fulfills the above two structural conditions:

$$\frac{1}{2}b^2(1-\rho^2)\frac{d^2 p}{d\rho^2} - [k\bar{\rho} - (k-2b^2)]\frac{dp}{d\rho} + (k-b^2)p = 0$$

First, we examine the simplified case $\bar{\rho} = 0$:

$$\frac{1}{2}b^2(1-\rho^2)\frac{d^2 p}{d\rho^2} + (k-2b^2)\frac{dp}{d\rho} + (k-b^2)p = 0$$

The general solution for this ODE is:

$$p(x) = (1-\rho)^{\frac{k-b^2}{b^2}}[c_1(1+\rho)^{\frac{k-b^2}{b^2}} + c_2\ {}_2F_1(-\frac{k-b^2}{b^2}, \frac{k}{b^2}; -\frac{k-2b^2}{b^2}; \frac{1}{2}(1+\rho))]$$

where c_1, c_2 are constants and ${}_2F_1(a,b;c;z)$ is the hypergeometric function

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} $$

with $\alpha_n = \alpha(\alpha+1)...(\alpha+n-1)$

Since $\int_{-1}^{1} \rho(1-\rho)^{\frac{k-b^2}{b^2}}(1+\rho)^{\frac{k-b^2}{b^2}} d\rho = 0 = \bar{\rho}$, we can choose $c_2 = 0$. Then we choose

$$ c_1 = \frac{1}{\int_{-1}^{1} (1-\rho)^{\frac{k-b^2}{b^2}}(1+\rho)^{\frac{k-b^2}{b^2}} d\rho} = \frac{\Gamma(\frac{k}{b^2}+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(\frac{k}{b^2})} $$

so that $\int_{-1}^{1} p(\rho) d\rho = 1$

Next, we investigate the more general stationary Fokker-Plank equation

$$ \frac{1}{2} b^2 (1-\rho^2) \frac{d^2 p}{d\rho^2} - [k\bar{\rho} - (k-2b^2)] \frac{dp}{d\rho} + (k-b^2)p = 0. $$

The general solution is

$$ p(\rho) = (1-\rho)^{\frac{k(1-\bar{\rho})-b^2}{b^2}} [c_1 (1+\rho)^{\frac{k(1+\bar{\rho})-b^2}{b^2}} + c_2\ {}_2F_1(-\frac{k(1+\bar{\rho})-b^2}{b^2}, \frac{k(1-\bar{\rho})}{b^2}; -\frac{k(1+\bar{\rho})-2b^2}{b^2}; \frac{1}{2}(1+\rho))]. $$

For $\bar{\rho}$ =0, the solution should agree with the simplified one, thus we have $c_2 = 0$ and

$$ c_1 = \frac{1}{\int_{-1}^{1} (1-\rho)^{\frac{k(1-\bar{\rho})-b^2}{b^2}}(1+\rho)^{\frac{k(1+\bar{\rho})-b^2}{b^2}} d\rho} = \frac{\Gamma(\frac{k(1-\bar{\rho})}{b^2}+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(\frac{k(1-\bar{\rho})}{b^2})_2F_1(-\frac{k\bar{\rho}}{b^2}, -\frac{k\bar{\rho}}{b^2}+\frac{1}{2}; \frac{k(1-\bar{\rho})}{b^2}+\frac{1}{2}; 1)}. $$

Therefore, the asymptotic transition density is

$$ p(\rho) = \frac{\Gamma(\frac{k(1-\bar{\rho})}{b^2}+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(\frac{k(1-\bar{\rho})}{b^2})_2F_1(-\frac{k\bar{\rho}}{b^2}, -\frac{k\bar{\rho}}{b^2}+\frac{1}{2}; \frac{k(1-\bar{\rho})}{b^2}+\frac{1}{2}; 1)} (1-\rho)^{\frac{k(1-\bar{\rho})-b^2}{b^2}}(1+\rho)^{\frac{k(1+\bar{\rho})-b^2}{b^2}}. $$

With stationary correlation time series, the density of ρ_t conditional on ρ_s is

$$ p(\rho_t|\rho_s) = \frac{\Gamma(\frac{k(1-\bar{\rho})}{b^2}+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(\frac{k(1-\bar{\rho})}{b^2})_2F_1(-\frac{k\bar{\rho}}{b^2}, -\frac{k\bar{\rho}}{b^2}+\frac{1}{2}; \frac{k(1-\bar{\rho})}{b^2}+\frac{1}{2}; 1)} (1-\rho_t)^{\frac{k(1-\bar{\rho})-b^2}{b^2}}(1+\rho_t)^{\frac{k(1+\bar{\rho})-b^2}{b^2}}. $$

In addition, $\rho_{t_1}|\rho_{t_0}$, $\rho_{t_2}|\rho_{t_1}$, ..., $\rho_{t_m}|\rho_{t_{m-1}}$ are independent.

Thus, the conditional likelihood function is

$$L(k, b, \bar{\rho}) = \prod_{i=1}^{m} p(\rho_{t_i} | \rho_{i-1})$$

Given a set of data, we seek to maximize the above likelihood function. The solutions, k, b, $\bar{\rho}$, to this maximization problem are then used as parameter estimates for the stochastic differential equation in ρ. This yields a calibrated simulation for ρ.

4 Single Factor Model

Although the initial model is easy to work with and gives easily identifiable results which verify our intuition on the long σ and short ρ positions, it has deficiencies. Specifically, we have no reason to assume that all stocks have the same volatility or that they might share some common pairwise correlation.

Here we introduce a more sophisticated model. We suggest a single factor model in which we introduce a market factor Z and assume that all stocks are correlated to each other via correlation to the market. Further, we assume that each stock has an idiosyncratic noise component, W_t^i, with Z_t, W_t^i all independent. Mathematically, we have

$$\frac{dS_t^i}{S_t^i} = \mu_i dt + \sigma_i(\rho_i dZ_t + \sqrt{1 - \rho_i^2} dW_t^i), \tag{1}$$

where ρ_i is the correlation between stock i and the market factor. For a specific reference, we may use the S&P 500 index to specify the market factor Z.

We use the same technique as we used in the initial model to estimate the expectation of the payoff in this single-factor model; viz.,

$\{\ln R_t^i\}_{i=1}^N$ is multi-variate normally distributed, with the expectation being $\{\mu_i - \sigma_i^2/2\}_{i=1}^N$, and covariance matrix

$$t \begin{pmatrix} \sigma_1^2 & \rho_1\rho_2\sigma_1\sigma_2 & \cdots & \rho_1\rho_N\sigma_1\sigma_N \\ \rho_2\rho_1\sigma_2\sigma_1 & \sigma_2^2 & \cdots & \rho_2\rho_N\sigma_2\sigma_N \\ \vdots & \vdots & \ddots & \vdots \\ \rho_N\rho_1\sigma_N\sigma_1 & \rho_N\rho_2\sigma_N\sigma_2 & \cdots & \sigma_N^2 \end{pmatrix}.$$

We also have that $\ln R_t^i - \frac{1}{N}\sum_k(\ln R_t^k)$ is a one dimensional normal distribution, i.e.,

$$\ln R_t^i - \frac{1}{N}\sum_k(\ln R_t^k) \sim (m_i, \sqrt{\beta_i t}),$$

where

$$m_i = \mu_i - \frac{1}{2}\sigma_i^2 - \frac{1}{N}\sum_k(\mu_k - \frac{1}{2}\sigma_k^2)$$

and

$$\beta_i = (1 - \frac{2}{N} + \frac{2}{N}\rho_i^2)\sigma_i^2 - \frac{2}{N}\rho_i\sigma_i\vec{\rho}\cdot\vec{\sigma} + \frac{1}{N^2}(\vec{\sigma}\cdot\vec{\sigma} + (\vec{\rho}\cdot\vec{\sigma})^2 + \sum_k \rho_k^2\sigma_k^2).$$

It follows that

$$E\left(\left|\ln R_t^i - \frac{1}{N}\sum_k(\ln R_t^k)\right|\right) = m_i(2\mathbf{F}(\frac{m_i}{\sqrt{\beta_i t}}) - 1) + \sqrt{\frac{2\beta_i t}{\pi}}e^{-\frac{m_i^2}{2\beta_i t}} \sim \sqrt{\frac{2\beta_i t}{\pi}} + \sqrt{\frac{1}{2\pi\beta_i t}}m_i^2$$

where $\mathbf{F}$ is the c.d.f of a standard normal distribution.

The expected payoff is therefore

$$E(D) \sim \frac{1}{N}\sum_i\left[m_i(2\mathbf{F}(\frac{m_i}{\sqrt{\beta_i t}}) - 1) + \sqrt{\frac{2\beta_i t}{\pi}}e^{-\frac{m_i^2}{2\beta_i t}}\right] \sim \frac{1}{N}\sum_i\left(\sqrt{\frac{2\beta_i t}{\pi}} + \sqrt{\frac{1}{2\pi\beta_i t}}m_i^2\right)$$

There are several facts we can see from the above results:
First, the payoff is long $\sqrt{\beta_i}$,

$$\beta_i = \sigma_i^2(1 - \frac{1}{N\sigma_i^2}\cdot(\text{sum of i-th row of }\Sigma) + \frac{1}{N^2\sigma_i^2}\cdot(\text{sum of all elements in }\Sigma))$$

while

$$\frac{1}{N\sigma_i^2}\cdot(\text{sum of i-th row of }\Sigma) - \frac{1}{N^2\sigma_i^2}\cdot(\text{sum of all elements in}\Sigma)$$

could be considered as a weighted correlation, $\beta_i \sim \sigma_i^2(1-\bar{\rho})$ for some $\bar{\rho}$, and the option is long $\sqrt{\beta} \sim \sigma\sqrt{1-\bar{\rho}}$, which matches our previous model.

Second, we could see from the second term in the asymptotic behavior of the expected payoff that the option is also long m_i^2, the dispersion of the variance.

4.1 Single Factor Model with Markovian Correlation

4.1.1 Assumption

Here we provide an alternate method to describe how ρ_t^i behaves with respect to time t. In this analysis, we consider the change of ρ_t^i as a Markovian variable. We have that ρ_t^i satisfies the following equation:

$$\Delta\rho_t^i = \Delta\rho^i(S_t) = \Delta\rho_j^i, S_t = j, \quad (2)$$

$$\Delta\rho_t^i = \frac{\rho_t^i}{\rho_{t-\Delta t}^i} - 1, \quad (3)$$

$$X_t = 1, 2, \ldots, J. \quad (4)$$

Where X_t is a J-state Markovian variable and has the following probability distribution:

$$\mathbf{P}\{S_t = k|S_{t-\Delta t} = j\} = p_{jk}. \quad (5)$$

$\mathbf{P} = [p_{jk}]$ is a $J \times J$ transition matrix of the change of correlation ρ and p_{jk} denotes the probability of state S changing from j to k.

4.1.2 Algorithm for Simulation under These Dynamics

We do not seek solutions of the SDE involving S and dynamic ρ in this analysis. Instead, we consider the case that all correlations ρ_t^i in (1) are independent of time t, and obtain the familiar analytic solution:

$$S^i(T) = S^i(0)exp\left\{\left(\mu^i - \frac{1}{2}{\sigma^i}^2\right)T + \sigma^i\left(\rho^i Z(T) + \sqrt{1-{\rho^i}^2}W^i(T)\right)\right\} \quad (6)$$

We use this solution to incorporate dynamic correlation, and solve problems involving these dynamic quantities numerically. Our updating scheme is illustrated below.
From time t to $t + \Delta t$:
[1]Get a random number StateDet which satisfies the uniform distribution $\mathbf{U}(0, 1)$.
[2]According to StateDet and the transition probability matrix $\mathbf{P}$ (see 5) of the change of ρ, we can replace the state S_t by the new value $S_{t+\Delta t}$.
[3]From $S_{t+\Delta t}$, we determine the new value of $\Delta\rho^i_{t+\Delta t}$, then obtain $\rho^i_{t+\Delta t}$.
[4]Then by using (6) to get the new stock price:

$$S^i(t + \Delta t) = S^i(t)exp\left\{\left(\mu^i - \frac{1}{2}{\sigma^i}^2\right)\Delta t + \sigma^i\left(\rho^i_{t+\Delta t}Z(t + \Delta t) + \sqrt{1 - {\rho^i_{t+\Delta t}}^2}W^i(t + \Delta t)\right)\right\}.$$

4.2 Single Factor Model with Stochastic Correlation

Here we use the same dynamic of ρ as we used in the initial model, i.e.,

$$d\rho^i_t = k^i(\bar{\rho}^i - \rho^i_t)dt + b^i\sqrt{1 - \rho_i^2}dZ_t.$$

With the assumed dynamic of the correlations, we try to derive a PDE to price options involving stocks and the index, which we assume follows a regular Black Scholes dynamic

$$dI = \mu I dt + \sigma_I I dZ_t.$$

We start by considering an option whose underlying assets are the index I and a stock R,

$$\frac{dR}{R} = \mu dt + \sigma\rho dZ_t + \sigma\sqrt{1 - \rho^2}dW_t$$

$$d\rho = k(\bar{\rho} - \rho)dt + b\sqrt{1 - \rho^2}dZ_t$$

Since the random component of I is common to both ρ and I, we find that

$$d\rho = \left[k(\bar{\rho} - \rho) - \frac{\mu}{\sigma_I}b\sqrt{1 - \rho^2}\right]dt + \frac{b\sqrt{1 - \rho^2}}{\sigma_I I}dI \tag{7}$$

We see from the above that ρ is in fact a deterministic function of t and I, and the above equation is essentially a first-order PDE that can be solved by using the method of characteristic curves.

Next, suppose f is a function of t, I and R. Then we have the following dynamic

$$\begin{aligned} df = &\left[\frac{\partial f}{\partial t} + \mu I\frac{\partial f}{\partial I} + \mu R\frac{\partial f}{\partial R} + \frac{1}{2}\sigma_I^2 I^2\frac{\partial^2 f}{\partial I^2} + \frac{1}{2}\sigma^2 R^2\frac{\partial^2 f}{\partial R^2} + \rho\sigma_I\sigma IR\frac{\partial^2 f}{\partial I\partial R}\right]dt \\ &+ \left(\sigma_I I\frac{\partial f}{\partial I} + \sigma\rho R\frac{\partial f}{\partial R}\right)dZ_t \\ &+ \sigma\sqrt{1 - \rho^2}R\frac{\partial f}{\partial R}dW_t \end{aligned}$$

Using standard financial reasoning, we may obtain a risk free portfolio by being
long one option,

short $\dfrac{\partial f}{\partial R}$ of the stock, and
short $\dfrac{\partial f}{\partial I}$ of the index.

We note that if we assume that $\frac{\partial f}{\partial \rho}$ is nonzero, we still obtain similar results, but must refer to the total derivative $\frac{df}{dI}$ rather than the standard partial derivative. In particular, in this case one obtains a hedge in the index of

$$\frac{\partial f}{\partial I} + \frac{\partial \rho}{\partial I}\frac{\partial f}{\partial \rho} = \frac{\partial f}{\partial I} + \frac{b\sqrt{1-\rho^2}}{\sigma_I I}\frac{\partial f}{\partial \rho},$$

but this is exactly the total derivative of f with respect to I. This result is reassuring, as we are only witness to the total derivative in practice.

Again, by standard financial reasoning, we obtain the following PDE

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma_I^2 I^2 \frac{\partial^2 f}{\partial I^2} + \rho\sigma_I \sigma I R \frac{\partial^2 f}{\partial I \partial R} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 f}{\partial R^2} + rR\frac{\partial f}{\partial R} + rI\frac{\partial f}{\partial I} - rf = 0$$

with ρ an explicit (in the case that we can get a closed-form solution of the 1st order PDE (1)) or an implicit function of t and I.

The above equation is in fact a Black-Schole's equation with two underlyings which are correlated by ρ. It can be used to price options involving indicies and stocks.

5 Simulation Results

In this section, we provide simulation results that we compare with historical information used in this study.We will demonstrate our results in the following way:

1. Calibration from historial data.

2. Payoff distribution, Expected Payoff and Discounted Price for different models.

3. Comparisons between historical data and simulation data.

We set $N = 28$, $days = 252$, $T = 1$, $\mu = 0.05$, $\sigma = 0.2$ and $trials = 1000$ in all of the simulations. Where N is the total number of stocks; $days$ is the number of days for simulation per year; T is the period of time (years) and $trials$ is the number of repeated experiments .

5.1 Calibration from historical data

Using the analysis shown in previous sections, we obtain the calibrations shown in Figure 1 under the single factor model (SFM) with stochastic correlation. We also implement a the Markovian model suggested above with number of states, $J = 7$. These state values are calibrated to historical data. We find that the mean value of b for all stocks in our analysis during one year under SFM with stochastic is $\mathbf{b} = -0.15854$ and the mean value of k for all stocks is $\mathbf{k} = -0.08249$. These data are used as the inputs of our numerical simulations.

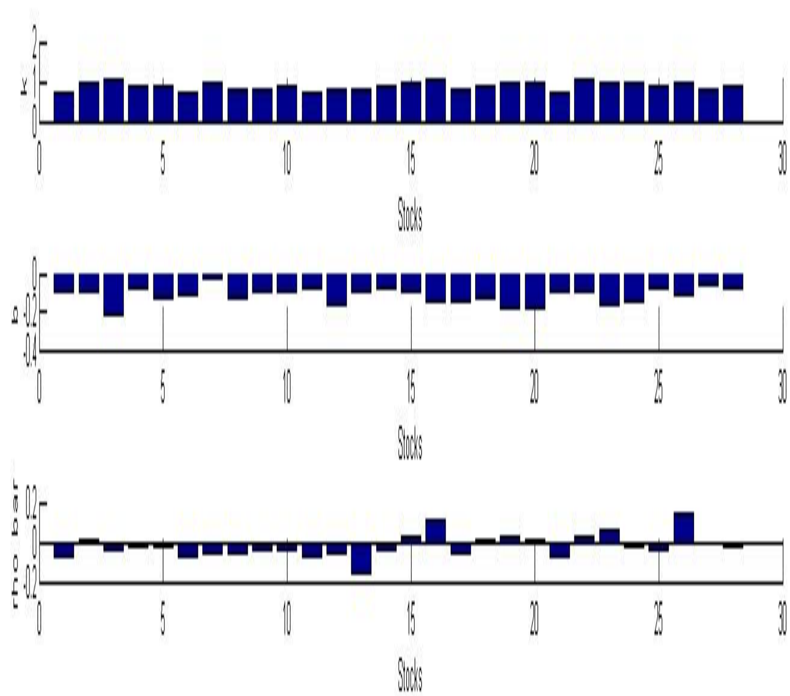

Figure 1: The calibration from the historical data under SFM with stochastic correlation ρ.

	B-S model		Single factor model	
	constant ρ	dynamical ρ	Markovian ρ	stochastic ρ
E(D)	0.1653	NaN	0.1657	0.1660
$DiscountedPrice$	0.1573	NaN	0.1576	0.1578

Table 1: The expected payoff for different models.Parameters: B-S model: $\rho_{const} = 5.6E - 5$

5.2 Payoff Distribution and Expected Payoff

Using the calibrated values suggested above, we compute the payoff distribution and expected payoff of the dispersion portion of the option presented. We use an initial value of correlation between the stocks and the Market of $\rho_{\mathbf{0}} = \bar{\rho}$, the trailing mean correlation just prior to the beginning of the simulation.

1. Payoff distribution graphs for different models.

   Remark:
   The payoff distribution of SFM with Markov ρ is very similar to that of SFM with stochastic ρ.
   The three models incorporating dynamic correlation have distribution graphs with a fat tail.

2. Expected payoff and Discounted Price.
   Remark:
   We did not obtain results for the B-S Model using dynamic correlation. This is a result of the time constraint for this report.
   The constant ρ_{const} is chosen as the mean value of ρ of SFM with Markovian correlation for all stocks during one year. Since it is very small, the difference of the three models in Table 1 is not large.
   From the data displayed in Table 1, we find that the expected payoff and discounted price of the B-S model is always less than those of the single factor models. We suggest that this is because the basket option prices react asymmetrically to positive and negative correlations, where a change in negative correlations has a higher impact on the option price than a change in positive correlations of the same magnitude.(see [4]). And furthermore, the expected payoff increase when correlations decrease.

3. The simulation results compared with the mathematical results.
   Recall the first order approximation of the expected payoff of the toy model with constant correlation, the formula is:

$$\mathbf{E}(D) \approx \sqrt{\frac{2}{\pi}}\sigma\sqrt{(1-1/N)\,T}\sqrt{1-\rho}.$$

   Our simulation results (see Figure 3) demonstrate the correctness of mathematical analysis, and the relative error between the simulation ones and the first order approximation is about 0.05.

5.3 Comparisons between historical data and simulation data

In this subsection, we want to exhibit the difference between the simulation results and historical data. And what we are interested in is whether the simulation results keep the distribution of the historical data.

1. **Log**(Return)

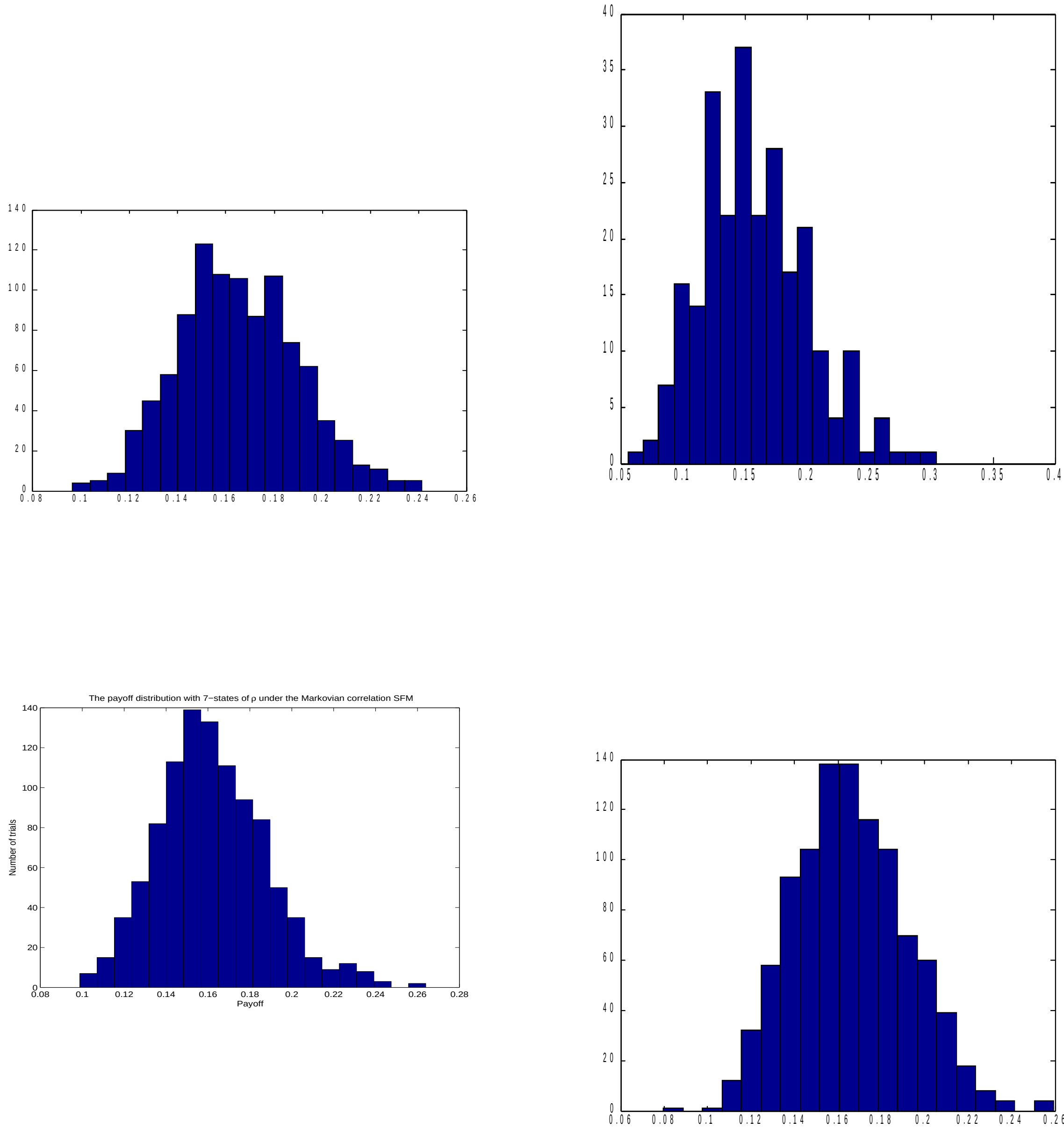

Figure 2: The payoff distribution of (a) Black-Scholes model with constant ρ;(b) Black-Scholes model with dynamical ρ (3)SFM with Markovian ρ;(d) SFM with stochastic ρ .

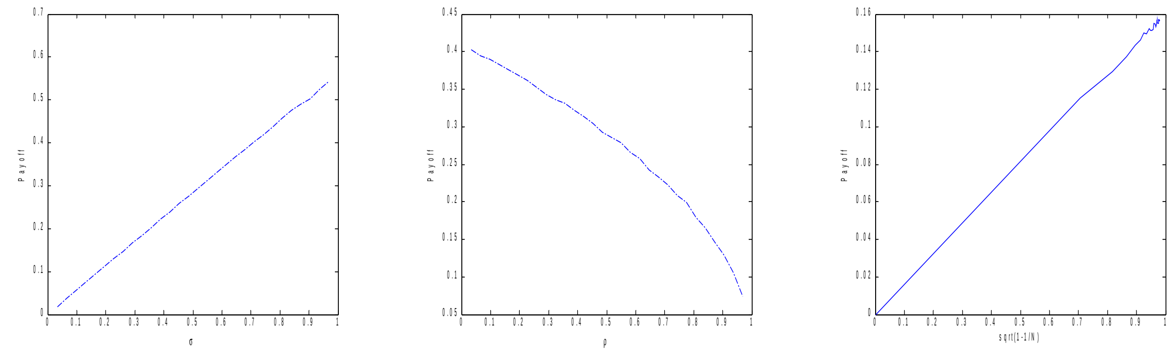

Figure 3: The relationship between the Expected payoff and parameters (a) σ, (b) ρ and (c)N under the Toy model with constant ρ.

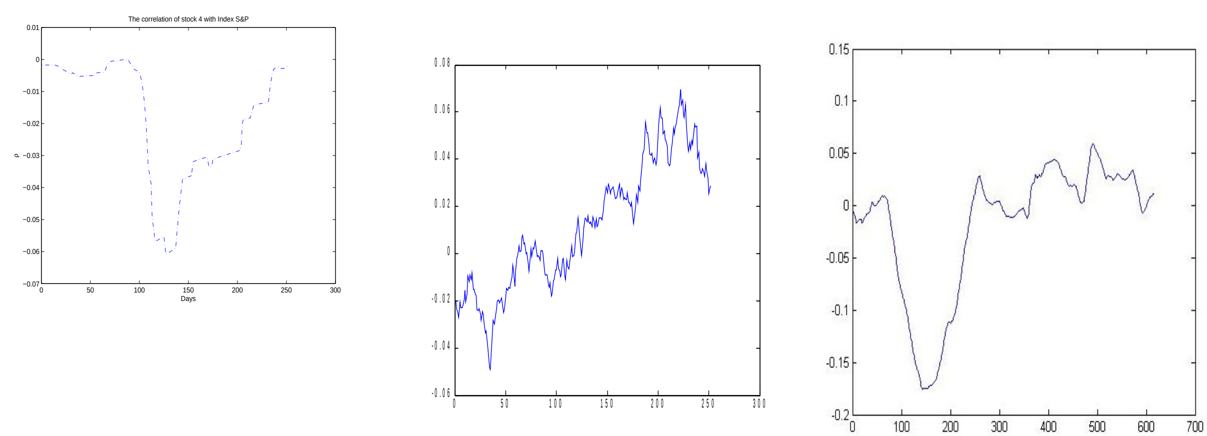

Figure 4: The path of ρ of stock 4 under (a) SFM with Markov ρ, (b) SFM with stochastic ρ and (c) historical ρ.

Remark: From Figure 4, we find that the path of ρ under the SFM with Markovian correlation has a very similar shape with that of the historical graph, which shows the advantage of simulating correlation by Markovian variables.

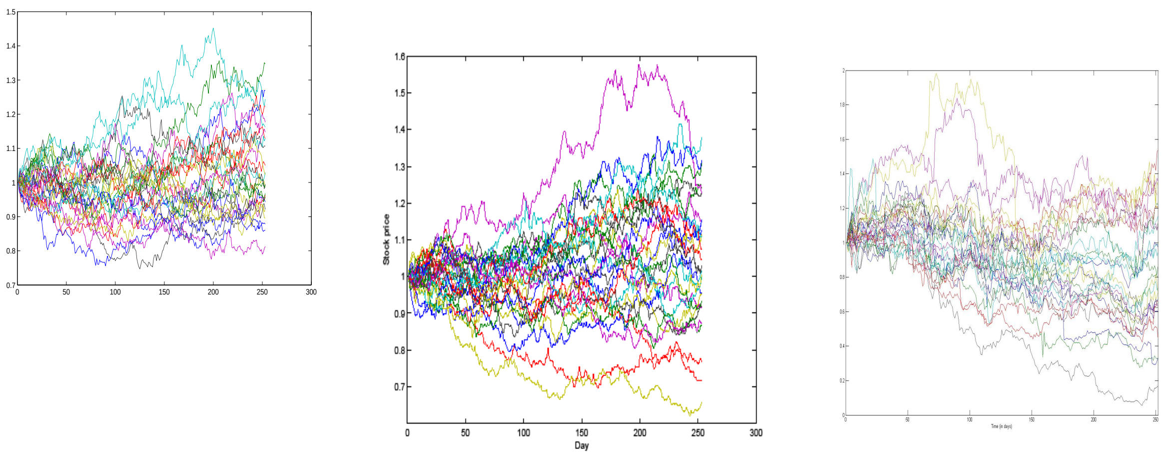

Figure 5: The stocks' prices of stock 4 under (a)SFM with Markov ρ, (b)SFM with stochastic ρ and (c)historical ρ.

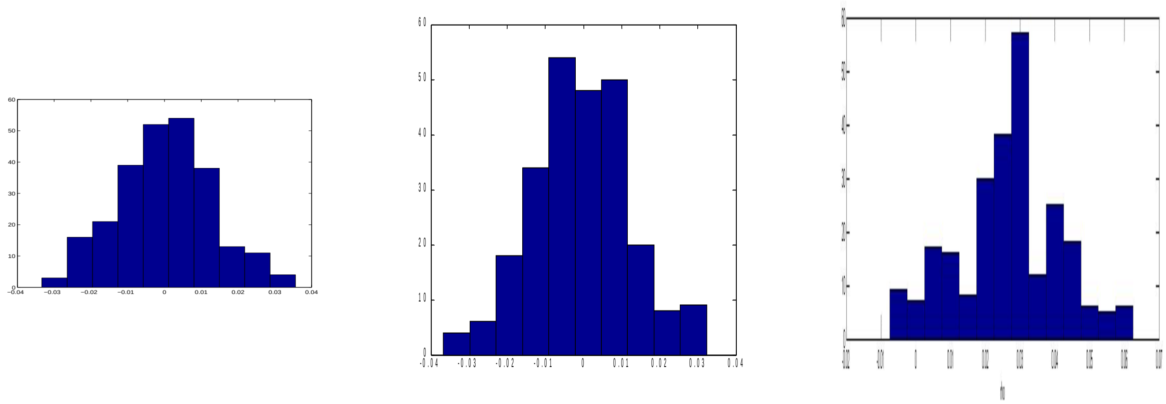

Figure 6: The distribution of log returns of stock 4 under (a) SFM with Markov ρ, (b) SFM with stochastic ρ and (c) historical.

Remark: The distributions of these three looks similar to each other. All of them have a thick tail on the distribution, especially the graphs of SFM with stochastic ρ and historical ρ.

6 Future Work

Using the correlated Black-Scholes PDE exhibited in the final analysis, we have derived a method to obtain an implied correlation between any two underlyings that we assume follow a geometric Brownian motion. Our intention is to reinterpret the volatility smile in this context. That is, we seek to assume that correlation to the market factor provides information regarding market sentiment. And, since we observe stationary dynamics in correlation, there is some mild hope that these results will have lasting duration. In addition, these implied correlations may be used to help price options. In particular, we may gain pricing insight for the dispersion option just presented.

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