Sparse solution of integral equation

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Outline of the talk

1. Sparse solution of underdetermined integral equations

2. Gaussian quadrature design for inner products

3. Their connections

4. Applications
1. First kind integral equation for $u$ (1-D):

$$\int_a^b G(k, x)u(x)dx = s(k), \quad u \in L^2[a, b]$$

with kernel $G$ non-singular, often continuous

2. Alternative format: $2n$ equations

$$\int_a^b v_j(x)u(x)dx = b_j, j = 1 : 2n \quad \text{or} \quad V(2n, x)u(x) = b(2n)$$

with

$$V(2n, x) = \begin{bmatrix}
- - - - v_1(x) - - - - \\
- - - - v_2(x) - - - - \\
- - - - - - - - - - - - - \\
- - - - - - - - - - - - - \\
- - - - v_{2n}(x) - - - - \\
\end{bmatrix} : L^2[a, b] \mapsto C^{2n}$$

**flat matrix** of size $2n$-by-$[a, b]$, or $2n$-by-$\infty$
1. First kind integral equation for $u$ (1-D):
\[ \int_{a}^{b} G(k, x) u(x) dx = s(k), \quad u \in L^2[a, b] \]

2. Alternative format: $2n$ equations
\[ \int_{a}^{b} v_j(x) u(x) dx = b_j, j = 1 : 2n \quad \text{or} \quad V(2n, x) u(x) = b(2n) \]

3. Inherently underdetermined: Uniqueness a big issue

4. Central problem in many applications
1. Recover \( u(x) \) from its Fourier coefficients \( s(k) \)

\[
 s(k) = \int_{-\pi}^{\pi} e^{ikx} u(x) dx, \quad k = -n + 1 : n
\]

... underdetermined IE \( G(k, x)u(x) = s(k) \)

2. Determine \( n \) photons arrival times \( t_j \) and energies \( c_j \) from signal \( s(t) \) measured at detector

\[
 s(t) = \sum_{j=1}^{n} c_j G(t - t_j), \quad t \in [0, T]
\]

- \( G \) is detector’s response function
- Let \( u(t) = \sum_{j=1}^{n} c_j \delta(t - t_j) \), we obtain
  underdetermined IE \( G(t - \tau)u(\tau) = s(t) \)
1. Approximate \( u \) by subspace \( \mathcal{B} \) of dim \( M \):

\[
u(x) \sim \sum_{j=1}^{M} c_j B_j(x)
\]

2. If \( u \in \mathcal{B} \) with \( M \leq 2n \), \( u \) is uniquely determined by IE

\[
\int_{a}^{b} v_j(x)u(x)dx = b_j, \quad j = 1 : 2n
\]

3. Don’t know subspace \( \mathcal{B} \)

4. ... will assume \( u \) is arbitrary \( L^2 \) function
1. The size of the representation is $M$

$$u(x) \sim \sum_{j=1}^{M} c_j B_j(x)$$

- Number of parameters is also $M$

2. Alternatively, $u$ can be represented by its values sampled at $M$ locations $\{x_j, u(x_j), j = 1 : M\}$

- A weak representation of size $M$, and with $2M$ parameters
1. Will use \( \{x_j, u(x_j), j = 1 : n\} \) as our representation for sparse or parsimonious solution of IE

2. Solution is sparse for its representation size is shortest

3. The \( 2n \) unknowns \( \{x_j, u(x_j), j = 1 : n\} \) to be determined by the \( 2n \) equations
\[ V(2n, x)u(x) = b(2n) \]

4. This is only possible if \( \{x_j\} \) are Gaussian nodes
The quadrature approach to sparse solution

To design a G-quadrature is to solve IE for sparse solution

\[
\int_{a}^{b} v_j(x)u(x)dx = b_j, \quad j = 1 : 2n \quad \text{or} \quad V(2n, x)u(x) = b(2n)
\]

1. Kernel \( v_j(x) \) - 2n functions to be integrated

2. RHS \( b_j \) - exact integrals to be matched by quadrature

3. Unknown \( u \) - weight function (may not be positive definite)

- Design quadrature \( \{x_j, w_j, 1 \leq j \leq n\} \) with data \( \{b_j\} \)
- Recover \( u(x_j) \) from \( w_j \)
Properties of the quadrature approach

1. Exact if \( u \) is “truly sparse”: \( u(x) = \sum_{j=1}^{m} c_j \delta(x - a_j), \ m \leq n. \)

   ... Quadrature nodes and weights will be \( a_j, c_j \)

2. No Gaussian point \( x_j \) is placed where \( u(x) \) vanishes

   Example: Integral equation \( \exp[-a^2(x - y)^2]u(y) = b(x) \)
IE for $u$: $\exp[-a^2(x - y)^2]u(y) = b(x)$
Example: \( b(x) \) is \( u(x) \) blurred by Gaussian kernel

IE for \( u \):
\[
\exp[-a^2(x-y)^2]u(y) = b(x)
\]
Example: No Gaussian node is wasted where \( u = 0 \)

Design a G-quadrature for IE: \( \exp[-a^2(x - y)^2]u(y) = b(x) \)
Properties of quadrature approach to sparse solution

1. Exact if \( u \) is “truly sparse”: 
   \[ u(x) = \sum_{j=1}^{m} c_j \delta(x - x_j), \quad m \leq n. \]

2. No Gaussian point \( x_j \) is placed where \( u(x) \) vanishes

3. \( \{w_j\} \textbf{ jump sharply where } u(x) \textbf{ jumps} - \textbf{No Gibbs} \)

   Example: Recover discon function from Fourier coeff ...

Example: IE with a discontinuous solution $u(x)$

Design a quadrature for IE $\exp(ikx)u(x) = s(k)$
$u(x)$ recovered from quadrature nodes $x_j$ weights $w_j$
Properties of quadrature approach to sparse solution

1. Exact if $u$ is “truly sparse”: $u(x) = \sum_{j=1}^{m} c_j \delta(x - x_j)$, $m \leq n$.
2. No Gaussian point $x_j$ is wasted where $u(x)$ vanishes
3. $\{w_j\}$ jump sharply where $u(x)$ jumps - No Gibbs

4. Convergence: Quadrature weights converge to weight function $u$ - in weak star topology - as $n$ grows

5. Invariance of solution as IE is reformulated:

$$V(2n, x)u(x) = b(2n), \text{ transformed to }$$
$$A(2n, 2n)V(2n, x)u(x) = A(2n, 2n)b(2n), \quad A \text{ invertible}$$
Outline of the talk

1. Sparse solution of underdetermined integral equations

2. Gaussian quadrature design for inner products

3. Their connections

4. Applications
1-D: Given 2n functions \( \{f_i(x)\} \), find n-term quadrature
\[
\int_a^b w(x) f_i(x) \, dx \approx \sum_{j=1}^n w_j f_i(x_j), \ x_j \in (a, b)
\]

2-D: Given 3n functions \( \{f_i(x, y)\} \), find n-term quadrature
\[
\int_D w(x) f_i(x, y) \, dx \, dy \approx \sum_{j=1}^n w_j f_i(x_j, y_j), \ (x_j, y_j) \in D
\]
1. Old idea: Given weight \( u \), construct orthogonal polynomials.
   \( n \) roots as quadrature nodes to integrate \( 2n \) functions \( \{ f_i \} \)

2. New idea: Given \( n \) basis functions \( \{ f_i \} \), find \( n \)-term quadrature to integrate their \( n^2 \) inner products

\[
(f_i, f_j) = \int_a^b w(x)f_i(x)f_j(x) \, dx
\]

2.1. Let \( F(n, x) \) be the \( n \)-by-\([a, b]\) matrix, whose i-th row is \( f_i(x) \)

2.2. Gramian matrix \( B(n, n) = F(n, x)w(x)F(x, n) = F(n, x) \cdot F(x, n) \)
Design a $n$ term quadrature \{${x}_j, {w}_j , j = 1 : n$\} to integrate the $n^2$ inner products exactly:

$$\int_{a}^{b} w(x) f_i(x) f_k(x) \, dx = \sum_{j=1}^{n} w_j f_i(x_j) f_k(x_j)$$

... or, in matrix form

$$F(n, x) \cdot F(x, n) = \sum_{j=1}^{n} F(n, x_j) {w}_j F(x_j, n)$$

... or more compactly

$$B(n, n) = F(n, \{x_j\}) \{w_j\} F(\{x_j\}, n)$$

- We say the quadrature is (exact) for the Gramian
- Also say: Design a quadrature for the Gramian
Let $Q(n, x)$ be $n$ orthonormal basis for $\text{Span}\{F(n, x)\}$.

$$F(n, x) = \begin{bmatrix} - & - & - & f_1(x) & - & - & - & \cdots & \cdots & - & - & - & f_n(x) & - & - & - \end{bmatrix}, \quad Q(n, x) = \begin{bmatrix} - & - & - & q_1(x) & - & - & - & \cdots & \cdots & - & - & - \end{bmatrix}$$

**Theorem.** There is a $n$ term quadrature $\{x_j, w_j, j = 1 : n\}$, exact for the Gramian matrix

$$B(n, n) = F(n, x) \cdot F(x, n)$$

iff the $n$ columns

$$Q(n, \{x_j\}) = \begin{bmatrix} q_1(x_1) & q_1(x_2) & \cdots & q_1(x_n) \\ q_2(x_1) & q_2(x_2) & \cdots & q_2(x_n) \\ \cdots & \cdots & \cdots & \cdots \\ q_n(x_1) & q_n(x_2) & \cdots & q_n(x_n) \end{bmatrix}$$

are orthogonal
Minimal function $\mu(x)$

- Difficult problem: Selecting $n$ orthogonal columns out of infinite number of columns of $Q(n, x)$

- Made easier by asking a bit more: If there is a very simple function $\mu(x)$ such that the $n$-term quadrature also integrates another Gramian matrix

$$A(n, n) = F(n, x) \cdot \mu(x) F(x, n)$$

**Theorem.** ... such a quadrature exists iff

$$\lambda_j(AB^{-1}) = \mu(x_j)$$

$$v_j(AB^{-1}) = \begin{bmatrix} f_1(x_j) \\ f_2(x_j) \\ \vdots \\ f_n(x_j) \end{bmatrix} = F(n, x)|_{x=x_j}$$
**Polynomial case** : Old = New

**Old approach**: Integrating 2n polynomials: $1, x, x^2, \ldots, x^{2n-1}$

**New approach**: $F(x, n) = [1, x, x^2, \ldots, x^{n-1}]$, $\mu(x) = x$

1. Gram matrix $B = F(n, x) \cdot F(x, n)$, namely $B_{i,j} = (x^i, x^j)$
   integrating polynomials of degree up to 2n-2

2. Gram matrix $A = F(n, x) \cdot xF(x, n)$, namely $A_{i,j} = (x^i, xx^j)$
   integrating poly of deg up to 2n-1

**Old = New**: A n-term quadrature accurate for $A$ and $B$
   integrates poly of deg up to 2n-1
Typical minimal functions $\mu(x)$

- Polynomials: $x$
- Exponentials $\exp(ikx)$, $k \in [-b, b]$ $x$ or $e^{ihx}$
- Trig Polys $\exp(ikx)$, $k = -m$ $m$ $\cos(x)$
- Power functions $x^k$, $k \in (-0.5, 0.5)$: $\log(x)$
- Bessel functions: $J_k(x)$, $0 \leq k < n$ $1/x$
**Minimal function definition**

A *typical method to grow a family of functions*:

1. Multiply the family members by a function $\mu$
2. $\mu$ may or may not be in the family
3. The resulting functions are linearly combined with existing family members to produce a new function:

$$f_{n+1}(x) = \alpha(1,n)F(n,x)\mu(x) + \beta(1,n)F(n,x)$$

where the row vectors $\alpha, \beta$ are LinComb coeffts

- In 3-term recursion: $\mu$ is the multiplier function
- Obviously $\text{Span}\{F(n,x)\mu(x)\} \neq \text{Span}\{F(n,x)\}$
Minimal function defined for $n$ functions

... since $\text{Span}\{F(n, x)\mu(x)\} \neq \text{Span}\{F(n, x)\}$ ...

**Definition** $\mu(x)$ is minimal if the new stuff it created in $F(n, x)\mu(x)$ is of rank 1, namely

$$\dim\{\text{Span}\{F(n, x)\mu(x)\} - \text{Span}\{F(n, x)\} = 1$$

- $\mu(x)$ introduces new blood into the span
- $\mu(x)$ pushes old family members out of the clan
- $\mu(x)$ - the multiplier, pusher
For a family of functions, such as $G(k, x) = \exp(kx)$, with $k$ a continuous parameter in $(k_1, k_2]$ ...

we expect $\mu(x) = \exp(hx)$ for tiny $h > 0$ to push $k_2$ to $k_2 + h$
... in the limit:
\[
\mu(x) = \left\{ \frac{\partial}{\partial k} \log G(k, x) \right\}_{k_2}
\]

- For $G(k, x) = \exp(kx)$, min-function is $\mu(x) = x$
- For $G(k, x) = k^x$, $k \in (0, a]$, min-function is also $\mu(x) = x$
Outline of the talk

1. Sparse solution of underdetermined integral equations

2. Gaussian quadrature design for inner products

Summary:

2.1. G-quadrature for inner product of n functions $F(n, x)$

2.2. Two Gramians

$B(n, n) = F(n, x) \cdot F(x, n)$

$A(n, n) = F(n, x) \cdot \mu(x) F(x, n)$

2.3. Eigen decomposition on quotient matrix $AB^{-1}$
Outline of the talk

1. Sparse solution of underdetermined integral equations

2. Gaussian quadrature design for inner products

3. Their connections: Sparse solution via Gramians

4. Applications
1. Sparse solution of IE $V(2n, x)u(x) = b(2n)$: Design n-term quadrature for $V$ with $u$ as weight function

2. $F(n, x)$ is called a factor space of $V(2n, x)$ if
   \[ \text{Span}\{F(n, x) \times F(n, x)\} = \text{Span}\{V(2n, x)\} \]

   namely,
   \[ \text{Span}\{f_i(x)f_j(x), \ 1 \leq i, j \leq n\} = \text{Span}\{v_l(x), \ l = 1 : 2n\} \]

3. Consequence of factor space
   3.1. Quadrature problem for $V(2n, x)$ is equivalent to that for Gramian $B(n, n) = F(n, x) \cdot F(x, n)$
   3.2. Gramians $B$ and $A$ can be contructed from $b(2n)$
Factor space

\[ \text{Span}\{F(n, x) \times F(n, x)\} = \text{Span}\{V(2n, x)\} \]

Pairwise product in \( F \) expressed by \( V \): \( f_i(x)f_j(x) = \sum_{l=1}^{2n} \alpha_{l}^{(i,j)}v_l(x) \)

- LinComb coefficients \( \alpha_{l}^{(i,j)} \) will guide \( b(2n) \) into \( B_{ij} \)

\[ B_{ij} = \sum_{l=1}^{2n} \alpha_{l}^{(i,j)}b_l \]

- Similarly, Gramian \( A(n, n) = F(n, x) \cdot \mu(x)F(x, n) \) may also be constructed out of \( b(2n) \)
Outline of the talk

1. Sparse solution of underdetermined integral equations
2. Gaussian quadrature design for inner products
3. Their connections: Sparse solution via Gramians

Summary:
3.1. $V(2n, x)u(x) = b(2n)$, Sparse solution by G-quadrature
3.2. Given $V(2n, x)$, determine its factor space $F(n, x)$
3.3. Given $F(n, x)$, determine its minimal function $\mu(x)$
3.4. Fold $b(2n)$ to construct Gramians $A$ and $B$
3.5. Eigen decomposition on the quotient $AB^{-1}$

- Not all integral equations can be solved this way
Types of integral equations that can be solved for sparse solution via the Gramians:

1. Exponential kernels
2. Convolution kernels
3. Polynomials, power functions
4. Bessel and Hankel functions $J_m(x), H_m(x)$

5. Continuous kernel $G(k, x) = r(x) \exp(s(x)t(k))$
6. $G(k, x) = \phi(kx), \phi$ solution of Sturm-Liouville $(-L + k^2)\phi = 0$

7. Varieties of LinEqns $Au = f$, with $A$ m-by-n matrix
Outline of the talk

1. Sparse solution of underdetermined integral equations

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4. Applications
1. Factor space and minimal function - Exacting requirements

2. They are naturally provided in sensing/imaging problems

2.1. Scattering matrix $S(\theta, \theta') = G_{bv}[I - qG_{vv}]^{-1}qG_{vb}$

2.2. Its Born approximation: $S(\theta, \theta') = G_{bv} q G_{vb}$

- The scattering matrix is a Gramian
- Where is the minimal function?
Find a pair of Gramians

- Consider Born case first: $S(\theta, \theta') = G_{bv} q G_{vb}$

1. For horizontally polarized EM waves: $S_H(\theta, \theta') = G_{bv} q_h G_{vb}$

2. For vertically polarized EM waves: $S_V(\theta, \theta') = G_{bv} q_v G_{vb}$

3. The ratio $q_h : q_v$ is our minimal function

4. Quotient $Q = S_h S_v^{-1}$ for imaging dominant scatterers in $q$

4.1. More appropriately: Gen-eig problem for pair $(S_h, S_v)$
Acoustic case

- For scalar wave, no polarization game to play to create a second Gramian to pair with the first $S(\theta, \theta') = G_{bv} q G_{vb}$

- Three methods to create a second Gramian:

1. $\partial_{\theta} S(\theta, \theta')$ - Owl rocking its head
2. $\partial_{k} S(\theta, \theta')$ - Chirpping
3. Motion of scatterers in medium $q$ - Doppler
4. Or their linear combinations
Nonlinear case

- Born approximation: \( S(\theta, \theta') = G_{bv} q G_{vb} \), \( q \) is diagonal matrix

- Multiple scattering: \( S'(\theta, \theta') = G_{bv}[I - qG_{vv}]^{-1}qG_{vb} \)

  \( q \) messed up and replaced by dense matrix \([I - qG_{vv}]^{-1}q\)

- Our quadrature design also works for matrix weights, tensor weights

... their utility in real applications under investigation.
Conclusion

1. Introduced weak formulation for sparse solution of IE

2. Quadrature for inner prod, designed by eigen decomp

3. Inner prod quadratures connected to classical ones

4. Applications in sensing, imaging, inverse scattering

4.1. Forming Gramians $A, B$, and performing eigen decomp on $AB^{-1}$: A fundamental signal processing operation