Dyadic harmonic analysis and weighted inequalities

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Workshop for Women in Analysis and PDE
IMA Institute for Mathematics and its Applications
Minneapolis, MN
May 30 to June 2, 2012

General Outline

- Lecture 1 Overview - with emphasis on the Hilbert transform
- Lecture 2 Overview - A_2 conjecture
- Lecture 3 A study case: dyadic paraproduct
- Lecture 4 Loose ends: Bellman functions and others

Outline Lecture 1

In this lecture we introduce the main objects of study with a focus on the classical Hilbert transform.

- Weighted inequalities
- A_p weights
- Hilbert transform H
- Dyadic intervals, Haar functions
- Martingale transform
- Petermichl's Haar shift operator Ш
- H as average of Шs
- Why are L^p estimates important?

Weighted inequalities

Question (Two-weights L^p-inequalities for operator T)

Is there a constant $C_p(u,v) > 0$ such that

$$\|Tf\|_{L^p(v)} \leq C_p(u,v)\,\|f\|_{L^p(u)}, \quad \textit{for all } f \in L^p(u)?$$

- *The weights u, v are a.e. positive locally integrable functions on $\mathbb{R}^d$.*
- $f \in L^p(u)$ *iff* $\|f\|_{L^p(u)} := (\int |f(x)|^p u(x)\,dx)^{1/p} < \infty$.
- *Operator* $T : L^p(u) \to L^p(v)$.

- Goal 1: given operator T, identify and classify weights u, v for whom the operator T is bounded from $L^p(u)$ to $L^p(v)$).
- Goal 2: understand nature of constant $C_p(u,v)$.

We concentrate on *one-weight L^2 inequalities*: $u = v = w$, and $p = 2$, for Calderón-Zygmund singular integral operators.

Question (One-weight L^2 inequality for operator T)

Is there a constant $C(w) > 0$ such that

$$\|Tf\|_{L^2(w)} \leq C(w)\,\|f\|_{L^2(w)}, \text{ for all } f \in L^2(w)?$$

In particular we study one-weight inequalities in $L^2(w)$, for the Hilbert transform $T = H$, and even more specifically, for simpler dyadic operators such as

- the martingale transform T_σ,
- Petermichl's Haar shift operator Ш ("Sha"),
- the dyadic paraproduct π_b,
- the dyadic square function S^d.

For all these operators to be bounded in $L^2(w)$, the weight w must belong to the Muckenhoupt A_2-class.

A_p weights

Definition

A weight w is in the *Muckenhoupt* A_p class if its A_p characteristic, $[w]_{A_p}$ is finite, where,

$$[w]_{A_p} := \sup_Q \left(\frac{1}{|Q|} \int_Q w \, dx \right) \left(\frac{1}{|Q|} \int_Q w^{-1/(p-1)} dx \right)^{p-1}, \quad 1 < p < \infty \,,$$

the supremum is over all cubes in $\mathbb{R}^d$ with sides parallel to the axes.

Note that a weight $w \in A_2$ if and only if

$$[w]_{A_2} := \sup_Q \left(\frac{1}{|Q|} \int_Q w \, dx \right) \left(\frac{1}{|Q|} \int_Q w^{-1} \, dx \right) < \infty.$$

Example

In $\mathbb{R}$, $w(x) := |x|^\alpha$, $w \in A_p \Leftrightarrow -1 < \alpha < p - 1$.

History

- (Hunt-Muckenhoupt-Wheeden, 1973) Hilbert transform is bounded on $L^p(w)$ if and only if $w \in A_p$.
- (Coifman-Fefferman, 1974) Same holds for convolution type singular integral with standard kernels.
- (A_2 conjecture) Linear (in weight A_2 characteristic) estimate for arbitrary Calderón-Zygmund operator T.

Theorem (Hytönen, Annals 2012)

$$\|Tf\|_{L^2(w)} \leq C_T [w]_{A_2} \|f\|_{L^2(w)},$$

H as a Fourier multiplier

Definition

On Fourier side the *Hilbert transform* is defined by

$$\widehat{Hf}(\xi) := -i\,\mathrm{sgn}(\xi)\,\widehat{f}(\xi),$$

where $\mathrm{sgn}(\xi) = 1$ if $\xi > 0$, $\mathrm{sgn}(\xi) = -1$ if $\xi < 0$, and is zero at $\xi = 0$.

The absolute value of the symbol $m_H(\xi) := -i\,\mathrm{sgn}(\xi)$ is 1 a.e., and Plancherel's identity used twice implies that $H : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and that it is an isometry,

$$\|Hf\|_2 = \|\widehat{Hf}\|_2 = \|\widehat{f}\|_2 = \|f\|_2.$$

Recall: $\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i\xi x}dx$ for f in the Schwartz class.

H as a singular integral operator

Definition

$$Hf(x) := \text{p.v.}\frac{1}{\pi}\int \frac{f(y)}{x-y}\,dy := \lim_{\epsilon\to 0}\frac{1}{\pi}\int_{|x-y|>\epsilon}\frac{f(y)}{x-y}\,dy.$$

The Hilbert transform is given on Fourier side by

$$\widehat{Hf}(\xi) = m_H(\xi)\,\widehat{f}(\xi), \quad \text{where} \quad m_H(\xi) = -i\,\text{sgn}(\xi).$$

Multiplication on Fourier side corresponds to convolution on space

$$Hf(x) = K_H * f(x),$$

where K_H, is the inverse Fourier transform of the multiplier m_H.

$$K_H(x) := (m_H)^{\vee}(x) := \int_{\mathbb{R}} m_H(\xi)e^{2\pi i x\xi}d\xi = \text{p.v.}\frac{1}{\pi x}.$$

Recall: $f * g(x) = \int_{\mathbb{R}} f(y)g(x-y)dy = g * f(x)$.

Boundedness properties of H

Integrable kernel implies boundedness on $L^p(\mathbb{R})$ by Hausdorff-Young's inequality for $p \geq 1$: if $g \in L^1(\mathbb{R}), f \in L^p(\mathbb{R})$ then $\|g * f\|_p \leq \|g\|_1 \|f\|_p$.
But K_H is not in $L^1(\mathbb{R})$, despite this fact,

Properties

- *H is bounded on $L^p(\mathbb{R})$ for all $1 < p < \infty$ (Marcel Riesz 1927):*

$$\|Hf\|_p \leq C_p \|f\|_p.$$

- *H is not bounded on L^1, is of weak-type (1,1) (Kolmogorov 1927).*
- *H is not bounded on L^∞, is bounded on BMO (Fefferman 1971?).*

Example

A calculation shows $H\chi_{[0,1]}(x) = (1/\pi) \log |x|/|x-1|$.
$\log |x|$ is in BMO but not in L^∞.

Properties shared by all Calderón-Zygmund singular integral operators.

Calderón-Zygmund singular integral operators

Definition

A *Calderón-Zygmund operator* is an operator T defined on some dense subspace of $L^2(\mathbb{R}^d)$, and that has a kernel representation

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y)f(y)dy, \;\; x \notin \operatorname{supp} f,$$

where K is a standard kernel (with smoothness parameter α):
K is a function defined on $\mathbb{R}^d \times \mathbb{R}^d$ minus the diagonal, such that

- (size condition) $|K(x,y)| \leq \frac{C_1}{|x-y|^d}$,
- (cancellation) For all $x, x', y \in \mathbb{R}^d$ with $|x-y| > 2|x-x'|$,
$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leq C_2 \frac{|x-x'|^\alpha}{|x-y|^{d+\alpha}}.$$

$T(1)$ Theorem

Departure point for the CZ theory is boundedness on L^2.

- Standard estimates plus L^2 imply weak $(1,1)$ estimates (Calderón-Zygmund decomposition).
- Interpolation then gives L^p for $1 < p < 2$.
- Duality gives $p > 2$.

To deduce boundedness on L^2:

- Plancherel: bounded Fourier multiplier (convolution operators).
- Cotlar's lemma (almost-orthogonality).
- $T(1)$ theorem:

Theorem (David-Journé 1984)

A CZ operator can be extended to be a bounded operator in $L^2(\mathbb{R}^d)$ if and only if $T(1), T^*(1) \in BMO$, and T is weakly bounded.

Symmetries for H

Properties

- *Convolution* $\Leftrightarrow$ *H commutes with* translations
  $\tau_h(Hf) = H(\tau_h f)$ *where* $\tau_h f(x) := f(x-h)$.
- *Homogeneity of kernel* $\Leftrightarrow$ *H commutes with* dilations
  $D_\delta(Hf) = H(D_\delta f)$ *where* $D_\delta f(x) = f(\delta x)$.
- *Kernel odd* $\Leftrightarrow$ *H anticommutes with* reflections
  $(\tilde{Hf}) = -H(\tilde{f})$ *where* $\tilde{f}(x) := f(-x)$.

A linear and bounded operator T in $L^2(\mathbb{R})$ that commutes with translations, dilations, and anticommutes with reflections must be a constant multiple of the Hilbert transform: $T = cH$.
Using this principle, (Stefanie Petermichl 2000) showed that we can write H as a suitable "average of dyadic operators".

Dyadic intervals

Definition

The *standard dyadic intervals* $\mathcal{D}$ is the collection of intervals of the form $[k2^{-j}, (k+1)2^{-j})$, for all integers $k, j \in \mathbb{Z}$.

They are organized by generations: $\mathcal{D} = \cup_{j\in\mathbb{Z}}\mathcal{D}_j$, where $I \in \mathcal{D}_j$ iff $|I| = 2^{-j}$. Each generation is a partition of $\mathbb{R}$. They satisfy

Properties

- *Nested:* $I, J \in \mathcal{D}$ *then* $I \cap J = \emptyset, \quad I \subseteq J,$ *or* $J \subset I$.
- *One parent: if* $I \in \mathcal{D}_j$ *then there is a unique interval* $\tilde{I} \in \mathcal{D}_{j-1}$ *(the parent) such that* $I \subset \tilde{I}$, *and* $|\tilde{I}| = 2|I|$.
- *Two children: There are exactly two disjoint intervals* $I_r, I_l \in \mathcal{D}_{j+1}$ *(the right and left children), such that* $I = I_r \cup I_l$, *and* $|I| = 2|I_r| = 2|I_l|$.

Random dyadic grids on $\mathbb{R}$

Definition

A dyadic grid in $\mathbb{R}$ is a collection of intervals, organized in generations, each of them being a partition of $\mathbb{R}$, that have the nestedness and two children per interval properties.

For example, the shifted and rescaled regular dyadic grid will be a dyadic grid. However these are NOT all possible dyadic grids. The following parametrization will capture ALL dyadic grids on $\mathbb{R}$.

Lemma

For each scaling or dilation parameter r with $1 \leq r < 2$, and the random parameter β with $\beta = \{\beta_i\}_{i\in\mathbb{Z}}$, $\beta_i = 0, 1$, let $x_j = \sum_{i<-j} \beta_i 2^i$, the collection of intervals $D^{r,\beta} = \cup_{j\in\mathbb{Z}} \mathcal{D}_j^{r,\beta}$ is a dyadic grid. Where

$$\mathcal{D}_j^{r,\beta} := r\mathcal{D}_j^{\beta}, \qquad \textit{and} \qquad \mathcal{D}_j^{\beta} := x_j + \mathcal{D}_j.$$

Random dyadic grids were

- introduced by Nazarov, Treil and Volberg in their study of CZ singular integrals on non-homogeneous spaces [NTV 2003],
- utilized by Hytönen in his representation theorem [Hytonen 2012].

The advantage of this parametrization is that there is a very natural probability space, say $(\Omega, \mathbb{P})$ associated to the parameters, and averaging here means calculating the expectation in this probability space, that is $\mathbb{E}_\Omega f = \int_\Omega f(\omega)\, d\mathbb{P}(\omega)$.

Haar basis

Definition

Given an interval I, its associated *Haar function* is defined to be

$$h_I(x) := |I|^{-1/2}\big(\chi_{I_r}(x) - \chi_{I_l}(x)\big),$$

where $\chi_I(x) = 1$ if $x \in I$, zero otherwise.

Properties

- $\|h_I\|_2 = 1$, *and it has zero integral* $\int h_I = 0$.
- $\{h_I\}_{I\in\mathcal{D}}$ *is a complete orthonormal system in* $L^2(\mathbb{R})$ *(Haar 1910).*
- *The Haar basis is an unconditional basis in* $L^p(\mathbb{R})$ *for* $1 < p < \infty$.
- *The Haar basis is an unconditional basis in* $L^p(w)$ *if* $w \in A_p$ *(Treil-Volberg 1996).*

Unconditional basis

Definition

A basis $\{\psi_n\}_{n\in\mathbb{N}}$ in a Banach space X is an *unconditional basis* if for all $x \in X$, $x = \sum_{n\in\mathbb{N}} a_n\psi_n$ then for all choices of signs $\epsilon_n = \pm 1$, the series $\sum_{n\in\mathbb{N}} \epsilon_n a_n \psi_n$ converges in X.

Example

The trigonometric functions $\{e^{2\pi i n\theta}\}_{n\in\mathbb{Z}}$ form an unconditional basis in $L^p(\mathbb{T})$ *only* if $p = 2$.
Wavelets are unconditional bases in various functional spaces.

We can define an operator, the *martingale transform*, whose boundedness implies the unconditionality of the Haar basis in L^p, $p > 1$.

Martingale transform

Definition (The Martingale transform)

$$T_\sigma f(x) := \sum_{I\in\mathcal{D}} \sigma_I \langle f, h_I\rangle h_I(x), \qquad \text{where} \quad \sigma_I = \pm 1.$$

- Martingale transform is a good model for CZ singular operators.
- Unconditionality of the Haar basis on $L^p(\mathbb{R})$ follows from boundedness of T_σ on $L^p(\mathbb{R})$ with norm independent on the choice of signs,

$$\sup_\sigma \|T_\sigma f\|_p \leq C_p \|f\|_p.$$

  Burkholder (1984) found the optimal constant C_p.
- Unconditionality on $L^p(w)$ when $w \in A_p$ follows from the boundedness of the martingale transform on $L^p(w)$ (Treil-Volberg 1996). Sharp linear bounds on $L^2(w)$ (Wittwer 2000).

Petermichl's dyadic shift operator

Definition

Petermichl's dyadic shift operator Ш (pronounced "Sha") associated to the standard dyadic grid $\mathcal{D}$ is defined for functions $f \in L^2(\mathbb{R})$ by

$$Ш f(x) := \sum_{I \in \mathcal{D}} \langle f, h_I \rangle H_I(x),$$

where $H_I = 2^{-1/2}(h_{I_r} - h_{I_l})$.

- Ш is an isometry on $L^2(\mathbb{R})$, i.e. $\|Ш f\|_2 = \|f\|_2$.
- Notice that $Ш h_I(x) = H_I(x)$. The profiles of h_I and H_I can be viewed as a localized sine and cosine. First indication that the dyadic shift operator maybe a good dyadic model for the Hilbert transform.
- More evidence comes from the way Ш interacts with translations, dilations and reflections.

Symmetries for Sha

Denote by $Ш_{r,\beta}$ Petermichl's shift operator associated to the random dyadic grid $\mathcal{D}_{r,\beta}$.

Properties

The following symmetries for the family of shift operators $\{Ш_{r,\beta}\}_{(r,\beta)\in\Omega}$ *hold,*

- *Translation* $\tau_h(Ш_{r,\beta}f) = Ш_{r,\tau_{-h}\beta}(\tau_h f)$.
- *Dilation* $D_\delta(Ш_{r,\beta}f) = Ш_{D_\delta(r,\beta)}(D_\delta f)$.
- *Reflection* $\widetilde{Ш_{r,\beta}f} = -Ш_{r,\tilde{\beta}}(\tilde{f})$.

where $\tilde{\beta}_i = 1-\beta_i$, $\tau_{-h}\beta \in \{0,1\}^{\mathbb{Z}}$, and $D_\delta(r,\beta) \in [1,2)\times\{0,1\}^{\mathbb{Z}} = \Omega$.

Petermichil's representation theorem for H

Each Shift Dyadic Operator does not have the symmetries that characterize the Hilbert transform, but an average over all random dyadic grids does.

Theorem (Petermichl 2000)

$$\mathbb{E}_{\Omega} Ш_{r,\beta} = \int_{\Omega} Ш_{r,\beta} \, d\mathbb{P}(r,\beta) = cH,$$

- Result follows once one verifies that $c \neq 0$ (which she did!).
- Similar representation works for the *Beurling-Ahlfors* (Petermichl-Volberg 2002), *Riesz transforms* (Petermichl 2008).
- There is a representation valid for ALL Calderón-Zygmund singular integral operators (Hytönen 2010).

L^p-boundedness of the Hilbert Transform

Estimates for the Hilbert transform H follow from uniform estimates for Petermichl's shift operators.

Theorem (Riesz 1927)

Hilbert transform is bounded on L^p for $1 < p < \infty$.

$$\|Hf\|_p \leq C_p \|f\|_p.$$

Proof.

Suffices to check that

$$\sup_{(r,\beta)\in\Omega} \|Ш_{r,\beta} f\|_p \leq C_p \|f\|_p.$$

Case $p = 2$ follows from orthonormality of the Haar basis. □

Proof (continuation).

First rewrite Petermichl's shift operator in the following manner, where $\tilde{I}$ is the parent of I in the dyadic grid $\mathcal{D}^{r,\beta}$,

$$Ш_{r,\beta} f = \sum_{I \in \mathcal{D}_{r,\beta}} \frac{1}{\sqrt{2}} \operatorname{sgn}(I, \tilde{I}) \langle f, h_{\tilde{I}} \rangle h_I,$$

where $\operatorname{sgn}(I, \tilde{I}) = 1$ if $I = \tilde{I}_r$, and -1 if $I = \tilde{I}_l$. By Plancherel (remember that each parent has two children),

$$\|Ш_{r,\beta} f\|_2^2 = \sum_{I \in \mathcal{D}_{r,\beta}} \frac{|\langle f, h_{\tilde{I}} \rangle|^2}{2} = \|f\|_2^2.$$

Minkowski Integral Inequality then shows that

$$\|\mathbb{E} Ш_{r,\beta} f\|_2 \leq \mathbb{E} \|Ш_{r,\beta} f\|_2 = \|f\|_2.$$

Case $p \neq 2$ follows from L^p estimates for the dyadic square function. □

Why are we interested in these estimates?

- Boundedness of H on $L^p(\mathbb{T})$ implies convergence on $L^p(\mathbb{T})$ of the partial Fourier sums.
- Hf is the boundary value of the harmonic conjugate of the Poisson extension of a function $f \in L^p(\mathbb{R})$.
- To show that wavelets are unconditional bases on several functional spaces.
- Boundedness properties of Riesz transforms (singular operators on $\mathbb{R}^d$) have deep connections to PDEs.
- Boundedness of the Beurling transform (a singular integral operator in $\mathbb{R}^2$) on $L^p(w)$ for $p > 2$ and with linear estimates on $[w]_{A_p}$, has important consequences in the theory of quasiconformal mappings (Astala-Iwaniecz-Saksman '01, Petermichl-Volberg '02).

Outline Lecture 2

In this lecture we discuss:

- Weighted inequalities for the Hilbert transform
- Linear estimates on $L^2(w)$ for dyadic operators
- Estimates in $L^p(w)$ via sharp extrapolation theorem
  - Dyadic square function
  - Maximal function
- Haar shift operators of complexity (m, n)
- Dyadic paraproducts
- Boundedness properties for Haar shift operators
- Hytönen's theorem (A_2 conjecture).

Boundedness of H on weighted L^p

Theorem (Hunt-Muckenhoupt-Wheeden 1972)

$$w \in A_p \Leftrightarrow \|Hf\|_{L^p(w)} \leq C_p(w)\|f\|_{L^p(w)}.$$

Dependence of the constant on $[w]_{A_p}$ was found 30 years later.

Theorem (Petermichl 2007)

$$\|Hf\|_{L^p(w)} \leq C_p[w]_{A_p}^{\max\{1,\frac{1}{p-1}\}}\|f\|_{L^p(w)}.$$

Sketch of the proof.

For $p = 2$ suffices to find uniform (on the grids) linear estimates for Petermichl's shift operator on $L^2(w)$. For $p \neq 2$ a sharp extrapolation theorem, that we will discuss later, automatically gives the result from the *linear estimate* on $L^2(w)$. □

Chronology of first Linear Estimates on $L^2(w)$

- *Maximal function* (Steve Buckley 1993)
- *Martingale transform* (Janine Wittwer 2000)
- *Dyadic square function* (Hukovic,Treil,Volberg; and Wittwer 2000)
- *Beurling transform* (Petermichl, Volberg 2002)
- *Hilbert transform* (Stephanie Petermichl 2003, published 2007)
- *Riesz transforms* (Stephanie Petermichl 2008)
- *Dyadic paraproduct in* $\mathbb{R}$ (Oleksandra Beznosova 2008)

Estimates based on Bellman functions and bilinear Carleson estimates (except for maximal function).
The Bellman function method was introduced to harmonic analysis by Nazarov, Treil and Volberg. With their students and collaborators have been able to use this method to obtain a number of astonishing results not only in this area see [VasV, V] and references.

Sharp extrapolation

$L^p(w)$ inequalities are deduced from linear bounds on $L^2(w)$: Rubio de Francia extrapolation theorem.

Theorem (Sharp Extrapolation Theorem)

If for all $w \in A_r$ there is $\alpha > 0$, and $C > 0$ such that

$$\|[Tf\|_{L^r(w)} \leq C[w]_{A_r}^{\alpha}\|f\|_{L^r(w)} \text{ for all } f \in L^r(w).$$

then for each $1 < p < \infty$ and for all $w \in A_p$, there is $C_{p,r} > 0$

$$\|[Tf\|_{L^p(w)} \leq C_{p,r}[w]_{A_p}^{\alpha \max\{1, \frac{r-1}{p-1}\}}\|f\|_{L^p(w)} \text{ for all } f \in L^p(w).$$

Dragicevic-Grafakos-P-Petermichl '05, new proof Duoandikoetxea '11.

Example

Sharp extrapolation from $r = 2$, $\alpha = 1$, is sharp for the martingale, Hilbert, Beurling-Ahlfors and Riesz transforms for all $1 < p < \infty$.

Dyadic square function

It is sharp for the dyadic square function only when $1 < p \leq 2$ ("sharp" DGPPet, "only" Lerner '07).

Definition (The dyadic square function)

$$(S^d f)^2(x) := \sum_{I\in\mathcal{D}} |m_I f - m_{\tilde{I}} f|^2 \chi_I(x) = \sum_{I\in\mathcal{D}} \frac{|\langle f, h_I\rangle|^2}{|I|}\chi_I(x),$$

where $m_I f = (1/|I|)\int_I f$, $\tilde{I}$ is the parent of I.

- S^d is an isometry on $L^2(\mathbb{R})$.
- S^d is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$ furthermore

$$\|S^d f\|_p \sim \|f\|_p.$$

This plays the role of Plancherel's theorem in L^p (Littlewood-Paley theory). It implies boundedness of T_σ (and Ш) on L^p

$$\|T_\sigma f\|_p \sim \|S^d(T_\sigma f)\|_p = \|S^d f\|_p \sim \|f\|_p.$$

Dyadic square function - weighted estimates

- S^d is bounded on $L^2(w)$ if $w \in A_2$

$$\|S^d f\|_{L^2(w)} \le C[w]_{A_2} \|f\|_{L^2(w)} \quad \text{(HukTV, Witt 2000).}$$

Extrapolation will give boundedness on L^p and on weighted-L^p.

- S^d is bounded on $L^p(w)$ if $w \in A_p$

$$\|S^d f\|_{L^p(w)} \le C_p [w]_{A_p}^{\max\{\frac{1}{2}, \frac{1}{p-1}\}} \|f\|_{L^p(w)} \quad \text{(CrMPz 2010).}$$

This power is optimal. It corresponds to sharp extrapolation starting at $r = 3$ with square root power instead of starting at $r = 2$ with linear power.

The Hardy-Littlewood maximal function

Definition

The *Hardy-Littlewood maximal function*

$$Mf(x) := \sup_{I \ni x} \frac{1}{|I|} \int_I |f(y)|\, dy.$$

- M is bounded on $L^p(\mathbb{R})$ for $1 < p$.
- M is not bounded on $L^1(\mathbb{R})$, but it is of weak-type $(1,1)$ (Hardy-Littlewood 1930).
- M is bounded on $L^p(w)$ if and only if $w \in A_p$ (Muckenhoupt 1972).

Buckley's Theorem

The optimal dependence on the A_p-characteristic of the weight was discovered 20 years later.

Theorem (Buckley 1993)

Let $w \in A_p$ *and* $1 < p$ *then*

$$\|Mf\|_{L^p(w)} \leq C_p [w]_{A_p}^{\frac{1}{p-1}} \|f\|_{L^p(w)}.$$

- This estimate is key in the proof of the sharp extrapolation theorem.
- Observe that if we start with Buckley's estimate for M on $L^r(w)$ then sharp extrapolation will give the right power for all $1 < p \leq r$, however for $p > r$ it will simply give $\frac{1}{r-1}$ which is bigger than the correct power $\frac{1}{p-1}$.

Haar shift operators

Definition

A *Haar shift operator of complexity* (m, n) is

$$Ш_{m,n} f(x) := \sum_{L \in \mathcal{D}} \sum_{I \in \mathcal{D}_m(L), J \in \mathcal{D}_n(L)} c^L_{I,J} \langle f, h_I \rangle h_J(x),$$

where the coefficients $|c^L_{I,J}| \leq \frac{\sqrt{|I|\,|J|}}{|L|}$, and $\mathcal{D}_m(L)$ denotes the dyadic subintervals of L with length $2^{-m}|L|$.

- The cancellation property of the Haar functions and the normalization of the coefficients ensures that $\|Ш_{m,n} f\|_2 \leq \|f\|_2$.
- T_σ is a Haar shift operator of complexity $(0, 0)$.
- $Ш$ is a Haar shift operator of complexity $(0, 1)$.
- The dyadic paraproduct π_b is not one of these.

The dyadic paraproduct

Definition

A locally integrable function $b \in BMO^d$ iff for all $J \in \mathcal{D}$ there is a constant $C > 0$ such that for all $J \in \mathcal{D}$

$$\int_J |b(x) - m_J b|^2 dx = \sum_{I \in \mathcal{D}(J)} |\langle b, h_I \rangle|^2 \leq C|J|.$$

($b \in BMO^d$ iff $\{|\langle b, h_I\rangle|^2\}_{I \in \mathcal{D}}$ is a *Carleson sequence*.)

Definition

The *dyadic paraproduct* associated to $b \in BMO^d$ is

$$\pi_b f(x) := \sum_{I \in \mathcal{D}} m_I f \, \langle b, h_I \rangle h_I(x),$$

where $m_I f = \frac{1}{|I|} \int_I f(x)\, dx = \langle f, \chi_I / |I| \rangle$.

The adjoint of the dyadic paraproduct

Definition

The *adjoint of the dyadic paraproduct* is

$$\pi_b^* f(x) = \sum_{I \in \mathcal{D}} \langle f, h_I \rangle \langle b, h_I \rangle \frac{\chi_I(x)}{|I|}.$$

- These are bounded operators in $L^p(\mathbb{R})$ (if and only if $b \in BMO^d$).
- Formally, $fb = \pi_b f + \pi_b^* f + \pi_f b$.
- In $T(1)$ theorem: $T = T_0 + \pi_{T1} + \pi_{T^*1}^*$.

Estimates for Shift Operators

- Michael Lacey, Stefanie Petermichl and Mari Carmen Reguera (2010) proved the A_2 conjecture for the Haar shift operators of arbitrary complexity (with constant depending exponentially in the complexity). They don't use Bellman functions. They use a *corona decomposition* and a *two-weight theorem for "well localized operators"* of Nazarov, Treil and Volberg.
- Cruz-Uribe, Martell, and Pérez (2010) recover all results for Haar shift operators. No Bellman functions, no two-weight results. Instead they use a local median oscillation introduced by Lerner. The method is very flexible, they get new results such as the sharp bounds for the square function for $p > 2$, for the dyadic paraproduct, also for vector-valued maximal operators, and two-weight results as well. Dependence on complexity is exponential.

The A_2 conjecture (now Theorem)

Theorem (Hytönen 2010)

Let $1 < p < \infty$ *and let* T *be any Calderón-Zygmund singular integral operator in* $\mathbb{R}^n$*, then there is a constant* $c_{T,n,p} > 0$ *such that*

$$\|Tf\|_{L^p(w)} \leq c_{T,n,p}\,[w]_{A_p}^{\max\{1,\frac{1}{p-1}\}}\|f\|_{L^p(w)}.$$

Sketch of the proof.

- Enough to show $p = 2$ thanks to sharp extrapolation.
- Prove a representation theorem in terms of Haar shift operators of arbitrary complexity and paraproducts on random dyadic grids.
- Prove linear estimates on $L^2(w)$ with respect to the A_2 characteristic for Haar shift operators and with polynomial dependence on the complexity (independent of the dyadic grid).

□

Hytönen's Representation theorem

Theorem (Hytönen's Representation Theorem 2010)

Let T be a Calderón-Zygmund singular integral operator, then

$$Tf = \mathbb{E}_\Omega \left(\sum_{(m,n)\in\mathbb{N}^2} a_{m,n} Ш^{r,\beta}_{m,n} f + \pi^{r,\beta}_{T1} f + (\pi^{r,\beta}_{T^*1})^* f \right),$$

with $a_{m,n} = e^{-(m+n)\alpha/2}$, α is the smoothness parameter of T.

- $Ш^{r,\beta}_{m,n}$ are Haar shift operators of complexity (m,n),
- $\pi^{r,\beta}_{T1}$ a dyadic paraproduct,
- $(\pi^{r,\beta}_{T^*1})^*$ the adjoint of the dyadic paraproduct ,

All defined on random dyadic grid $\mathcal{D}^{r,\beta}$.

Recent progress

Active area of research!

- There is a nice survey by Lacey up to 2010.
- Since the appearance of Hytönen's theorem several simplifications of the argument have appeared [HytPzTV,NV,T,L2,T,Hyt3... 2011-12].
- There are already extension to metric spaces with geometric doubling condition [NRezV 2011], and to the solution of the two weight bump conjecture [NRTV, CrRV, HytPz, Lerner - 2012].
- Also mixed $A_p - A_\infty$ estimates.
- Different attempts to get rid of one or more components of the proofs: randomness, Bellman functions, Haar shift operators [LHyt, HytLPz, Le - 2012].

Outline Lecture 3

In this lecture we discuss:

- Ingredients in the proof of the A_2 conjecture for the dyadic paraproduct.
  - Weighted or disbalanced Haar basis.
  - Carleson and w-Carleson sequences.
    - The Little Lemma
    - The α Lemma
  - Weighted maximal function.
  - Weighted Carleson Lemma.
- The Proof

Weighted or disbalanced Haar basis

Definition

Given weight v and interval I, the *weighted Haar function* h_I^v is

$$h_I^v(x) := \frac{1}{\sqrt{v(I)}} \sqrt{\frac{v(I_-)}{v(I_+)}} \chi_{I_+}(x) - \frac{1}{\sqrt{v(I)}} \sqrt{\frac{v(I_+)}{v(I_+)}} \chi_{I_-}(x).$$

- $\{h_I^v\}_{I \in \mathcal{D}}$ is an orthonormal system in $L^2(v)$.
- There exist sequences α_I^v, β_I^v such that

$$h_I(x) = \alpha_I^v h_I^v(x) + \beta_I^v \frac{\chi_I(x)}{\sqrt{|I|}}$$

(i) $|\alpha_I^v| \leq \sqrt{m_I v}$,
(ii) $|\beta_I^v| \leq \frac{|\Delta_I v|}{m_I v}$, and $\Delta_I v := m_{I_+} v - m_{I_-} v$.

Carleson and w-Carleson sequences

Definition

Let $\lambda_I, \sigma_I > 0$, and v a weight.

- $\{\lambda_I\}_{I\in\mathcal{D}}$ is a *Carleson sequence with intensity* $Q > 0$, if

$$\sum_{I\in\mathcal{D}(J)} \lambda_I \leq Q|J|, \quad \text{for all} \quad J \in \mathcal{D}.$$

- $\{\sigma_I\}_{I\in\mathcal{D}}$ is a v-*Carleson sequence with intensity* $Q > 0$, if

$$\sum_{I\in\mathcal{D}(J)} \sigma_I \leq Qv(J) \quad \text{for all} \quad J \in \mathcal{D}.$$

Example

$b \in BMO^d$, then $\{\langle b, h_I\rangle^2\}_{I\in\mathcal{D}}$ is Carleson with intensity $\|b\|^2_{BMO^d}$.

Beznosova's Little Lemma

To create v-Carleson sequences from a given Carleson sequences we have the following lemma.

Lemma (Beznosova 2008)

Let v be a weight, such that v^{-1} is a a weight as well. Let $\{\lambda_I\}_{I\in\mathcal{D}}$ be a Carleson sequence with intensity Q, then for all $J \in \mathcal{D}$

$$\sum_{I\in\mathcal{D}(J)} \frac{\lambda_I}{m_I v^{-1}} \leq 4Q\, v(J).$$

"The sequence $\{\frac{\lambda_I}{m_I v^{-1}}\}_{I\in\mathcal{D}}$ is a v-Carleson sequence with intensity $4Q$."

The proof uses a Bellman function argument.

Example (▸ Sigma1)

$b \in BMO^d$, $b_I = \langle b, h_I\rangle$ then $\{b_I^2/m_I v^{-1}\}_{I\in\mathcal{D}}$ is a v-Carleson sequence.

Algebra of Carleson sequences

Lemma

Given a weight v. Let λ_I and γ_I be two v-Carleson sequences with intensities A and B respectively then for any $c, d > 0$

- *$c\lambda_I + d\gamma_I$ is a v-Carleson sequence with intensity at most $cA + dB$.*
- *$\sqrt{\lambda_I \gamma_I}$ is a v-Carleson sequence with intensity at most $\sqrt{AB}$.*

The proof is a simple exercise. Sigma2

The α-Lemma

Lemma (Beznosova 2008)

If $w \in A_2$ *and* $0 < \alpha < 1/2$, *then the sequence*

$$\mu_I := (m_I w)^\alpha (m_I w^{-1})^\alpha |I| \left(\frac{|\Delta_I w|^2}{m_I^2 w} + \frac{|\Delta_I w^{-1}|^2}{m_I^2 w^{-1}} \right) \quad I \in \mathcal{D}$$

is a Carleson sequence with Carleson intensity at most $C_\alpha [w]_{A_2}^\alpha$ *for any* $\alpha \in (0, 1/2)$.

By α- Lemma, and algebra of Carleson sequences ▸ Sigma2, if $w \in A_2^d$:

- $\{\nu_I := |\Delta_I w^{-1}|^2 (m_I w)^2 |I|\}_{I \in \mathcal{D}}$ is Carleson with intensity $[w]_{A_2}^2$.
- If $b \in BMO^d$, and $b_I := |\langle b, h_I \rangle|$, then $\{b_I \sqrt{\nu_I}\}_{I \in \mathcal{D}}$ is Carleson with $B = C[w]_{A_2} \|b\|_{BMO}$.

Weighted maximal function

Definition

The *weighted dyadic maximal function* M_v^d, v a weight, is defined by

$$M_v^d f(x) := \sup_{I: x\in I\in\mathcal{D}} \frac{1}{v(I)} \int_I |f(y)|v(y)\,dy.$$

Lemma

For all $f \in L^p(v)$ *there is a constant* $C(p) > 0$ *such that*

$$\|M_v^d f\|_{L^p(v)} \leq C(p)\|f\|_{L^p(v)}.$$

Moreover $C(p) \sim p'$ *at least for* p *near* 1.

Let I be a dyadic interval, then

$$\frac{m_I(|f|v)}{m_I v} \leq \inf_{x\in I} M_v^d f(x).$$

Weighted Carleson Lemma

Lemma

If v is a weight, $\{\sigma_L\}_{L\in\mathcal{D}}$ is a v-Carleson sequence with intensity Q, and F is a positive measurable function on $\mathbb{R}$, then

$$\sum_{L\in\mathcal{D}} \sigma_L \inf_{x\in L} F(x) \leq Q \int_{\mathbb{R}} F(x)v(x)\,dx.$$

In applications F involves the dyadic weighted maximal function M_v^d

- $F(x) = [M_v^d f(x)]^2$: the RHS becomes $\|M_v^d f\|_{L^2(v)}^2 \leq C\|f\|_{L^2(v)}^2$.
- $F(x) = [M_v^d f^p(x)]^{2/p}$: the RHS becomes $\|M_v^d f^p\|_{L^{2/p}(v)}^{2/p}$, this is useful if $1 < p < 2$ so that $2/p > 1$. In that case:

$$\|M_v^d f^p\|_{L^{2/p}(v)}^{2/p} \leq c(2/p)'\|f^p\|_{L^{2/p}(v)}^{2/p} = c\frac{2}{2-p}\|f\|_{L^2(v)}^2.$$

▸ Sigma1 ▸ Sigma2

A_2 conjecture for the dyadic paraproduct

We present a proof of Beznosova's theorem, for $b \in BMO^d$, $w \in A_2^d$,

$$\|\pi_b f\|_{L^2(w)} \leq C\|b\|_{BMO^d}[w]_{A_2^d}\|f\|_{L^2(w)}.$$

The proof uses

- Same ingredients introduced by Oleksandra.
- An argument by Nazarov and Volberg that yields polynomial in the complexity bounds for Haar shift operators. In joint work with Jean Moraes (2011) we extended their result to *paraproducts with arbitrary complexity*.

Suffices by duality to prove:

$$|\langle \pi_b(fw), gw^{-1}\rangle| \leq C\|b\|_{BMO^d}[w]_{A_2}\|f\|_{L^2(w)}\|g\|_{L^2(w^{-1})}$$

Proof of A_2 conjecture for dyadic paraproduct

$$|\langle \pi_b(fw), gw^{-1}\rangle| \leq \sum_{I\in\mathcal{D}} |b_I| m_I(|f|w)|\langle gw^{-1}, h_I\rangle| \leq \Sigma_1 + \Sigma_2,$$

where we replace $h_I = \alpha_I^{w^{-1}} h_I^{w^{-1}} + \beta_I^{w^{-1}} \frac{\chi_I}{\sqrt{|I|}}$, to get the two sums:

$$\Sigma_1 := \sum_{I\in\mathcal{D}} |b_I| m_I(|f|w)|\langle gw^{-1}, h_I^{w^{-1}}\rangle| \sqrt{m_I w^{-1}}$$

$$\Sigma_2 := \sum_{I\in\mathcal{D}} |b_I| m_I(|f|w) m_I(|g|w^{-1}) \frac{|\Delta_I w^{-1}|}{m_I w^{-1}} \sqrt{|I|}$$

Sigma1 Sigma2

First sum

proof

$$\Sigma_1 \leq \sum_{I\in\mathcal{D}} \frac{|b_I|}{\sqrt{m_I w^{-1}}} \frac{m_I(|f|w)}{m_I w} |\langle g, h_I^{w^{-1}}\rangle_{w^{-1}}| m_I w \, m_I w^{-1}$$

$$\leq [w]_{A_2} \sum_{I\in\mathcal{D}} \frac{|b_I|}{\sqrt{m_I w^{-1}}} \inf_{x\in I} M_w f(x) |\langle g, h_I^{w^{-1}}\rangle_{w^{-1}}|$$

$$\leq [w]_{A_2} \left(\sum_{I\in\mathcal{D}} \frac{|b_I|^2}{m_I w^{-1}} \inf_{x\in I} M_w^2 f(x) \right)^{\frac{1}{2}} \left(\sum_{I\in\mathcal{D}} \left| \langle g, h_I^{w^{-1}}\rangle_{w^{-1}} \right|^2 \right)^{\frac{1}{2}}$$

Use Weighted Carleson Lemma with $F(x) = M_w^2 f(x)$ and $v = w$, and w-Carleson sequence $b_I^2/m_I w^{-1}$ by Little Lemma.

$$\Sigma_1 \leq [w]_{A_2} \|b\|_{BMO^d} \left(\int_{\mathbb{R}} M_w^2 f(x) w(x) dx \right)^{\frac{1}{2}} \|g\|_{L^2(w^{-1})}$$

$$\leq C[w]_{A_2} \|b\|_{BMO^d} \|f\|_{L^2(w)} \|g\|_{L^2(w^{-1})}$$

Second sum

proof Using similar arguments that we used for Σ_1

$$\Sigma_2 \le \sum_{I\in\mathcal{D}} |b_I| \frac{m_I(|f|w)}{m_I w} \frac{m_I(|g|w^{-1})}{m_I w^{-1}} \sqrt{|\Delta_I w^{-1}|^2 (m_I w)^2 |I|}$$
$$\le \sum_{I\in\mathcal{D}} |b_I|\sqrt{\nu_I} \inf_{x\in I} M_w f(x)\, M_{w^{-1}} g(x),$$

where $|b_I|^2$ and ν_I are Carleson sequences with intensities $\|b\|^2_{BMO^d}$ and $[w]^2_{A_2}$ Alpha Lemma then by algebra CS the sequence $|b_I|\sqrt{\nu_I}$ is Carleson sequence with intensity $\|b\|_{BMO^d}[w]_{A_2}$.
Using Weighted Carleson Lemma with $F(x) = M_w f(x)\, M_{w^{-1}} g(x)$, $v = 1$,

$$\Sigma_2 \le [w]_{A_2} \|b\|_{BMO^d} \int_{\mathbb{R}} M_w f(x) M_{w^{-1}} g(x) dx.$$

To finish use Cauchy-Schwarz and $w^{1/2}(x)\, w^{-1/2}(x) = 1$,

$$\begin{aligned}
\Sigma_2 &\leq [w]_{A_2} \|b\|_{BMO^d} \int_{\mathbb{R}} M_w f(x) M_{w^{-1}} g(x) dx \\
&\leq [w]_{A_2} \|b\|_{BMO^d} \left(\int_{\mathbb{R}} M_w^2 f(x) w(x) dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} M_{w^{-1}}^2 g(x) w^{-1}(x) dx \right)^{\frac{1}{2}} \\
&= [w]_{A_2} \|b\|_{BMO^d} \|M_w f\|_{L^2(w)} \|M_{w^{-1}} g\|_{L^2(w^{-1})} \\
&\leq C[w]_{A_2} \|b\|_{BMO^d} \|f\|_{L^2(w)} \|g\|_{L^2(w^{-1})}.
\end{aligned}$$

We are done!!

Outline Lecture 4

- Bellman function proof of the Little Lemma
  - Induction on scales
  - The Bellman function
- Bellman function proof of the α Lemma
- Proof of the Weighted Carleson Lemma

Beznosova's Little Lemma

Lemma (Beznosova 2008)

Let v be a weight, such that v^{-1} is a a weight as well. Let $\{\lambda_I\}_{I\in\mathcal{D}}$ be a Carleson sequence with intensity Q, then for all $J \in \mathcal{D}$

$$\sum_{I\in\mathcal{D}(J)} \frac{\lambda_I}{m_I v^{-1}} \leq 4Q\, v(J).$$

"The sequence $\{\frac{\lambda_I}{m_I v^{-1}}\}_{I\in\mathcal{D}}$ is a v-Carleson sequence with intensity $4Q$."

The proof uses a Bellman function argument, which we now describe.

Proof of the Little Lemma

The first lemma encodes what now is called an *induction on scales argument*. If we can find a Bellman function with certain properties, then we will solve our problem by induction on scales.

Lemma (Induction on scales)

Suppose there exists a real valued function of 3 variables $B(x) = B(u, v, l)$, *whose domain* $\mathfrak{D}$ *contains points* $x = (u, v, l)$

$$\mathfrak{D} = \{(u, v, l) \in \mathbb{R}^3 : u, v > 0, \quad uv \geq 1 \quad \text{and} \quad 0 \leq l \leq 1\},$$

whose range is given by $0 \leq B(x) \leq u$, *and such that the following convexity property holds: for all* $x_\pm \in \mathfrak{D}$ *such that* $x - \frac{x_+ + x_-}{2} = (0, 0, \alpha)$ *we have*

$$B(x) - \frac{B(x_+) + B(x_-)}{2} \geq \frac{1}{4v}\alpha.$$

Then the Little Lemma holds.

Induction on scales

Proof.

Fix a dyadic interval J. Let $u_J = m_J w$, $v_J = m_J(w^{-1})$ and $l_J = \frac{1}{|J|Q}\sum_{I\in D(J)}\lambda_I$, then $x_J := (u_J, v_J, l_J) \in \mathfrak{D}$. Let $x_\pm := x_{J^\pm} \in \mathfrak{D}$.

$$x_J - \frac{x_{J^+} + x_{J^-}}{2} \;=\; (0,0,\alpha_J), \quad \text{where} \;\; \alpha_J := \frac{1}{|J|Q}\lambda_J.$$

Then, by the size and convexity conditions, and $|J^+| = |J^-| = |J|/2$,

$$|J|\, m_J w \;\geq\; |J|\, B(x_J) \geq |J^+|B(x_{J^+}) + |J^-|B(x_{J^-}) + \frac{\lambda_J}{4Qm_J(w^{-1})}.$$

Repeat for $|J^+|B(x_{J^+})$ and $|J^-|B(x_{J^-})$, use that $B \geq 0$ on $\mathfrak{D}$ to get:

$$m_J w \geq \frac{1}{4|J|Q}\sum_{I\in D(J)}\frac{\lambda_I}{m_I(w^{-1})}.$$

The Bellman function

Lemma (Beznosova 2008)

The function

$$B(u,v,l) := u - \frac{1}{v(1+l)}$$

is defined on $\mathfrak{D}$, $0 \leq B(x) \leq u$ *for all* $x = (u,v,l) \in \mathfrak{D}$ *and on* $\mathfrak{D}$*:*

$$(\partial B/\partial l)(u,v,l) \;\geq\; 1/(4v),$$

$$-(du,dv,dl)\, d^2B(u,v,l)\,(du,dv,dl)^t \;\geq\; 0,$$

where $d^2B(u,v,l)$ *denotes the Hessian matrix of the function* B *evaluated at* (u,v,l)*. Moreover, these imply the dyadic convexity condition* $B(x) - \frac{B(x_+)+B(x_-)}{2} \;\geq\; \alpha/(4v)$.

Differential convexity implies dyadic convexity

Proof.

Differential conditions can be check by direct calculation.
By the Mean Value Theorem and some calculus,

$$B(x) - \frac{B(x_+) + B(x_-)}{2} = \frac{\partial B}{\partial l}(u, v, l')\alpha - \frac{1}{2}\int_{-1}^{1}(1 - |t|)b''(t)dt \geq \frac{1}{4v}\alpha.$$

where

$$b(t) := B(x(t)), \quad x(t) := \frac{1+t}{2}x_+ + \frac{1-t}{2}x_-, \quad -1 \leq t \leq 1.$$

Note that $x(t) \in \mathfrak{D}$ whenever x_+ and x_- do, since $\mathfrak{D}$ is a convex domain and $x(t)$ is a point on the line segment between x_+ and x_-, and l' is a point between l and $\frac{l_+ + l_-}{2}$. □

Proof Alpha Lemma

Beznosova 2008.

- Use the Bellman function method.
- Figure out the domain, range and convexity conditions needed to run an induction on scale arguments that will yield the inequality.
- Verify that the Bellman function $B(u,v) = (uv)^{\alpha}$ satisfies those conditions (or at least a differential version).

□

Weighted Carleson Lemma

Lemma

Let v be a weight, $\{\alpha_L\}_{L\in\mathcal{D}}$ a v-Carleson sequence with intensity Q, and F a positive measurable function on $\mathbb{R}$, then

$$\sum_{L\in\mathcal{D}} \alpha_L \inf_{x\in L} F(x) \leq Q \int_{\mathbb{R}} F(x)v(x)\,dx.$$

Proof.

Assume that $F \in L^1(v)$ otherwise the first statement is automatically true. Setting $\gamma_L = \inf_{x\in L} F(x)$, we can write

$$\sum_{L\in\mathcal{D}} \gamma_L \alpha_L = \sum_{L\in\mathcal{D}} \int_0^\infty \chi(L,t)\,dt\,\alpha_L = \int_0^\infty \Big(\sum_{L\in\mathcal{D}} \chi(L,t)\,\alpha_L\Big)dt,$$

where $\chi(L,t) = 1$ for $t < \gamma_L$ and zero otherwise, and by the MCT.

Proof Weighted Carleson Lemma

Proof (continuation).

Define $E_t = \{x \in \mathbb{R} \,:\, F(x) > t\}$.

- Since F is assumed a v-measurable function then E_t is a v-measurable set for every t.
- Since $F \in L^1(v)$ we have, by Chebychev's inequality, that the v-measure of E_t is finite for all real t.
- Moreover, there is a collection of maximal disjoint dyadic intervals $\mathcal{P}_t$ that will cover E_t except for at most a set of v-measure zero.
- $L \subset E_t$ if and only if $\chi(L,t) = 1$.

All together we can rewrite the integrand in previous page as

$$\sum_{L \in \mathcal{D}} \chi(L,t)\alpha_L = \sum_{L \subset E_t} \alpha_L \leq \sum_{L \in \mathcal{P}_t} \sum_{I \in \mathcal{D}(L)} \alpha_I \leq B \sum_{L \in \mathcal{P}_t} v(L) = Bv(E_t),$$

□

Proof Weighted Carleson Lemma

Proof (continuation).

$$\sum_{L\in\mathcal{D}} \chi(L,t)\alpha_L = \sum_{L\subset E_t} \alpha_L \leq \sum_{L\in\mathcal{P}_t} \sum_{I\in\mathcal{D}(L)} \alpha_I \leq B \sum_{L\in\mathcal{P}_t} v(L) = Bv(E_t),$$

we used in the second inequality the fact that $\{\alpha_J\}_{I\in\mathcal{D}}$ is a v-Carleson sequence with intensity B.
Thus we can estimate

$$\sum_{L\in\mathcal{D}} \gamma_L \alpha_L \leq B \int_0^\infty v(E_t)dt = B \int_{\mathbb{R}} F(x)\, v(x)\, dx.$$

where the last equality follows from the layer cake representation. □

Thanks ;-)

;-)

THANKS FOR YOUR PATIENCE!!!!