Perspectives on linear and bilinear pseudodifferential operators

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Workshop for Women in Analysis and PDEs
IMA, May 30 - June 2, 2012

Outline

- Classes of symbols for linear and bilinear ΨDOs.
- A sample of motivations for bilinear ΨDOs.
- Comparison between linear and bilinear results.
- New results presented are joint work with A. Bényi, F. Bernicot, D. Maldonado and R.H. Torres.

Linear pseudodifferential operators

$$T_\sigma f(x) = \int_{\mathbb{R}^n} \sigma(x,\xi)\hat{f}(\xi)e^{2\pi i\xi\cdot x}\, d\xi, \qquad x \in \mathbb{R}^n.$$

Multipliers:

$$\sigma(x,\xi) = m(\xi) \Longrightarrow \widehat{T_\sigma(f)} = m\,\hat{f}$$

Pointwise multiplication:

$$\sigma(x,\xi) = b(x) \Longrightarrow T_\sigma(f) = b\,f$$

Linear partial differential operators:

$$L = \sum_{|\alpha|\le m} a_\alpha(x)\,\partial^\alpha,\ \ \sigma(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\,(2\pi i\xi)^\alpha \Longrightarrow L(f) = T_\sigma(f)$$

Formally:

$$T_\sigma f(x) = \int_{\mathbb{R}^n} k(x,y)f(y)\,dy, \qquad k(x,y) = \int_{\mathbb{R}^n} \sigma(x,\xi)e^{2\pi i(x-y)\xi}d\xi$$

Linear Hörmander classes and Mihlin multipliers

$$T_\sigma f(x) = \int_{\mathbb{R}^n} \sigma(x,\xi)\hat{f}(\xi)e^{2\pi i\xi\cdot x}\,d\xi, \qquad x \in \mathbb{R}^n.$$

Hörmander's classes $S^m_{\rho,\delta}$, $m \in \mathbb{R}$ and $0 \le \delta \le \rho \le 1$:

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{m+\delta|\alpha|-\rho|\beta|}, \quad x,\xi \in \mathbb{R}^n,$$

"for all" multiindices α, $\beta \in \mathbb{N}_0^n$.

Mihlin multipliers:

$$|\partial_\xi^\beta \sigma(\xi)| \le C_\alpha |\xi|^{-|\beta|}, \quad \xi \in \mathbb{R}^n \setminus \{0\},$$

"for all" multiindices $\beta \in \mathbb{N}_0^n$.

Bilinear pseudodifferential operators

$$T_\sigma(f,g)(x) = \int_{\mathbb{R}^{2n}} \sigma(x,\xi,\eta)\hat{f}(\xi)\hat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\,d\xi\,d\eta, \qquad x \in \mathbb{R}^n.$$

Pointwise multiplication:

$$\sigma(x,\xi,\eta) = b(x)\,(2\pi i\xi)^\alpha(2\pi i\eta)^\beta \Longrightarrow T_\sigma(f,g) = b\,D^\alpha f\,D^\beta g$$

Formally:

$$T_\sigma(f,g)(x) = \int_{\mathbb{R}^{2n}} k(x,y,z)f(y)g(z)\,dy\,dz,$$

$$k(x,y,z) = \int_{\mathbb{R}^{2n}} \sigma(x,\xi,\eta)e^{2\pi i(x-y)\cdot\xi}e^{2\pi i(x-z)\cdot\eta}d\xi d\eta.$$

$$T_\sigma(f,g)(x) = \int_{\mathbb{R}^{2n}} \sigma(x,\xi,\eta)\hat{f}(\xi)\hat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\,d\xi\,d\eta, \qquad x \in \mathbb{R}^n.$$

Bilinear Hörmander's classes $BS^m_{\rho,\delta}$, $m \in \mathbb{R}$, $0 \le \delta \le \rho \le 1$:

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)| \le C_{\alpha,\beta,\gamma}(1+|\xi|+|\eta|)^{m+\delta|\alpha|-\rho(|\beta|+|\gamma|)}, \quad x,\xi,\eta \in \mathbb{R}^n,$$

"for all" multiindices α, β, $\gamma \in \mathbb{N}_0^n$.

Coifman-Meyer multipliers:

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi,\eta)| \le C_{\alpha,\beta}(|\xi|+|\eta|)^{-(|\alpha|+|\beta|)}, \quad \xi,\eta \in \mathbb{R}^n,$$

"for all" multiindices α, $\beta \in \mathbb{N}_0^n$.

A sample of motivations and applications

- Commutators
- Fractional Leibniz rule
- Paraproducts

Commutators: Coifman and Meyer, 1978

- If $\sigma \in S^1_{1,0}$ and $A \in Lip_1(\mathbb{R}^n)$ then $[T_\sigma, A] := T_\sigma A - AT_\sigma$ is bounded on $L^2(\mathbb{R}^n)$.

$$[T_\sigma, A] \text{ relates to } BS^0_{1,0}.$$

- If $\sigma \in S^{\rho-1}_{\rho,\rho}$, $0 < \rho < 1$, and $A \in C^1(\mathbb{R}^n)$ then $\nabla[T_\sigma, A]$ is bounded on $L^2(\mathbb{R}^n)$.

$$\nabla[T_\sigma, A] \text{ relates to } BS^0_{\rho,\rho}.$$

Fractional Leibniz rule

For $m \geq 0$ define $J^m := (1-\Delta)^{m/2}$, as

$$\widehat{J^m(f)}(\xi) := \langle \xi \rangle^m \hat{f}(\xi), \qquad \langle \xi \rangle = (1+|\xi|^2)^{1/2}.$$

For $p > 1$, set

$$\|f\|_{W^{m,p}} := \|J^m f\|_{L^p}.$$

Fractional Leibniz rule:

$$\|fg\|_{W^{m,p}} \lesssim \|f\|_{W^{m,p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_1}} \|g\|_{W^{m,p_2}}$$

for $1 < p_1, p_2, p < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.

Kato-Ponce, Christ-Weinstein.

Fractional Leibniz rule by frequency decoupling

$\phi : \mathbb{R} \to [0,1]$ supported in $[-2,2]$ and $\phi(t) + \phi(1/t) = 1, \; t > 0$.

$$\begin{aligned}
J^m(fg)(x) &= \int\int \langle \xi + \eta \rangle^m \, \hat{f}(\xi)\hat{g}(\eta) \, e^{2\pi i x\cdot(\xi+\eta)} \, d\xi d\eta \\
&= \int\int \langle \xi + \eta \rangle^m \phi\left(\tfrac{\langle \eta \rangle}{\langle \xi \rangle}\right) \, \hat{f}(\xi)\hat{g}(\eta) \, e^{2\pi i x\cdot(\xi+\eta)} \, d\xi d\eta \\
&+ \int\int \langle \xi + \eta \rangle^m \phi\left(\tfrac{\langle \xi \rangle}{\langle \eta \rangle}\right) \, \hat{f}(\xi)\hat{g}(\eta) \, e^{2\pi i x\cdot(\xi+\eta)} \, d\xi d\eta \\
&= \int\int \frac{\langle \xi + \eta \rangle^m}{\langle \xi \rangle^m} \phi\left(\tfrac{\langle \eta \rangle}{\langle \xi \rangle}\right) \, \langle \xi \rangle^m \hat{f}(\xi) \, \hat{g}(\eta) \, e^{2\pi i x\cdot(\xi+\eta)} \, d\xi d\eta \\
&+ \int\int \frac{\langle \xi + \eta \rangle^m}{\langle \eta \rangle^m} \phi\left(\tfrac{\langle \xi \rangle}{\langle \eta \rangle}\right) \, \hat{f}(\xi) \, \langle \eta \rangle^m \hat{g}(\eta) \, e^{2\pi i x\cdot(\xi+\eta)} \, d\xi d\eta \\
&=: T_{\sigma_1}(J^m f, g)(x) + T_{\sigma_2}(f, J^m g)(x).
\end{aligned}$$

Fractional Leibniz rule by frequency decoupling (cont.)

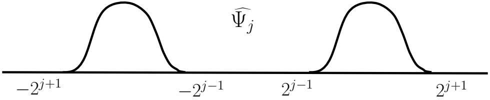
$$J^m(fg)(x) = T_{\sigma_1}(J^m f, g)(x) + T_{\sigma_2}(f, J^m g)(x)$$

- $\sigma_1(\xi,\eta) := \frac{\langle \xi+\eta\rangle^m}{\langle \xi\rangle^m}\phi\left(\frac{\langle\eta\rangle}{\langle\xi\rangle}\right)$ and $\sigma_2(\xi,\eta) := \frac{\langle \xi+\eta\rangle^m}{\langle \eta\rangle^m}\phi\left(\frac{\langle\xi\rangle}{\langle\eta\rangle}\right)$.
- σ_1 and σ_2 belong to $BS^0_{1,0}$ for m large.
- If $\sigma \in BS^0_{1,0}$ then $T_\sigma : L^{p_1} \times L^{p_2} \to L^p$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.
- It follows that

$$\begin{aligned}
\|fg\|_{W^{m,p}} &= \|J^m(fg)\|_{L^p} \le \|T_{\sigma_1}(J^m f, g)\|_{L^p} + \|T_{\sigma_2}(f, J^m g)\|_{L^p} \\
&\lesssim \|J^m f\|_{L^{p_1}}\|g\|_{L^{p_2}} + \|f\|_{L^{p_1}}\|J^m g\|_{L^{p_2}} \\
&= \|f\|_{W^{m,p_1}}\|g\|_{L^{p_2}} + \|f\|_{L^{p_1}}\|g\|_{W^{m,p_2}}, \qquad \text{for } m \text{ large.}
\end{aligned}$$

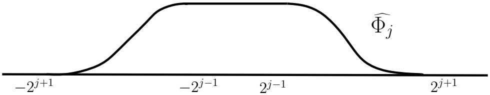
- Interpolate with the case $m = 0$.

$$fg = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \Delta_j(f)\Delta_k(g), \qquad \Delta_j(f) = \Psi_j * f.$$



Paraproduct of f and g:

$$\Pi(f, g) := \sum_{j\in\mathbb{Z}} \sum_{k\leq j} \Delta_j(f)\Delta_k(g) = \sum_{j\in\mathbb{Z}} \Delta_j(f)S_j(g), \qquad S_j(g) = \Phi_j * g.$$



The bilinear symbol of Π

We have

$$\Pi(f,g) = \sum_{j\in\mathbb{Z}} (\Psi_j * f)(\Phi_j * g).$$

Using the Fourier transform

$$\Pi(f,g)(x) = \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^{2n}} \widehat{\Psi_j}(\xi)\widehat{\Phi_j}(\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\, d\xi d\eta.$$

Thus, Π is a bilinear multiplier with symbol

$$\sigma_\Pi(\xi,\eta) = \sum_{j\in\mathbb{Z}} \widehat{\Psi_j}(\xi)\widehat{\Phi_j}(\eta).$$

σ_Π is a Coifman-Meyer multiplier.

Comparison between linear and bilinear results

- Kernels
- Symbolic calculus
- Boundedness on Lebesgue spaces

Linear in $\mathbb{R}^{2n}$ vs. bilinear in $\mathbb{R}^n$

$\sigma(x,\xi,\eta)$, $x,\xi,\eta \in \mathbb{R}^n$, $X = (x_1,x_2) \in \mathbb{R}^{2n}$, $\zeta = (\xi,\eta) \in \mathbb{R}^{2n}$,

$$\Sigma(X,\zeta) = \sigma\left(\frac{x_1+x_2}{2},\xi,\eta\right).$$

$$\sigma \in BS^m_{\rho,\delta}(\mathbb{R}^n) \Longrightarrow \Sigma \in S^m_{\rho,\delta}(\mathbb{R}^{2n})$$

The distributional bilinear kernel $k(x,y,z)$ of the bilinear operator T_σ is given by

$$k(x,y,z) = K((x,x),(y,z)), \quad x,y,z \in \mathbb{R}^n,$$

where $K(X,Y)$ is the distributional linear kernel associated to the linear operator T_Σ.

Blowup of the kernels

- Linear case, $\sigma \in S^m_{\rho,\delta}$:

  $$|\partial_x^\alpha \partial_y^\beta k(x,y)| \lesssim |x-y|^{-(m+|\alpha+\beta|+n)/\rho}, \quad m+|\alpha+\beta|+n>0.$$

- Bilinear case, $\sigma \in BS^m_{\rho,\delta}$:

  $$|\partial_x^\alpha \partial_y^\beta \partial_z^\gamma k(x,y,z)| \lesssim (|x-y|+|x-z|+|y-z|)^{-(m+|\alpha+\beta+\gamma|+2n)/\rho},$$

  $$m+|\alpha+\beta+\gamma|+2n>0.$$

Blowup of the kernels: $m = 0$ and $\rho = 1$

- Linear case, $\sigma \in S^0_{1,\delta}$:

$$|\partial_x^\alpha \partial_y^\beta k(x,y)| \lesssim |x-y|^{-(|\alpha+\beta|+n)}.$$

- Bilinear case, $\sigma \in BS^0_{1,\delta}$:

$$|\partial_x^\alpha \partial_y^\beta \partial_z^\gamma k(x,y,z)| \lesssim (|x-y|+|x-z|+|y-z|)^{-(|\alpha+\beta+\gamma|+2n)}.$$

These are Calderón-Zygmund kernels.

$m = 0,\ \rho = 1,\ 0 \leq \delta < 1$

- $\sigma \in S^0_{1,\delta}, 0 \leq \delta < 1 \Longrightarrow T_\sigma$ is a Calderón-Zygmund operator.

$$T_\sigma : L^p \to L^p, \qquad 1 < p < \infty.$$

- $\sigma \in BS^0_{1,\delta}, 0 \leq \delta < 1 \Longrightarrow T_\sigma$ is a bilinear Calderón-Zygmund operator.

$$T_\sigma : L^{p_1} \times L^{p_2} \to L^p, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2},\ 1 < p_1,\ p_2 < \infty.$$

Bilinear Calderón-Zygmund theory: Christ-Journé, Coifman-Meyer, Grafakos-Torres, Kenig-Stein.

$m = 0,\ \rho = \delta = 1$

- $\sigma \in S^0_{1,1} \implies T_\sigma$ has a C-Z kernel.

  Not every T_σ is bounded on Lebesgue spaces.

- $\sigma \in BS^0_{1,1} \implies T_\sigma$ has a bilinear C-Z kernel.

  Not every T_σ is bounded on Lebesgue spaces. Bényi-Torres '03.

Symbolic calculus

- Linear:

$$\delta < 1,\ \sigma \in S^m_{\rho,\delta} \Rightarrow T^*_\sigma = T_{\sigma^*} \text{ with } \sigma^* \in S^m_{\rho,\delta}$$

(Hörmander '65/'67)

- Bilinear: For $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$,

$$\sigma \in BS^m_{\rho,\delta} \Rightarrow T^{*j}_\sigma = T_{\sigma^{*j}} \text{ with } \sigma^{*j} \in BS^m_{\rho,\delta},\ j = 1, 2$$

where

$$\langle T_\sigma(f,g), h\rangle = \langle T^{*1}_\sigma(h,g), f\rangle = \langle T^{*2}_\sigma(f,h), g\rangle, \quad f,\ g \in \mathcal{C}^\infty_c(\mathbb{R}^n)$$

Bényi-Maldonado-N.-Torres '10 (general case)
Bényi-Torres '03 ($m = \delta = 0,\ \rho = 1$)

Linear Hörmander's classes: L^p-theory for $0 \le \delta \le \rho \le 1$

- Hömander '65, Calderón-Vaillancourt '71, $\delta < 1$,

$$T_\sigma : L^2 \to L^2, \qquad \sigma \in S^0_{\rho,\delta}.$$

- Fefferman-Stein '72, Fefferman '73, $0 \le \delta \le \rho < 1$,

$$T_\sigma : H^1 \to L^1 \quad \text{and} \quad T_\sigma : L^\infty \to BMO, \qquad \sigma \in S^{n(\rho-1)/2}_{\rho,\delta}.$$

- Stein, $0 < \delta = \rho < 1$,

$$T_\sigma : L^1 \to L^{1,\infty}, \qquad \sigma \in S^{n(\rho-1)/2}_{\rho,\delta}.$$

- Calderón-Zygmund theory applies for $S^0_{1,\delta}$, $0 \le \delta < \rho = 1$.

Linear Hörmander's classes: L^p-theory for $0 \leq \delta \leq \rho \leq 1$

Full result for $1 < p < \infty$:

$$T_\sigma : L^p \to L^p, \qquad \sigma \in S^m_{\rho,\delta}, \qquad m \leq n(\rho - 1)|\tfrac{1}{p} - \tfrac{1}{2}|, \; \delta < 1.$$

This result is sharp:

If $m > n(\rho - 1)|\frac{1}{p} - \frac{1}{2}|$ there exists $\sigma \in S^m_{\rho,\delta}$ such that T_σ is unbounded on L^p. Hardy-Littlewood-Hirschman-Wainger.

Bilinear Hörmander classes: symbols of order zero

- Given $1 \leq p_1, p_2, p < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, there exist symbols $\sigma \in BS^0_{0,0}$, even multipliers, such that their associated bilinear pseudo-differential operators do not have the mapping property

$$T_\sigma : L^{p_1} \times L^{p_2} \to L^p.$$

  (Bényi-Torres '04)

- Unboundedness carries on to the classes $BS^0_{\rho,\delta}$ for all $0 < \delta, \rho < 1$ (Bényi-Bernicot-Maldonado-N.-Torres '11)

- $BS^0_{1,\delta}$, $0 < \delta < 1$, give rise to bilinear Calderón-Zygmund operators.

- $BS^0_{1,1}$ gives rise to operators with bilinear Calderón-Zygmund kernels.

Unboundedness for $BS^0_{0,0} \Longrightarrow$ unboundedness for $BS^0_{\rho,\delta}$

Fix $\delta, \rho, p_1, p_2, p$ as above. Suppose, on the contrary, that T_σ is bounded from $L^{p_1} \times L^{p_2}$ into L^p for all $\sigma \in BS^0_{\rho,\delta}$.

Step 1: Show that if $\sigma \in BS^0_{\rho,\delta}$ is x-independent then T_σ satisfies

$$\|T_\sigma(f,g)\|_{L^p} \lesssim \|\sigma\|_{L^\infty}\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}, \quad f, g \in \mathcal{S}. \tag{1}$$

Step 2: Use step 1 and the fact that there are x-independent symbols in $BS^0_{0,0}$ that are not bounded from $L^{p_1} \times L^{p_2}$ into L^p to get a contradiction.

Proof of step 1:

Consider an x-independent symbol $\sigma \in BS^0_{\rho,\delta}$ and, for multi-indices β, γ, set

$$C_{\beta,\gamma}(\sigma) := \sup_{\xi,\eta\in\mathbb{R}^n} |\partial_\xi^\beta \partial_\eta^\gamma \sigma(\xi,\eta)|(1+|\xi|+|\eta|)^{\rho(|\beta|+|\gamma|)}.$$

(Note that $C_{0,0}(\sigma) = \|\sigma\|_{L^\infty}$)

For $\lambda > 0$ define $\sigma_\lambda(\xi,\eta) := \sigma(\lambda\xi,\lambda\eta)$, $\xi,\eta \in \mathbb{R}^n$. Then, for all multi-indices β,γ and $0 < \lambda < 1$, we have

$$C_{\beta,\gamma}(\sigma_\lambda) \le \lambda^{(1-\rho)(|\beta|+|\gamma|)} C_{\beta,\gamma}(\sigma).$$

Unboundedness $BS^0_{0,0} \Longrightarrow$ unboundedness $BS^0_{\rho,\delta}$ (cont.)

Let f, $g \in \mathcal{S}$ and define $f_\lambda(x) := f\left(\frac{x}{\lambda}\right)$ and $g_\lambda(x) := g\left(\frac{x}{\lambda}\right)$, then

$$T_\sigma(f,g)(x) = T_{\sigma_\lambda}(f_\lambda, g_\lambda)(\lambda x).$$

$$\begin{aligned}
\|T_\sigma(f,g)\|_{L^p} &= \|T_{\sigma_\lambda}(f_\lambda, g_\lambda)(\lambda \cdot)\|_{L^p} = \lambda^{-\frac{n}{p}}\|T_{\sigma_\lambda}(f_\lambda, g_\lambda)\|_{L^p} \\
&\lesssim \lambda^{-\frac{n}{p}} \left(\sup_{|\beta|, |\gamma| \le N} C_{\beta,\gamma}(\sigma_\lambda) \right) \|f_\lambda\|_{L^{p_1}} \|g_\lambda\|_{L^{p_2}} \qquad \text{(for some N)} \\
&= \lambda^{-\frac{n}{p} + \frac{n}{p_1} + \frac{n}{p_2}} \left(\sup_{|\beta|, |\gamma| \le N} C_{\beta,\gamma}(\sigma_\lambda) \right) \|f\|_{L^{p_1}} \|g\|_{L^{p_2}} \\
&\lesssim \left(\sup_{|\beta|, |\gamma| \le N} \lambda^{(1-\rho)(|\beta|+|\gamma|)} C_{\beta,\gamma}(\sigma) \right) \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}.
\end{aligned}$$

Letting $\lambda \to 0$,

$$\|T_\sigma(f,g)\|_{L^p} \lesssim C_{0,0}(\sigma)\|f\|_{L^{p_1}}\|g\|_{L^{p_2}} \quad f \in L^{p_1}, g \in L^{p_2},$$

which is (1).

Unboundedness $BS^0_{0,0} \Longrightarrow$ unboundedness $BS^0_{\rho,\delta}$ (cont.)

Proof of Step 2:

- $\tau \in BS^0_{0,0}$ x-independent such that T_τ is not bounded from $L^{p_1} \times L^{p_2}$ into L^p.
- $\varphi \in \mathcal{C}_0^\infty$ supported in $\{(\xi,\eta) : |(\xi,\eta)| \leq 2\}$, $\varphi(0,0) = 1$.
- $\tau_\varepsilon(\xi,\eta) := \varphi(\varepsilon\,\xi, \varepsilon\,\eta)\tau(\xi,\eta)$, $\epsilon > 0$.
- $\tau_\varepsilon \in BS^0_{\rho,\delta}(\mathbb{R}^n)$ and $\|\tau_\varepsilon\|_{L^\infty} \lesssim \|\tau\|_{L^\infty}$ for all $\varepsilon > 0$.

By (1),

$$\|T_{\tau_\varepsilon}(f,g)\|_{L^p} \lesssim \|\tau\|_{L^\infty}\|f\|_{L^{p_1}}\|g\|_{L^{p_2}} \quad f,\, g \in \mathcal{S}, \qquad \text{for all } \varepsilon > 0.$$

As $\varepsilon \to 0$, $T_{\tau_\varepsilon}(f,g) \to T_\tau(f,g)$ pointwise; Fatou Lemma yields

$$\|T_\tau(f,g)\|_{L^p} \lesssim \|\tau\|_{L^\infty}\|f\|_{L^{p_1}}\|g\|_{L^{p_2}} \quad f,\, g \in \mathcal{S},$$

a contradiction.

Theorem 1 (BBMNT'11)

Let $0 \le \delta \le \rho \le 1$, $\delta < 1$, $1 \le p_1, p_2 \le \infty$, *p given by* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$,

$$m(p_1, p_2) := n(\rho - 1)\left(\max\{\tfrac{1}{2}, \tfrac{1}{p_1}, \tfrac{1}{p_2}, 1 - \tfrac{1}{p}\} + \max\{\tfrac{1}{p} - 1, 0\}\right),$$

and $\sigma \in BS^m_{\rho,\delta}$ *with* $m < m(p_1, p_2)$.

(i) *If* $p \ge 1 \ge \rho > 0$, *there exist N such that*

$$\|T_\sigma(f,g)\|_{L^p} \lesssim \|\sigma\|_N \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}, \quad f \in L^{p_1}, g \in L^{p_2}.$$

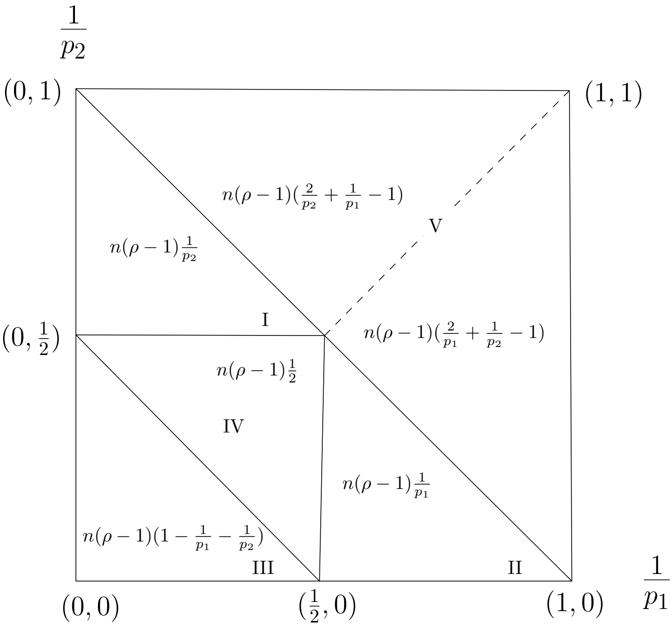
(ii) *If* $0 < \rho$, $p < 1$, $p_1 \ne 1$ *and* $p_2 \ne 1$, *there exist N such that*

$$\|T_\sigma(f,g)\|_{L^p} \lesssim \|\sigma\|_N \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}, \quad f \in L^{p_1}, g \in L^{p_2}.$$

(iii) *If* $0 < \rho$, $p < 1$ *and* $p_1 = 1$ *or* $p_2 = 1$, *there exist N such that*

$$\|T_\sigma(f,g)\|_{L^{p,\infty}} \lesssim \|\sigma\|_N \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}, \quad f \in L^{p_1}, g \in L^{p_2}.$$

Visualizing the values of $m(p_1, p_2)$



About $\|\sigma\|_N$

$$\|\sigma\|_N = \sup_{\substack{|\alpha|\le N \\ |\beta|,|\gamma|\le N}} \sup_{x,\xi,\eta\in\mathbb{R}^n} \frac{|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)|}{(1+|\xi|+|\eta|)^{m+\delta|\alpha|-\rho(|\beta|+|\gamma|)}}.$$

Lemma. *Let $0 < p \le \infty$, $1 \le p_1, p_2 < \infty$, $0 \le \delta$, $\rho \le 1$ and suppose T_σ is bounded from $L^{p_1} \times L^{p_2}$ into L^p for all $\sigma \in BS^m_{\rho,\delta}$. Then there exist $N \in \mathbb{N}_0$ such that*

$$\|T_\sigma\| \lesssim \|\sigma\|_N \quad \textit{for all } \sigma \in BS^m_{\rho,\delta}.$$

$(BS^m_{\rho,\delta}, \|\cdot\|_N)$ is a Banach space

Theorem 1: Remarks and a sketch of the proof

- Improves previous result (Michalowski-Rule-Staubach '11):

  $m < \tilde{m}(p_1, p_2) := n(\rho-1)\max\{\frac{1}{2}, (\frac{2}{p_1}-\frac{1}{2}), (\frac{2}{p_2}-\frac{1}{2}), (\frac{3}{2}-\frac{2}{p})\}, \;\; p \geq 1.$

$$m(p_1, p_2) - \tilde{m}(p_1, p_2) = \begin{cases} n(\rho-1)(\frac{1}{p_2}-\frac{1}{2}) & \text{region I} \\ n(\rho-1)(\frac{1}{p_1}-\frac{1}{2}) & \text{region II} \\ n(\rho-1)(\frac{1}{2}-\frac{1}{p}) & \text{region III} \\ 0 & \text{region IV} \end{cases}$$

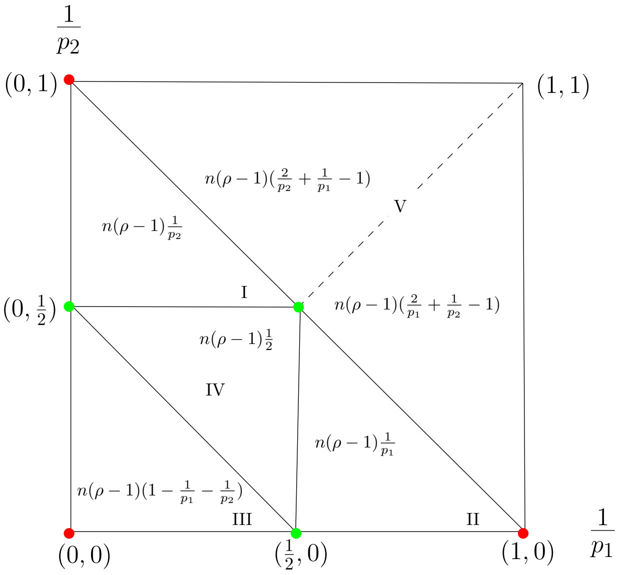
- Main idea of the proof: Based on previous lemma look at $T_\sigma(f,g)$ as a trilinear operator $T(\sigma, f, g)$ defined on $BS^m_{\rho,\delta} \times L^{p_1} \times L^{p_2}$ and use complex interpolation:

  - If $m_0, m_1 \in \mathbb{R}$, $0 \leq \rho < 1$ and $m = (1-\theta)\,m_0 + \theta\,m_1$ for some $\theta \in (0,1)$ then

$$\left(BS^{m_0}_{\rho,\rho}, BS^{m_1}_{\rho,\rho}\right)_{[\theta]} = BS^m_{\rho,\rho}.$$

A sketch of the proof: $p \geq 1$

Prove first the boundedness corresponding to red and green dots:



A sketch of the proof: $p \geq 1$ (cont.)

For red dots $m(p_1, p_2) = n(\rho - 1)$ and if $m < n(\rho - 1)$,

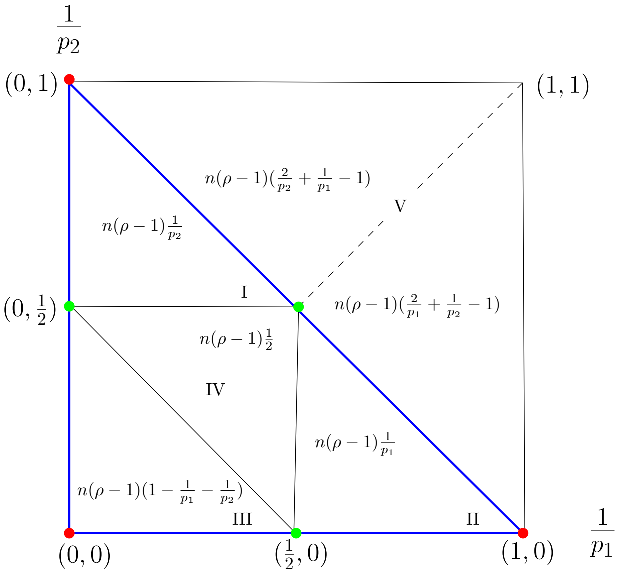
- $T : BS^m_{\rho,\rho} \times L^\infty \times L^\infty \to L^\infty$
- $T : BS^m_{\rho,\rho} \times L^1 \times L^\infty \to L^1$ (duality and symbolic calculus)
- $T : BS^m_{\rho,\rho} \times L^\infty \times L^1 \to L^1$ (duality and symbolic calculus)

For green dots $m(p_1, p_2) = \frac{n}{2}(\rho - 1)$ and if $m < \frac{n}{2}(\rho - 1)$,

- $T : BS^m_{\rho,\rho} \times L^2 \times L^2 \to L^1$ (Michalowski-Rule-Staubach '11)
- $T : BS^m_{\rho,\rho} \times L^2 \times L^\infty \to L^2$ (duality and symbolic calculus)
- $T : BS^m_{\rho,\rho} \times L^\infty \times L^2 \to L^2$ (duality and symbolic calculus)

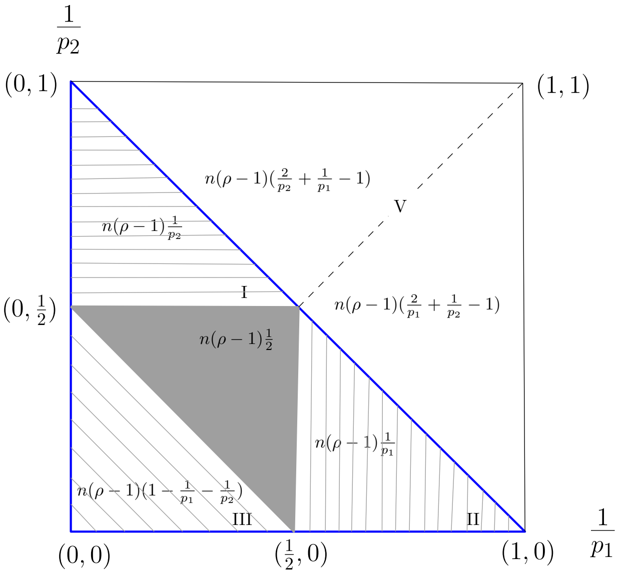
A sketch of the proof: $p \geq 1$ (cont.)

Use trilinear interpolation:



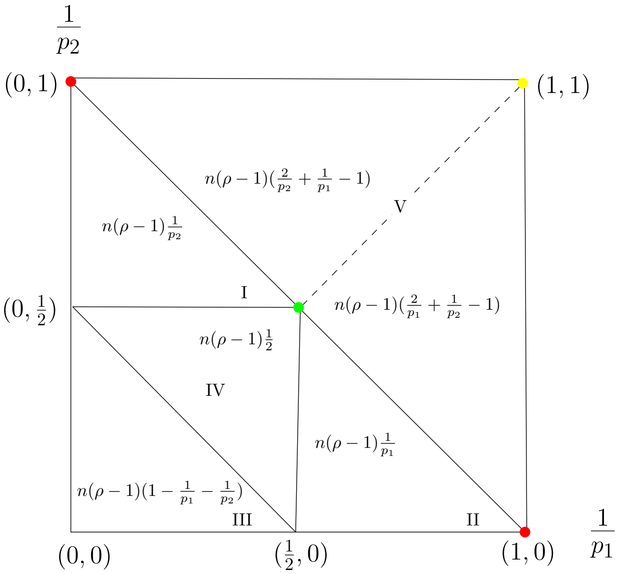
A sketch of the proof: $p \geq 1$ (cont.)

Use bilinear interpolation:



A sketch of the proof: $p < 1$

Needs the following endpoints:



A sketch of the proof: $p < 1$ (cont.)

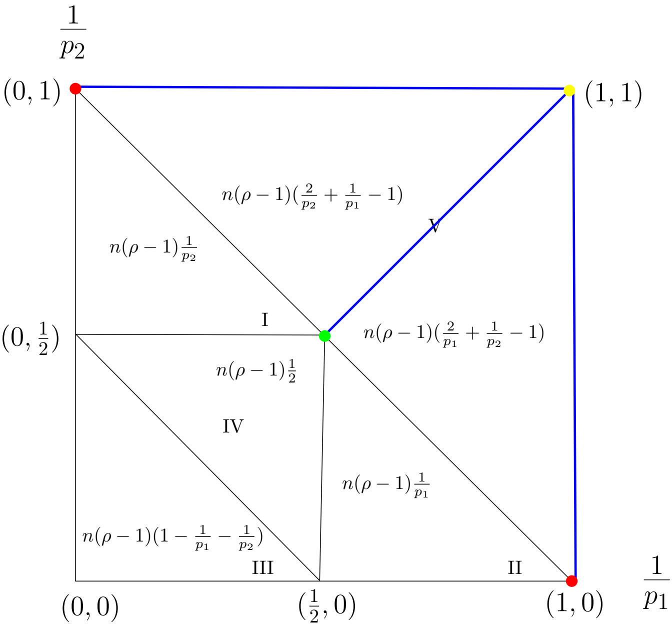
- $T : BS^m_{\rho,\rho} \times L^1 \times L^1 \to L^{\frac{1}{2},\infty}$ for $m < 2n(\rho - 1)$ (CZ theory)
- $T : BS^m_{\rho,\rho} \times L^1 \times L^\infty \to L^1$ for $m < n(\rho - 1)$
- $T : BS^m_{\rho,\rho} \times L^\infty \times L^1 \to L^1$ for $m < n(\rho - 1)$
- $T : BS^m_{\rho,\rho} \times L^2 \times L^2 \to L^1$ for $m < \frac{n}{2}(\rho - 1)$

By trilinear complex interpolation,

$$T : BS^m_{\rho,\rho} \times L^{p_1} \times L^{p_2} \to L^{p,\infty}, \quad m < m(p_1, p_2),$$

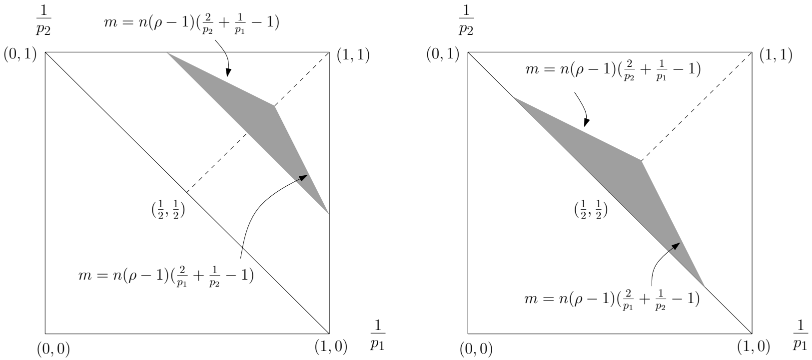
for $(\frac{1}{p_1}, \frac{1}{p_2})$ on the segments joining the yellow dot to the other three dots.

A sketch of the proof: $p < 1$ (cont.)



A sketch of the proof: $p < 1$ (cont.)

For $p_1 \neq 1$ and $p_2 \neq 1$, use bilinear real interpolation:



An endpoint result

When $p_1 = p_2 = \infty$ the previous stated result gives in particular:

if $m < n(\rho - 1)$ and $\sigma \in BS^m_{\rho,\delta}$, $0 \le \delta \le \rho \le 1$, then

$$\|T_\sigma(f,g)\|_{L^\infty} \lesssim \|\sigma\|_N \|f\|_{L^\infty} \|g\|_{L^\infty}, \quad f,\, g \in \mathcal{S}.$$

Theorem 2 (BBMNT '11)

If $\sigma \in BS^{n(\rho-1)}_{\rho,0}$, $0 \le \rho < \frac{1}{2}$, *then*

$$\|T_\sigma(f,g)\|_{BMO} \lesssim \|\sigma\|_N \|f\|_{L^\infty} \|g\|_{L^\infty}, \quad f,\, g \in \mathcal{S}.$$

Some comments on the proof of Theorem 2

We have to estimate $\frac{1}{|Q|}\int_Q |T_\sigma(f,g)(x) - T_\sigma(f,g)_Q|\,dx$, Q cube.
Let Q be a cube of diameter d.

$$T_\sigma(f,g)(x) = T_{\sigma_1}(f,g)(x) + T_{\sigma_2}(\phi f,\phi g)(x) + R(f,g)(x), \quad x \in Q,$$

where

$$\mathrm{supp}(\sigma_1) \subset \{(x,\xi,\eta) : |\xi| + |\eta| \le 2/d\}$$
$$\mathrm{supp}(\sigma_2) \subset \{(x,\xi,\eta) : |\xi| + |\eta| \ge 1/d\}$$

$$R(f,g)(x) = \phi^2(x)\,T_{\sigma_2}(f,g)(x) - T_{\sigma_2}(\phi f,\phi g)(x)$$

$$\phi \in \mathcal{S}, \quad \phi \ge 0, \quad \mathrm{supp}(\hat{\phi}) \subset \{z \in \mathbb{R}^n : |z| \le d^{-\rho}/8\}.$$
$$\phi \equiv 1 \text{ on } Q, \qquad \|\phi\|_{L^2} \lesssim d^{\frac{n\rho}{2}}.$$

Some comments on the proof of Theorem 2 (cont.)

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |T_\sigma(f,g)(x) - T_\sigma(f,g)_Q|\,dx \\
&\qquad \leq \frac{1}{|Q|}\int_Q |T_{\sigma_1}(f,g)(x) - T_{\sigma_1}(f,g)_Q|\,dx \\
&\qquad + \frac{2}{|Q|}\|T_{\sigma_2}(\phi f,\phi g)\|_{L^1(Q)} + 2\|R(f,g)\|_{L^\infty} \\
&\qquad \lesssim (\|\sigma_1\|_N + \|\sigma_2\|_N + \|\sigma_2\|_N)\,\|f\|_{L^\infty}\|g\|_{L^\infty}, \quad \text{(for some N)}
\end{aligned}$$

The last inequality requires some technical work. What follows is a rough idea for the estimation of the term corresponding to T_{σ_2}.

Some comments on the proof of Theorem 2 (cont.)

The inequality for T_{σ_2} is implied by

$$\|T_{\sigma_2}(\phi f, \phi g)\|_{L^2} \lesssim \|\sigma_2\|_N \, d^{\frac{n}{2}} \|f\|_{L^\infty} \|g\|_{L^\infty}, \quad f, g \in \mathcal{S}, \qquad (2)$$

since

$$\frac{1}{|Q|} \|T_{\sigma_2}(\phi f, \phi g)\|_{L^1(Q)} \leq \frac{1}{|Q|^{1/2}} \|T_{\sigma_2}(\phi f, \phi g)\|_{L^2(Q)}$$

Moreover, because ϕ satisfies $\|\phi\|_{L^2} \lesssim d^{\frac{\rho n}{2}}$, (2) can be reduced to proving that

$$\|T_{\sigma_2}\|_{L^2 \times L^2 \to L^2} \lesssim \|\sigma_2\|_N \, d^{\frac{n}{2} - \rho n}. \qquad (3)$$

Some comments on the proof of Theorem 2 (cont.)

Theorem 3 (BBMNT '11)

If $\sigma(x, \xi, \eta)$, $x, \xi, \eta \in \mathbb{R}^n$, *is a bilinear symbol such that*

$$C(\sigma) := \sup_{\substack{|\beta| \le [\frac{n}{2}]+1 \\ |\alpha| \le 2(2n+1)}} \sup_{\xi, y \in \mathbb{R}^n} \|\partial_\xi^\alpha \partial_y^\beta \sigma(y, \xi - \cdot, \cdot)\|_{L^2} < \infty,$$

then T_σ *maps continuously* $L^2 \times L^2$ *into* L^2 *with*

$$\|T_\sigma\|_{L^2 \times L^2 \to L^2} \lesssim C(\sigma).$$

Some comments on the proof of Theorem 2 (cont.)

By Theorem 3, the support of σ_2, and the fact that $\sigma_2 \in BS^m_{\rho,0}$ with $m = n(\rho - 1)$ and $0 < \rho < \frac{1}{2}$,

$$\|T_{\sigma_2}\|_{L^2\times L^2\to L^2} \lesssim \sup_{\substack{|\beta|\le[\frac{n}{2}]+1 \\ |\alpha|\le 2(2n+1)}} \sup_{y,\xi\in\mathbb{R}^n} \|\partial_\xi^\alpha \partial_y^\beta \sigma_2(y,\xi-\cdot,\cdot)\|_{L^2}$$

$$\lesssim \|\sigma_2\|_{K,M} \sup_{\xi\in\mathbb{R}^n} \|\chi_{\{|\xi-\eta|+|\eta|\ge d^{-1}\}}(\xi,\eta)\,(1+|\xi-\eta|+|\eta|)^m\,\|_{L^2(d\eta)}$$

$$\lesssim \|\sigma_2\|_{K,M} \left[\left(\int_{|\eta|\ge d^{-1}} |\eta|^{2m} d\eta\right)^{1/2} + \left(\int_{|\eta|\le d^{-1}} d^{-2m} d\eta\right)^{1/2}\right]$$

$$\lesssim \|\sigma_2\|_{K,M}\, d^{-m-\frac{n}{2}} = \|\sigma_2\|_{K,M}\, d^{\frac{n}{2}-\rho n},$$

where we have taken $K = [\frac{n}{2}] + 1$ and $M = 2(2n+1)$.

An account of linear and bilinear results

<table>
<tr><th></th><th>L^2- theory
$m = 0,\ 0 \le \delta \le \rho \le 1$</th><th>Symbolic Calculus
$m \in \mathbb{R},\ 0 \le \delta \le \rho \le 1$</th><th>$L^p$- theory
$m \in \mathbb{R},\ 0 \le \delta \le \rho \le 1$</th><th>$L^p$- theory
Multipliers</th></tr>
<tr><td>Linear</td><td>$\delta < \rho$, Hörmander '67
$\delta = \rho$, $\delta < 1$ Calderón-Vaillancourt '71</td><td>Kohn-Nirenberg '65
Hörmander '65 '67</td><td>Wainger '65
Fefferman-Stein '72
Fefferman '73</td><td>Mihlin '56</td></tr>
<tr><td rowspan="4">Bilinear</td><td>$\delta = \rho = 0$ fails
Bényi-Torres '04</td><td rowspan="2">$m = \delta = 0, \rho = 1$
Bényi-Torres '03</td><td rowspan="4">$m < m(p_1, p_2)$
$0 < \rho < 1$
BBMNT '11</td><td rowspan="4">Coifman-Meyer
Grafakos-Torres '02
C-Z theory*</td></tr>
<tr><td rowspan="2">$0 \le \delta < \rho = 1$
C-Z theory*</td></tr>
<tr><td rowspan="2">$m \in \mathbb{R},\ 0 \le \delta \le \rho \le 1$
BMNT '10</td></tr>
<tr><td>$0 < \rho < 1$ fails
BBMNT '11</td></tr>
</table>

*Bilinear C-Z theory: Christ-Journé, Coifman-Meyer, Grafakos-Torres, Kenig-Stein

BMNT=Bényi-Maldonado-N.-Torres BBMNT=Bernicot + BMNT

References

- A. Bényi, D. Maldonado, V. Naibo and R.H. Torres, *On the Hörmander classes of bilinear pseudodifferential operators*, Integral Equations and Operator Theory, **67** (3), (2010), 341–364.
- A. Bényi, F. Bernicot, D. Maldonado, V. Naibo and R.H. Torres, *On the Hörmander classes of bilinear pseudodifferential operators II*, preprint.

http://www.math.ksu.edu/∼vnaibo/research.html