Parallel Sections and Related Problems in Convex Geometry

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Definitions and notation

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A body K is convex if given any two points $P, Q \in K$, the segment $\overline{PQ}$ is contained in K.

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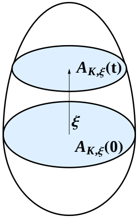

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General problem

Reconstruct K from information about the sections of K

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Proof.

A Fourier Analytical proof by Koldobsky is based on the fact that

$$A_{K,\xi}(0) = vol_{n-1}(K \cap \xi^{\perp}) = \frac{1}{\pi(n-1)}(\| \cdot \|_K^{-n+1})^{\wedge}(\xi).$$

□

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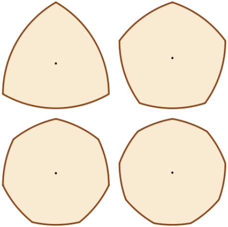

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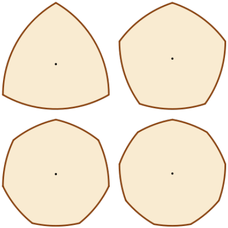

- The answer is also *no* in higher dimensions:
  For problem 1, Nazarov, Ryabogin, Zvavitch (2012).
  For problem 2, Gardner, Ryabogin, Yaskin, Zvavitch (2011).

The Busseman-Petty Problem (1956)

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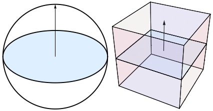

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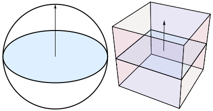

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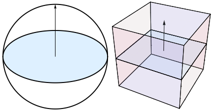

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- Gardner, Koldobsky, Schlumprecht (1999), analytic proof in all dimensions: Yes if $n = 2, 3, 4$; No for $n \geq 5$

The t-section problem

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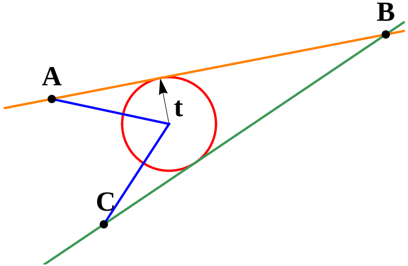
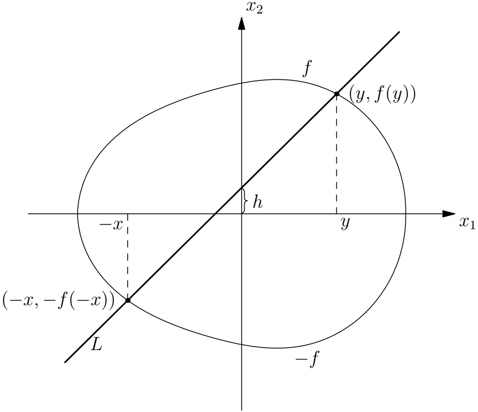


The t-section problem for bodies of revolution

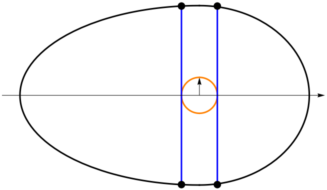
Nazarov, Ryabogin, Zvavitch (2012) prove the following formula for sections of bodies of revolution:

$$vol_{n-1}(K \cap H(L)) = \kappa_{n-2}\sqrt{1+s^2}\int_{-x}^{y}\left(f^2(x_1) - L^2(t,s,x_1)\right)^{(n-2)/2}\,dx_1.$$



Dimension 4

Let $t > 0$ and let K be a 4-dimensional body of revolution containing the ball of radius t centered at the origin.
We assume that all 3-dimensional t-sections of K have constant volume.



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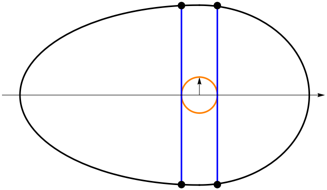

By dilating K, we may assume $(t, \sqrt{1-t^2}) \in \partial K$ (and then the point $(-t, \sqrt{1-t^2})$ must also be on ∂K).

Dimension 4

In dimension 4, the formula for sections becomes

$$vol_3(K \cap H(L)) = \pi \sqrt{1+s^2} \int_{-x}^{y} \left(f^2(x_1) - L^2(s,t,x_1) \right) \, dx_1$$

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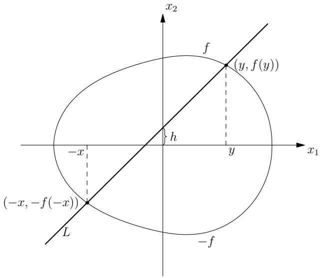

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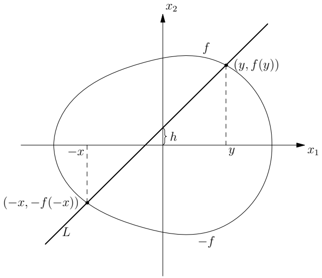

Differentiating with respect to s, we obtain a cubic equation in terms of x and y, that does not depend on f.

Iteration

$$(y^3 + x^3) \pm \frac{3t(1+2s^2)}{2s\sqrt{1+s^2}}\left(y^2 - x^2\right) + 3t^2(y+x) = 2\left(\frac{1-t^2}{1+s^2}\right)^{3/2}.$$

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- Given a point $(y, f(y)) \in \partial K$, we can solve the equation and find $(-x, -f(-x)) \in \partial K$. Then we can iterate, starting now at $(-x, -f(-x))$.

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- By assumption, ∂K contains four points that are on the unit sphere.

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- By assumption, ∂K contains four points that are on the unit sphere. If $\arccos(t)$ is an irrational multiple of π, the iteration will give us a dense set of points, both on ∂K and on the unit sphere, and hence $K = S^3$.

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Thank you!