The Cauchy Integral in $\mathbb{C}^n$

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Credits

- E. M. Stein

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Overview.

- Develop L^p-theory of Singular Integrals on

$$D \Subset \mathbb{C}^n$$

  - for D with **minimal** **boundary regularity**
  - modeled after **Cauchy Integral for a Lipschitz Curve** $\Gamma \subset \mathbb{C}$:

$$\mathbf{H}f(z) = \frac{1}{2\pi i}\int\limits_{w\in\Gamma} f(w)\frac{dw}{w-z}, \qquad z \in D$$

    $z \in bD$: Calderon (1978); Coifman-McIntosh-Meyer (1980s); David (1980s); David-Semmes (1990s); Semmes (1990s).

    Also: Jones; Wolfe; Melnikov-Verdera; Nazarov-Volberg-Treil; Tolsa;....

- Focus on Singular Integrals with **Holomorphic** kernel
- Applications to **Complex Function theory for** $D \Subset \mathbb{C}^n$

Complex Function Theory: Objects

$$D \Subset \mathbb{C}^n = \{z = (z_1, \ldots, z_n)\,,\ z_j = x_j + iy_j\}, \qquad n \geq 1$$

$$\vartheta(D) := \left\{ F : D \to \mathbb{C},\ \ \frac{\partial F}{\partial \bar{z}_j} := \left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)F = 0\,,\ j = 1, \ldots, n\right\}$$

$$1 \leq p < +\infty$$

- Bergman Space

$$\vartheta L^p(D) := \left\{ F \in \vartheta(D),\ \ \int\limits_{w \in D} |F(w)|^p dV(w) < +\infty \right\}$$

- D rectifiable: Hardy Space (aka *Smirnov Class*)

$$H^p(bD) := \left\{ F \in \vartheta(D),\ \ \sup_{\epsilon > 0} \int\limits_{w \in bD_\epsilon} |F(w)|^p d\sigma_\epsilon(w) < +\infty \right\}$$

Lemma

The Bergman Space

$$\vartheta L^p(D) = \left\{ F \in \vartheta(D), \quad \int\limits_{w \in D} |F(w)|^p dV(w) < +\infty \right\}$$

is a closed subspace of $L^p(D, dV)$.

Proof.

Cauchy formula on (poly)disc:

for any compact subset $\mathcal{K} \subset D$, for any $F \in \vartheta(D)$, for any $z \in \mathcal{K}$:

$$|F(z)| \ \leq \ C(\mathcal{K}) \, \|F\|_{L^1(\mathbb{P}(z,\delta))} \ \leq \ C(\mathcal{K}) \, \|F\|_{L^p(D)}$$

$$\delta := \frac{1}{2} \operatorname{dist}(\mathcal{K}, bD)$$

Lemma

The Hardy Space

$$H^p(bD) := \left\{ F \in \vartheta(D),\ \sup_{\epsilon>0} \int_{w\in bD_\epsilon} |F(w)|^p d\sigma_\epsilon(w) < +\infty \right\}$$

is a closed subspace of $L^p(bD, \sigma)$*.*

Proof.

Cauchy formula on polydisc + Co-Area Formula:

$$|F(z)| \ \leq\ C(\mathcal{K})\,\|F\|_{L^p(\mathbb{P}(z,\delta))} \ \leq C(\mathcal{K})\,\|F\|_{L^p(D_\epsilon)}$$
$$\leq\ \widetilde{C}(\mathcal{K}) \sup_{\epsilon>0} \left(\int_{w\in bD_\epsilon} |F(w)|^p d\sigma_\epsilon(w) \right)^{1/p}$$

for any compact subset $\mathcal{K} \subset D$, for any $F \in \vartheta(D)$, for any $z \in \mathcal{K}$. □

Complex Function Theory for $D \Subset \mathbb{C}^n$: Objects

$$D \Subset \mathbb{C}^n, \quad n \geq 1; \qquad p = 2$$

- Bergman Projection:

  $$\mathbf{B} : L^2(D, dV) \Rsh \vartheta L^2(D), \quad \mathbf{B}^2 = \mathbf{B}, \quad \mathbf{B}^* = \mathbf{B}, \quad \|\mathbf{B}\|_{L^2 \to L^2} = 1$$

- Szegő Projection:

  $$\mathbf{S} : L^2(bD, \sigma) \Rsh H^2(bD), \quad \mathbf{S}^2 = \mathbf{S}, \quad \mathbf{S}^* = \mathbf{S}, \quad \|\mathbf{S}\|_{L^2 \to L^2} = 1$$

given $D \Subset \mathbb{C}^n, \quad n \geq 1$

L^q-Regularity problem for Szegő Projection:

find (largest) $Q = Q(D) \in [2, +\infty]$ such that

S : $L^q(bD, \sigma) \to L^q(bD, \sigma)$ is **bounded** for $Q' < q < Q$

L^p-Regularity problem for Bergman Projection:

find (largest) $P = P(D) \in [2, +\infty]$ such that

B : $L^p(D, dV) \to L^p(D, dV)$ is **bounded for** $P' < p < P$

Motivation

- **Hörmander; Kohn**: The Bergman Projection is closely related to the canonical solution of

$$\bar{\partial} u = f$$

- **Bell** (n=1): The Szegő projection is closely related to the solution of

$$\Delta u = 0, \quad u\Big|_{bD} = f$$

- **Size of each of maximal (P', P) and (Q', Q) appears to be related to geometry and regularity of ambient domain D**

Focus on Bergman:

L^p-Regularity problem for

$$\mathbf{B} : L^2(D, dV) \to \vartheta L^2(D, dV), \quad \mathbf{B}^2 = \mathbf{B}, \quad \mathbf{B}^* = \mathbf{B}, \quad \|\mathbf{B}\|_{L^2 \to L^2} = 1$$

History $\quad D \Subset \mathbb{C} \quad (n = 1)$

$n = 1 \qquad D \Subset \mathbb{C}$ simply connected

- L.L. – Stein (2004):
  - If D is of class C^1, then $\quad P = +\infty$

    (Also true for Vanishing Chord-Arc)

("VCA" $\iff \sigma(W, Z) = (1 + o(1))|W - Z|, \quad W,\ Z \in bD$)

  - If D Lipschitz with constant M, then

$$P \geq 2\left(1 + \frac{\pi}{2 \arctan M}\right) > \ 4$$

  - If D rectifiable local graph, then $P \geq 4$

- Hedenmalm (2002): $P = 2 + \epsilon(D)$ for any D

Main tools in $\mathbb{C}$ (n=1)

$D \subset \mathbb{C}$ simply connected

- $D = \{\rho(w) < 0\} \in C^1$:

  Solid Cauchy: $$\mathbf{H}f(z) = \frac{1}{2\pi i}\int_{w\in D} f(w)\frac{\overline{\partial}\rho(w)\wedge dw}{[w - z - \rho(w)]^2} \quad z \in D$$

- $D \in \{$ Lipschitz; Rectifiable graph; "any"$\}$:

  Conformal Map: $\varphi : \mathbb{D}_1(0) \to D$

Obstacles in $\mathbb{C}^n$ $n \geq 2$

Example: a dimension-induced phenomenon:

"Every connected open set $S \subset X$ is convex"

- True for $X = \mathbb{R}$
- False for $X = \mathbb{R}^N$, $N \geq 2$

The L^p-theory for the Bergman projection for $D \Subset \mathbb{C}^n$, $n \geq 2$ is much less developed than corresponding theory for $D \Subset \mathbb{C}$ due to

- Dimension-induced obstructions ($\mathbb{C}^n$ vs. $\mathbb{C}$)
- Complex-Structure-induced obstructions ($\mathbb{C}^n$ vs. $\mathbb{R}^{2n}$)

These obstructions ultimately lead to the requirement that

$D \Subset \mathbb{C}^n$ be "**pseudoconvex**"

(Note: every $D \Subset \mathbb{C}$ is pseudoconvex)

Definition:

$D = \{\lambda(w) < 0\} \Subset \mathbb{C}^n$ is **Strongly Levi-Pseudoconvex** iff D is of class C^2, i.e. $\lambda \in C^2(\mathbb{C}^n)$ and

$$\sum_{j,k=1}^{n} \frac{\partial^2 \lambda}{\partial \zeta_j \partial \overline{\zeta}_k}(\zeta)\, v_j \overline{v_k} \geq c_0 |v|^2, \quad \zeta \in bD, \quad v \in T_\zeta^{\mathbb{C}}(bD)$$

for any defining function λ. Here

$$T_\zeta^{\mathbb{C}}(bD) = T_\zeta^{\mathbb{R}}(bD) \cap i\, T_\zeta^{\mathbb{R}}(bD)$$

Fact: If D is strongly Levi-pscvx then there is a special defining function ρ that is **strictly plurisubharmonic in D** i.e.

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial \zeta_j \partial \overline{\zeta}_k}(\zeta)\, v_j \overline{v_k} \geq c_0 |v|^2, \quad \zeta \in bD, \quad v \in \mathbb{C}^n$$

Example: the Siegel Upper Half Space:

$$D := \{z = (z_1, z_2) \in \mathbb{C}^2 \mid \operatorname{Im} z_2 < |z_1|^2\}$$

is strongly Levi-pseudoconvex but is not strongly convex because *bD* contains the real line

$$\{(0, x_2 + i0),\ x_2 \in \mathbb{R}\}$$

(Some) History $D \Subset \mathbb{C}^n, \quad n \geq 2$

Bekolle'-Bonami; Bell; Bonami-Lohoue'; Charpentier-DuPain; Fefferman; Halfpap-Nagel-Wainger; Krantz-Peloso; McNeal; McNeal-Stein; Nagel-Pramanik; Nagel-Rosay-Stein-Wainger; Phong-Stein.....

- Ligocka (1982): If is D strongly Levi-pseudoconvex and $D \in C^3$, then:

$$\mathbf{B} \quad \text{is bounded}: \ L^p \to L^p \text{ for } 1 < p < +\infty$$

- Zeytuncu (2011): there is $D_0 \Subset \mathbb{C}^2$ such that

$$\mathbf{B} \quad \text{is bounded}: \ L^p \to L^p \ \text{ only for } p = 2$$

(!!! see Hedenmalm $(n = 1)$!!!)

Focus on Ligocka (1982):

If is D strongly Levi-pseudoconvex and $D \in C^3$, then

$$\mathbf{B} \quad \text{is bounded : } \ L^p \to L^p \text{ for } 1 < p < +\infty$$

Ligocka's Strategy for $\mathbf{B}$, $\quad n \geq 1$

Compare B with suitable Cauchy-Fantappie' Integral H:

$$\mathbf{B} : L^2(D, dV) \ \Rsh \ \vartheta L^2(D), \qquad \mathbf{B}^2 = \mathbf{B}, \quad \mathbf{B}^* = \mathbf{B}$$

$$\mathbf{H} : L^2(D, dV) \to \vartheta L^2(D), \qquad \mathbf{H}^2 = \mathbf{H}, \quad (\mathbf{H}^* \neq \mathbf{H}):$$

$$\mathbf{BH} = \mathbf{H}$$

$$\mathbf{HB} = \mathbf{B} \quad \Rightarrow \quad \mathbf{BH}^* = \mathbf{B}$$

$$\mathbf{H} = \mathbf{B}\,[1 - (\mathbf{H}^* - \mathbf{H})]$$

$$\mathbf{A} := \mathbf{H}^* - \mathbf{H}$$

Goal:

1. Prove $\mathbf{H}$ bounded: $L^p(D, dV) \to L^p(D, dV)$, $1 < p < \infty$
2. Invert $\quad 1 - \mathbf{A}$ in $L^p(D, dV)$

- conclude $\mathbf{B} = \mathbf{H}\,(1 - \mathbf{A})^{-1}$ is bounded $L^p \to L^p$

Note: this strategy requires **H** with **holomorphic** kernel

Obstacles in $\mathbb{C}^n$, $n \geq 2$.

- "Canonical" Kernel for $D = \{\rho(w) < 0\}$:

$$H(w,z) = \left(\overline{\partial}_w \left(\sum_{j=1}^{n} \frac{(\overline{w}_j - \overline{z}_j)}{|w-z|^2 - \rho(w)} dw_j\right)\right)^n$$

Bochner-Martinelli-Ligocka kernel

- $n = 1$: $H(w,z) \ = \ \overline{\partial}_w \left(\dfrac{dw}{w - z - \rho(w)}\right)$ **Cauchy-Ligocka**
- $n \geq 2$: **B-M-L not holomorphic w.r.t.** $z \in D$**:**

need ad-hoc H

Classical Approach to implementing Ligocka's Strategy

$n \geq 2 \quad D \Subset \mathbb{C}^n$

- **Find ad-hoc** $\mathbf{H} : L^2(D, dV) \to \vartheta L^2(D), \quad \mathbf{H}^2 = \mathbf{H}$
- Show: $\mathbf{H}$ bounded $L^p(D, dV) \to L^p(D, dV), \quad 1 < p < \infty$
- Show: $\mathbf{A} := \mathbf{H}^* - \mathbf{H}$ **compact** in $L^2(D, dV)$;
- Obtain: $\mathbf{B} = \mathbf{H}(1 - \mathbf{A})^{-1}$ in $L^2(D, dV)$
- Conclude: $\mathbf{B} : L^p(D, dV) \to L^p(D, dV), \quad 1 < p < \infty$

Ligocka: $D \Subset \mathbb{C}^n$, strongly ψ-cvx, $D \in C^3$

Step 1: Construct ad-hoc **H**: an integral operator with kernel

$$H(w,z) := H_1(w,z) + H_2(w,z) \in \Lambda^{(w)}_{n,n}(D) \times \Lambda^{(z)}_{0,0}(D)$$

- $H_1(w,z)$ = **locally** holomorphic w.r.t. z
- $H_2(w,z) \in C(\overline{D} \times \overline{D})$ a **correction term** s.t.

$$\overline{\partial}_z (H_1(w,z) + H_2(w,z)) = 0, \quad (w,z) \in D \times U(\overline{D})$$

($\mathbf{H}_2 : L^p \to L^p$; $\mathbf{H}_2^* - \mathbf{H}_2$ compact on L^2 ... – so **ignore**)

Step 1: Construction of $H_1(w, z)$:

Let

- $D = \{\rho(w) < 0\}, \quad \rho \in C^3(\mathbb{C}^n)$ **strictly plurisubharmonic**.
- $\mathcal{P}_w(z) :=$ Levi polynomial of ρ:

$$\mathcal{P}_w(z) := \sum_j \frac{\partial \rho(w)}{\partial \zeta_j}(w_j - z_j) - \frac{1}{2}\sum_{j,k} \frac{\partial^2 \rho(w)}{\partial \zeta_j \partial \zeta_k}(w_j - z_j)(w_k - z_k)$$

- $\chi(w, z) = \chi(|w - z|)$ smooth cutoff on $\{|w - z| < \mu\}$
- $g(w, z) := \mathcal{P}_w(z)\chi(w, z) + |w - z|^2(1 - \chi(w, z)) - \rho(w)$

If

$$g(w,z) := \mathcal{P}_w(z)\chi(w,z) + |w-z|^2(1-\chi(w,z)) - \rho(w)$$

Then

- **basic inequality**:

$$2\mathrm{Re}\, g(w,z) \geq \begin{cases} -\rho(w) - \rho(z) + c|w-z|^2, & \text{if} \quad |w-z| < \mu \\ c > 0, & \text{if} \quad |w-z| \geq \mu \end{cases}$$

- **size estimate**:

$$|g(w,z)| \approx |\rho(w)| + |\rho(z)| + |\mathrm{Im}\langle \partial\rho(w), w-z\rangle| + |w-z|^2$$

- **symmetry estimate**: $|g(w,z)| \approx |g(z,w)|$
- **cancellation**: $|g(w,z) - \overline{g(z,w)}| \lesssim |w-z|^3$

Step 1: Construction of $H_1(w, z)$:

- $H_1(w,z) := \left[\overline{\partial}_w \left(\frac{\eta(w,z)}{g(w,z)}\right)\right]^n$

- $\eta(w,z) :=$

$$\sum_j \left[\left(\frac{\partial\rho(w)}{\partial\zeta_j} - \frac{1}{2}\sum_k \frac{\partial^2\rho(w)}{\partial\zeta_j\partial\zeta_k}(w_k - z_k)\right)\chi + (\overline{w}_j - \overline{z}_j)(1-\chi)\right] dw_j$$

$$\mathbf{H}_1 f(z) := \int\limits_{w\in D} f(w) \left[\overline{\partial}_w \left(\frac{\eta(w,z)}{g(w,z)}\right)\right]^n$$

Step 2: show $\mathbf{H}_1$ bounded: $L^p(D, dV) \to L^p(D, dV)$

- Comparison operator:

$$\Gamma(f)(z) = \int_D |g(w, z)|^{-n-1} f(w)\, dV(w), \quad z \in D$$

- Theorem: For $1 < p < \infty$, we have

$$\|\Gamma(f)\|_{L^p(D)} \leq c_p \|f\|_{L^p(D)}$$

Proof: size estimates and symmetry estimates for g. □

- Theorem: $\mathbf{H}_1$ is bounded: $L^p \to L^p$, $1 < p < \infty$

Proof: $|\mathbf{H}_1 f(z)| \lesssim \Gamma(|f|)(z), \quad z \in D$ □

Ligocka: $D \Subset \mathbb{C}^n$, strongly ψ-cvx, $D \in C^3$

Step 3: show $\mathbf{A}_1 := \mathbf{H}_1^* - \mathbf{H}_1$ compact in $L^2(D, dV)$.

- Proof: Cancellation: If $|w - z| \leq \mu$ then

$$|A_1(w,z)| = |\overline{H_1(z,w)} - H_1(w,z)| \lesssim \frac{|w-z|}{|g(w,z)|^{n+1}}$$

- Size estimates:

$$\frac{|w-z|}{|g(w,z)|^{n+1}} \lesssim \frac{1}{|g(w,z)|^{n+1-1/2}}$$

- $$\mathbf{A}_{1,\lambda} f(z) := \int\limits_{|g(w,z)|>\lambda} \frac{f(w)}{|g(w,z)|^{n+1-1/2}} dV(w), \quad \lambda < \mu$$
- $\mathbf{A}_{1,\lambda}$ is compact in $L^2(D, dV)$ for any $\lambda > 0$.
- $\|\mathbf{A}_1 - \mathbf{A}_{1,\lambda}\|_{L^2 \to L^2} \lesssim \lambda^{1/2}$
- $\mathbf{A}_{1,\lambda} \to \mathbf{A}_1$ as $\lambda \to 0$ $\square$

Objective

Ligocka: Why $D \in C^3$?

- η has $\dfrac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j}(w)$ (so $D \in C^2$ would seem OK), but
- $H_1(w,z) \approx (\overline{\partial}_w\, \eta)^n$ uses **three** derivatives of ρ.
- **Cancellation**: $|g(w,z) - \overline{g(z,w)}| \lesssim |w-z|^3$ uses $\rho \in C^3$.

Our current goal: deal with $D \in C^2$

$D \in C^2$ is **optimal** (minimal) in **strongly Levi-pseudconvex category**

(D strongly Levi-pseudconvex $\iff$ $D \in C^2$ **and** strongly pseudoconvex)

Results

$$D \Subset \mathbb{C}^n, \text{ strongly } \psi\text{-cvx}, \ D \in C^2$$

$$\text{Bergman :} \quad \mathbf{B}f(z) = \int\limits_{w\in D} f(w)\, B(w,z)\, dV(w) \quad z \in D$$

$$\text{Absolute Bergman :} \quad |\mathbf{B}|f(z) := \int\limits_{w\in D} f(w)\, |B(w,z)|\, dV(w) \quad z \in D$$

- Theorem (L. - Stein, 2011) $\mathbf{B}$ bdd: $L^p \to L^p$
- Theorem (L. - Stein, 2011): $\overline{\vartheta(\overline{D})}^{L^p} = \vartheta L^p(D)$

- Theorem (L. - Stein, 2011) : $|\mathbf{B}|$ bdd: $L^p \to L^p$

$$1 < p < \infty$$

Also, analogous results for

$$B^\sigma : L^2_\sigma(D) \Rsh \vartheta L^2_\sigma(D), \qquad (f, g)_\sigma := \int\limits_{w \in D} f(w)\overline{g}(w)\sigma(w)dV(w)$$

$$\sigma \in C(\overline{D}), \quad \sigma(w) > 0, \quad w \in D$$

$D \Subset \mathbb{C}^n$, strongly ψ-cvx, $D \in C^2$

A single **H** will no longer do. Instead:

Step 1: Construct ad-hoc family $\{\mathbf{H}_\epsilon\}_{\epsilon>0}$ with kernel

$$H_\epsilon(w,z) := H_{1,\epsilon}(w,z) + H_{2,\epsilon}(w,z) \in \Lambda^{(w)}_{n,n}(D) \times \Lambda^{(z)}_{0,0}(D)$$

- $H_{1,\epsilon}(w,z) = \left(\overline{\partial}_w\left(\frac{\eta_\epsilon}{g_\epsilon}\right)\right)^n$ locally holomorphic w.r.t. z
- $H_{2,\epsilon}(w,z) \in C(\overline{D} \times \overline{D})$ a correction term s.t.

$$\overline{\partial}_z(H_{1,\epsilon}(w,z) + H_{2,\epsilon}(w,z)) = 0, \quad \epsilon > 0, \ \ z \in U_w(\overline{D}), \ \ w \in D$$

$(\mathbf{H}_{2,\epsilon} : L^p \to L^p; \quad \mathbf{H}^*_{2,\epsilon} - \mathbf{H}_{2,\epsilon}$ compact on L^2 – so ignore$)$

Step 1: Construction of $H_{1,\epsilon}(w,z)$:

Let

- $D = \{\rho(w) < 0\}, \quad \rho \in C^2(\mathbb{C}^n)$ strictly plurisubharmonic.
- Given $\epsilon > 0$ choose $\tau^{\epsilon}_{j,k} \in C^2(\overline{D})$ s.t.

$$\sup_{w \in \overline{D}} \left| \frac{\partial^2 \rho(w)}{\partial \zeta_j \partial \zeta_k} - \tau^{\epsilon}_{j,k}(w) \right| \leq \epsilon \quad \text{for all} \quad 1 \leq j,\, k \leq n$$

Use $\tau^{\epsilon}_{j,k}(w)$ in place of $\dfrac{\partial^2 \rho(w)}{\partial \zeta_j \partial \zeta_k}$

- $H_{1,\epsilon}(w,z) := \left(\overline{\partial}_w \left(\dfrac{\eta_\epsilon(w,z)}{g_\epsilon(w,z)} \right) \right)^n$

$D \Subset \mathbb{C}^n$, strongly ψ-cvx, $D \in C^2$

If

$$g_\epsilon(w,z) := \mathcal{P}_{w,\epsilon}(z)\chi(w,z) + |w-z|^2(1-\chi(w,z)) - \rho(w)$$

Then for any $0 < \epsilon < \epsilon_0$

- **basic inequality**:

$$2\text{Re}\, g_\epsilon(w,z) \geq \begin{cases} -\rho(w) - \rho(z) + c|w-z|^2, & \textit{if} \quad |w-z| < \mu \\ c > 0, & \textit{if} \quad |w-z| \geq \mu \end{cases}$$

- **size estimate**:

$$|g_\epsilon(w,z)| \approx |\rho(w)| + |\rho(z)| + |\text{Im}\langle \partial\rho(w), w-z\rangle| + |w-z|^2$$

- **symmetry estimate**: $|g_\epsilon(w,z)| \approx |g_\epsilon(z,w)|$
- **cancellation**: $|g_\epsilon(w,z) - \overline{g_\epsilon(z,w)}| \leq c\epsilon|w-z|^2$

Step 2: show $\mathbf{H}_{1,\epsilon}$ bounded: $L^p(D, dV) \to L^p(D, dV)$ unif. in ϵ

- Comparison operators:

$$\Gamma_\epsilon\, (f)(z) = \int\limits_D |g_\epsilon(w,z)|^{-n-1} f(w)\, dV(w), \quad z \in D$$

- Theorem: For $1 < p < \infty$, **for any** $0 < \epsilon < \epsilon_0$ we have

$$\|\Gamma_\epsilon(f)\|_{L^p(D)} \leq c_p \|f\|_{L^p(D)}$$

  Proof: size estimates and symmetry estimates for g_ϵ. $\square$

- Theorem: $\mathbf{H}_{1,\epsilon}$ is bounded: $L^p \to L^p$, $\quad 1 < p < \infty$

  Proof: $|\mathbf{H}_{1,\epsilon} f(z)| \lesssim \Gamma_\epsilon(|f|)(z), \quad z \in D$ $\square$

$D \Subset \mathbb{C}^n$, strongly ψ-cvx, $D \in C^2$

Note: $\mathbf{A}_{1,\epsilon} := \mathbf{H}^*_{1,\epsilon} - \mathbf{H}_{1,\epsilon}$ **fails** compactness criteria for $L^2(D, dV)$.

- Proof: Cancellation: if $|w - z| \leq \mu$ then

$$|A_{1,\epsilon}(w,z)| = |\overline{H_{1,\epsilon}(z,w)} - H_{1,\epsilon}(w,z)| \lesssim \frac{\epsilon}{|g_\epsilon(w,z)|^{n+1}}$$

- $$\mathbf{A}_{1,\epsilon,\lambda} f(z) := \int\limits_{|g_\epsilon(w,z)|>\lambda} \frac{f(w)}{|g_\epsilon(w,z)|^{n+1}} dV(w), \quad \lambda < \mu$$
- $\mathbf{A}_{1,\epsilon,\lambda}$ is compact in $L^2(D, dV)$ for any $\lambda > 0$, **but**
- $\|\mathbf{A}_{1,\epsilon} - \mathbf{A}_{1,\epsilon,\lambda}\|_{L^2 \to L^2} \lesssim \lambda^0 = 1$
- $\mathbf{A}_{1,\epsilon,\lambda} \not\to \mathbf{A}_{1,\epsilon}$ as $\lambda \to 0$ □

$D \Subset \mathbb{C}^n$, strongly ψ-cvx, $D \in C^2$

What works instead:

Lemma For each $\epsilon > 0$, we have

$$\mathbf{H}^*_{1,\epsilon} - \mathbf{H}_{1,\epsilon} = \mathbf{E}_\epsilon + \mathbf{R}_\epsilon = \textbf{Essential Part + Remainder}$$

where

(a) $\|\mathbf{E}_\epsilon\|_{L^p \to L^p} \leq \epsilon\, c_p$, for $1 < p < \infty$

(b) Each Remainder has continuous kernel on $\overline{D} \times \overline{D}$ – hence

$$\mathbf{R}_\epsilon : L^1(D) \to C(\overline{D})$$

Proof:

- by cancellation: $|E_\epsilon(w,z)| \lesssim \epsilon |g_\epsilon(w,z)|^{-n-1}$
- $|\mathbf{E}_\epsilon f(z)| \lesssim \epsilon \Gamma_\epsilon(|f|)(z)$ □

Caveat: the norm of $\mathbf{R}_\epsilon$ may increase as $\epsilon \to 0$

$D \Subset \mathbb{C}^n$ strongly ψ-cvx, $D \in C^2$: Kerzman-Stein-Ligocka, revisited

Theorem: $\mathbf{B}$ is bounded: $L^p \to L^p$, $1 < p < \infty$.

Proof:

- $\mathbf{B}\left(1 - (\mathbf{H}_\epsilon^* - \mathbf{H}_\epsilon)\right) = \mathbf{H}_\epsilon$
- $\mathbf{H}_\epsilon^* - \mathbf{H}_\epsilon = \mathbf{E}_\epsilon + \mathbf{R}_\epsilon$
- $\mathbf{B}\left(1 - \mathbf{E}_\epsilon\right) = \mathbf{H}_\epsilon - \mathbf{B}\mathbf{R}_\epsilon$
- fix $1 < p < \infty$; choose $\epsilon = \epsilon(p)$ s.t. $\|\mathbf{E}_\epsilon\|_{L^p \to L^p} < 1$. Then
- $\mathbf{B} = \left(\mathbf{H}_\epsilon - \mathbf{B}\mathbf{R}_\epsilon\right)\left(1 - \mathbf{E}_\epsilon\right)^{-1}$
- Claim: $\mathbf{B}\mathbf{R}_\epsilon : L^p \to L^p$:
  - Proof. wlog: $1 < p \leq 2$. Then

$$L^p \hookrightarrow L^1 \to C(\overline{D}) \hookrightarrow L^\infty \hookrightarrow L^2 \to L^2 \hookrightarrow L^p \qquad \square$$

Absolute Bergman: Positive majorants.

- Definition. T is a bounded linear operator on L^p, we say that T has a **positive majorant** $\widehat{T}$, if $\widehat{T}$ is bounded linear operator on L^p s.t.

$$\begin{cases} \widehat{T}(f) \geq 0 \quad \text{if} \quad f \geq 0, \quad \text{and} \\ |T(f)(z)| \leq \widehat{T}(|f|)(z), \quad \text{for a.e. } z. \end{cases}$$

- $(\widehat{T_1 + T_2}) = \widehat{T_1} + \widehat{T_2}$
- $(\widehat{T_1 \circ T_2}) = \widehat{T_1} \circ \widehat{T_2}$
- $\| T_n - T\|_p \to 0$ and $\| \widehat{T}_n - S\|_p \to 0 \Rightarrow S = \widehat{T}$
- All of these grant

$$\widehat{(1 - T)^{-1}} = (1 - \widehat{T})^{-1}$$

- If $T : L^p \to C(\overline{D})$ then T has a positive majorant.

Absolute Bergman

- $\widehat{\mathbf{H}_\epsilon} = c\varGamma_\epsilon$
- $\widehat{\mathbf{H}^*_\epsilon} = c'\varGamma_\epsilon$
- $\widehat{\mathbf{E}_\epsilon} = c''\varGamma_\epsilon$
- $\widehat{(1 - \mathbf{E}_\epsilon)^{-1}} = (1 - \widehat{\mathbf{E}_\epsilon})^{-1}$
- $\mathbf{BR}_\epsilon$ has a positive majorant.

As a result: $\mathbf{B}$ has a positive majorant $\widehat{\mathbf{B}}$

And so does $|\mathbf{B}|$:

$$||\mathbf{B}|(f)(z)| \leq \mathcal{M}(\widehat{\mathbf{B}}(|f|))(z), \quad z \in D$$

$$\mathcal{M}(F)(z) := \frac{1}{V(\mathcal{B}_z)} \int_{\mathcal{B}_z} F(w)\, dV(w), \;\; \mathcal{B}_z = \{|w - z| < \frac{1}{2}\, d(z, bD)\}$$

So $|\mathbf{B}|$ is bounded: $L^p \to L^p$, $1 < p < \infty$.

Density for Bergman Space.

Theorem $\overline{\vartheta(\overline{D})}^{L^p} = \vartheta L^p(D)$

Proof

- $f \in \vartheta L^p(D) \Rightarrow \mathbf{H}_\epsilon(f) = f$

$$f_n(w) := \begin{cases} f(w) & if \quad w \in D_{1/n} := \{\rho < -1/n\} \\ 0 & if \quad w \in D \setminus D_{1/n} \end{cases}$$

- $\|f_n - f\|_p \to 0$
- $F_n := \mathbf{H}_\epsilon f_n \in \vartheta(D_{-1/n})$
- $D_{1/n} \Subset D \Subset D_{-1/n}$
- $\|F_n - f\|_p = \|\mathbf{H}_\epsilon(f_n - f)\|_p \lesssim \|f_n - f\|_p \to 0$ $\square$

Szegő Projection

Suppose

$$D \Subset \mathbb{C}^n \text{ strongly Levi-pseudoconvex }, D \in C^2$$

Then

1. Theorem (L. - Stein, 2011) **S** bdd: $L^p(bD) \to L^p(bD)$
2. Theorem (L. - Stein, 2011): $\overline{\vartheta(\overline{D})}^{L^p} = H^p(bD)$

$$1 < p < \infty$$

- Proofs for **S** are more difficult than proofs for **B** because
- There is no comparison operator Γ that will work for **S**. What works instead:
- $T(1)$-theorem for suitable space of homogeneous type: $\{bD;\ \mu;\ d\}$.
- Also, $|\mathbf{S}|$ **not** bounded: $L^p(bD, d\sigma) \to L^p(bD, d\sigma)$.