Long time average of mean field games

Alessio Porretta
Universita' di Roma Tor Vergata

IMA, 13/11/2012

joint works with P. Cardaliaguet, J.-M. Lasry and P.-L. Lions

Outlines of the talk

- "Long time behavior" of Mean Field Games: natural questions and setting. The ergodic problem and expected behavior.
- The long time average: energy estimates and time rescaled problem
- Exponential rate of convergence: two different approaches

  (i) the case of nonlocal coupling

  (ii) the case of local coupling.
- Links with optimal control problems in the long horizon

Long time behavior

Pb: What is the behavior of the MFG system when the horizon $T \to \infty$?

$$\begin{cases} -u_t - \Delta u + H(x, \nabla u) = F(x, m(t)) \,, & \text{in } (0,T) \\ m_t - \Delta m - \operatorname{div}(m\, H_p(x, \nabla u)) = 0 \,, & \text{in } (0,T) \\ m(x,0) = m_0(x) \,, \quad u(x,T) = G(x, m(T)) \,. \end{cases}$$

- To fix the ideas, we work in a periodic setting:

$$x \mapsto F(x,m) \,, \quad x \mapsto G(x,m) \,, \quad m_0(x) \quad \text{are } \mathbb{Z}^N\text{–periodic}$$

so u, m are $\mathbb{Z}^N$–periodic.

Long time behavior

Pb: What is the behavior of the MFG system when the horizon $T \to \infty$?

$$
\begin{cases}
-u_t - \Delta u + H(x, \nabla u) = F(x, m(t))\,, & \text{in } (0,T) \\
m_t - \Delta m - \operatorname{div}\,(m\, H_p(x, \nabla u)) = 0\,, & \text{in } (0,T) \\
m(x,0) = m_0(x)\,, \quad u(x,T) = G(x, m(T))\,.
\end{cases}
$$

- To fix the ideas, we work in a periodic setting:

$$
x \mapsto F(x,m)\,, \quad x \mapsto G(x,m)\,, \quad m_0(x) \quad \text{are } \mathbb{Z}^N\text{–periodic}
$$

  so u, m are $\mathbb{Z}^N$–periodic.

To simplify the presentation, I will consider a reference case:
$H(x, \nabla u) = \frac{1}{2}|\nabla u|^2$, and no coupling at $t = T$: $u(T) = G(x)$.

Long time behavior

Pb: What is the behavior of the MFG system when the horizon $T \to \infty$?

$$\begin{cases} -u_t - \Delta u + H(x, \nabla u) = F(x, m(t)), & \text{in } (0, T) \\ m_t - \Delta m - \text{ div } (m\, H_p(x, \nabla u)) = 0, & \text{in } (0, T) \\ m(x, 0) = m_0(x), \quad u(x, T) = G(x, m(T)). \end{cases}$$

- To fix the ideas, we work in a periodic setting:

$$x \mapsto F(x, m), \quad x \mapsto G(x, m), \quad m_0(x) \quad \text{are } \mathbb{Z}^N\text{–periodic}$$

so u, m are $\mathbb{Z}^N$–periodic.

To simplify the presentation, I will consider a reference case: $H(x, \nabla u) = \frac{1}{2}|\nabla u|^2$, and no coupling at $t = T$: $u(T) = G(x)$.

- We always assume that the coupling $F(x, \cdot)$ is monotone. As usual, two coupling regimes will be analyzed:

  (i) Nonlocal case: $F : \mathbb{R}^N \times L^1 \to \mathbb{R}$ is nonlocal and smoothing

  (ii) Local case: $F = F(x, m(t, x))$,

Long time behavior is a very natural question in the viewpoint of stochastic dynamics.

Recall: in the MFG model, each agent controls the same dynamics

$$dX_t = \alpha(t, X_t)dt + \sqrt{2}dB_t\,,$$

in order to minimize, among controls α, the cost

$$\inf_{\alpha} J(\alpha) := \mathbb{E}\left\{\int_0^T \frac{1}{2}|\alpha(X_s)|^2 + F(X_s, m(s))]ds + G(X_T)\right\}$$

where $m(t)$ is the probability measure associated to the state of players at time t. Namely, $m(t)$ is the distribution law of X_t corresponding to the optimal feedback

$$\alpha^*(t, x) = -Du(t, x)\,.$$

Long time behavior is a very natural question in the viewpoint of stochastic dynamics.

Recall: in the MFG model, each agent controls the same dynamics

$$dX_t = \alpha(t, X_t)dt + \sqrt{2}dB_t \,,$$

in order to minimize, among controls α, the cost

$$\inf_{\alpha} J(\alpha) := \mathbb{E}\left\{\int_0^T \frac{1}{2}|\alpha(X_s)|^2 + F(X_s, m(s))]ds + G(X_T)\right\}$$

where $m(t)$ is the probability measure associated to the state of players at time t. Namely, $m(t)$ is the distribution law of X_t corresponding to the optimal feedback

$$\alpha^*(t,x) = -Du(t,x)\,.$$

Pb: $\lim_{T\to\infty} \mathbb{E}\left\{\frac{1}{T}\int_0^T f(X_t)dt\right\}$? Is there an invariant measure ?

In the long horizon, agents are expected to behave in a way to minimize the average (ergodic) cost, regardless of the initial distribution.

Recall the case of a single equation (with no coupling):

$$\begin{cases} -u^T_t - \Delta u^T + \frac{1}{2}|\nabla u^T|^2 = F(x) & \text{in } (0,T) \\ u^T(x,T) = G(x) \\ \text{periodic b.c.} \end{cases}$$

Then one has:

(i) $\frac{u^T(x,0)}{T}$ converges uniformly to a constant $\bar{\lambda} \in \mathbb{R}$, which is the ergodic (minimal) cost

$$\bar{\lambda} = \inf_{\alpha} \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left\{ \int_0^T \frac{1}{2}|\alpha(X_s)|^2 + F(X_s)]ds \right\}$$

(ii) $u^T(x,0) - \bar{\lambda}T$ converges uniformly to some $\bar{u}$, periodic solution of the ergodic problem

$$\bar{\lambda} - \Delta \bar{u} + \frac{1}{2}|\nabla \bar{u}|^2 = F(x)\,.$$

([Bensoussan-Frehse], [Namah-Roquejoffre], [Barles-Souganidis]).

Typical arguments: comparison principles, gradient estimates, strong maximum principle.

As far as m is concerned, if one proves that $\nabla u^T \to \nabla \bar{u}$, then $m^T(t)$ converges to the unique invariant probability measure $\bar{m}$ associated to the process

$$dX_t = -\nabla \bar{u}(X_t)dt + \sqrt{2}dB_t$$

Anyway, the convergence of Du^T is essential to prove convergence of m^T.

(But for the coupled system $F = F(x, m^T)$, and the convergence of m^T will be needed for u^T as well....)

As far as m is concerned, if one proves that $\nabla u^T \to \nabla \bar{u}$, then $m^T(t)$ converges to the unique invariant probability measure $\bar{m}$ associated to the process

$$dX_t = -\nabla \bar{u}(X_t)dt + \sqrt{2}dB_t$$

Anyway, the convergence of Du^T is essential to prove convergence of m^T.

(But for the coupled system $F = F(x, m^T)$, and the convergence of m^T will be needed for u^T as well....)

Main (new) difficulties on the MFG system:

- the system is doubly coupled and one cannot use the arguments of the single equation: there are no standard/simple comparison arguments, gradient estimates, etc...

- forward-backward conditions: there is not just an evolution forward in time! Some boundary layer could appear at $t = 0$ or $t = T$.

One should reasonably reason on a large transient behavior $[\delta T, (1-\delta)T]$ inside the horizon $[0, T]$.

Good news: the ergodic problem for the Mean Field Games system is well posed.

It was proved by Lasry and Lions that there exists a unique couple $(\bar{u}, \bar{m})$ and a unique constant $\bar{\lambda}$ which solve

$$\begin{cases} \bar{\lambda} - \Delta \bar{u} + \frac{1}{2}|\nabla \bar{u}|^2 = F(x, \bar{m})\,, & \int_\Omega \bar{u} = 0 \\ -\Delta \bar{m} - \operatorname{div}(\bar{m}\nabla \bar{u}) = 0\,, & \int_\Omega \bar{m} = 1 \end{cases}$$

where $\Omega = [0,1]^N$ and the solutions are $\mathbb{Z}^N$–periodic.

Moreover, $\bar{u}$, $\bar{m}$ are smooth, $\bar{m} > 0$ (actually $\bar{m} = e^{-\bar{u}} / \left(\int_\Omega e^{-\bar{u}} dx\right)$).

Good news: the ergodic problem for the Mean Field Games system is well posed.

It was proved by Lasry and Lions that there exists a unique couple $(\bar{u}, \bar{m})$ and a unique constant $\bar{\lambda}$ which solve

$$\begin{cases} \bar{\lambda} - \Delta \bar{u} + \frac{1}{2}|\nabla \bar{u}|^2 = F(x, \bar{m}), & \int_\Omega \bar{u} = 0 \\ -\Delta \bar{m} - \operatorname{div}(\bar{m} \nabla \bar{u}) = 0, & \int_\Omega \bar{m} = 1 \end{cases}$$

where $\Omega = [0,1]^N$ and the solutions are $\mathbb{Z}^N$–periodic.

Moreover, $\bar{u}$, $\bar{m}$ are smooth, $\bar{m} > 0$ (actually $\bar{m} = e^{-\bar{u}} / \left(\int_\Omega e^{-\bar{u}} dx \right)$).

As in the non coupled case, now one would expect that $u^T / T \to \bar{\lambda}$ and $m^T(t) \to \bar{m}$ in the long time behavior. However, the way to express this convergence and the methods used are new and peculiar to this new system.

Convergence in average

A first look at the rescaled problem:

$$\upsilon^T(t,x) := u(tT,x) \quad ; \quad \mu^T(t,x) := m(tT,x) \qquad (t,x) \in [0,1] \times \mathbb{R}^N .$$

Now (υ^T, μ^T) solve

$$\begin{cases} -\frac{1}{T}\,\upsilon_t^T - \Delta \upsilon^T + \frac{1}{2}|\nabla \upsilon^T|^2 = F(x,\mu^T) & t \in (0,1) \\ \upsilon^T(x,1) = G(x,\mu(1)) \end{cases}$$

$$\begin{cases} \frac{1}{T}\,\mu_t^T - \Delta \mu^T - \text{ div } (\mu^T \nabla \upsilon^T) = 0 & t \in (0,1) \\ \mu^T(0) = m_0 \end{cases}$$

Convergence in average

A first look at the rescaled problem:

$$\upsilon^T(t,x) := u(tT,x) \quad ; \quad \mu^T(t,x) := m(tT,x) \qquad (t,x) \in [0,1] \times \mathbb{R}^N .$$

Now (υ^T, μ^T) solve

$$\begin{cases} -\frac{1}{T}\,\upsilon^T_t - \Delta \upsilon^T + \frac{1}{2}|\nabla \upsilon^T|^2 = F(x,\mu^T) & t \in (0,1) \\ \upsilon^T(x,1) = G(x,\mu(1)) \end{cases}$$

$$\begin{cases} \frac{1}{T}\,\mu^T_t - \Delta \mu^T - \text{ div } (\mu^T \nabla \upsilon^T) = 0 & t \in (0,1) \\ \mu^T(0) = m_0 \end{cases}$$

The first kind of result that we prove is the convergence in average:

- $\frac{\upsilon^T}{T}$ converges to $\bar{\lambda}(1-t)$ for every $t \in [0,1]$
- $\begin{cases} \upsilon^T - \int_\Omega \upsilon^T(y)dy \text{ converges to } \bar{u} \\ D\upsilon^T \text{ converges to } D\bar{u} \end{cases}$ (e.g. in $L^2((0,1)\times\Omega)$).
- μ^T converges to $\bar{m}$ (e.g. in $L^1((0,1)\times\Omega)$)

The type of convergence may vary according to local/nonlocal coupling

This result relies mainly on the uniqueness principle of the coupled structure.
Main ingredient: energy equality ([Lasry-Lions] $\rightarrow$ uniqueness).

Any couple of solutions (u_1, m_1) and (u_2, m_2) satisfy

$$-\frac{d}{dt}\int_\Omega (u_1 - u_2)(m_1 - m_2)dx =$$
$$\int_\Omega \frac{(m_1 + m_2)}{2}|Du_1 - Du_2|^2 + (F(x, m_1) - F(x, m_2))(m_1 - m_2)\ dx$$

This result relies mainly on the uniqueness principle of the coupled structure.
Main ingredient: energy equality ([Lasry-Lions] $\rightarrow$ uniqueness).

Any couple of solutions (u_1, m_1) and (u_2, m_2) satisfy

$$-\frac{d}{dt}\int_\Omega (u_1 - u_2)(m_1 - m_2)dx =$$
$$\int_\Omega \frac{(m_1 + m_2)}{2}|Du_1 - Du_2|^2 + (F(x, m_1) - F(x, m_2))(m_1 - m_2)\ dx$$

Rmk: Since $m_1 - m_2$ has zero mean, you can always keep the equality by adding a constant to u_1 or u_2 (ex.: the ergodic stationary solution $\bar{u}$ is a solution up to adding $\bar{\lambda}(T - t)$).

Apply the energy equality to (u, m) and $(\bar{u}, \bar{m})$ between 0 and T:

$$\int_0^T \int_\Omega \frac{(m+\bar{m})}{2}|Du - D\bar{u}|^2 + (F(x,m) - F(x,\bar{m}))(m-\bar{m})\,dx$$
$$= -\left[\int_\Omega (u-\bar{u})(m-\bar{m})dx\right]_0^T \quad \overset{?}{\leq} \quad C$$

Recall that $m,\ \bar{m} > 0$, and that $F(x,\cdot)$ is non decreasing.

Then, controlling the RHS, one controls the energy of the difference.

Then, if the RHS is bounded, one deduces

$$\frac{1}{T}\int_0^T \int_\Omega |Du - D\bar{u}|^2 + (F(x,m) - F(x,\bar{m}))(m-\bar{m})\,dx \leq \frac{C}{T}$$

Apply the energy equality to (u, m) and $(\bar{u}, \bar{m})$ between 0 and T:

$$\int_0^T \int_\Omega \frac{(m+\bar{m})}{2} |Du - D\bar{u}|^2 + (F(x,m) - F(x,\bar{m}))(m - \bar{m})\, dx$$
$$= -\left[\int_\Omega (u - \bar{u})(m - \bar{m}) dx\right]_0^T \quad \overset{?}{\leq} \quad C$$

Recall that $m,\ \bar{m} > 0$, and that $F(x, \cdot)$ is non decreasing.

Then, controlling the RHS, one controls the energy of the difference.

Then, if the RHS is bounded, one deduces

$$\frac{1}{T}\int_0^T \int_\Omega |Du - D\bar{u}|^2 + (F(x,m) - F(x,\bar{m}))(m - \bar{m})\, dx \leq \frac{C}{T}$$

Rescaling the functions this means

$$\int_0^1 \int_\Omega |Dv^T - D\bar{u}|^2 + (F(x,\mu^T) - F(x,\bar{m}))(\mu^T - \bar{m})\, dx \leq \frac{C}{T} \to 0$$

and in particular $Dv^T \to D\bar{u}$ in $L^2((0,1) \times \Omega)$.

In order to bound the RHS

$$-\left[\int_\Omega (u^T - \bar{u})(m^T - \bar{m})dx\right]_0^T$$

one is only concerned with the term

$$\int_\Omega (u^T(0) - \bar{u})(m_0 - \bar{m})dx$$

Typically, $u^T(0) \sim CT$. But we wish an estimate independent of T.

Note that, if we set $\langle u \rangle := \int u\,dx$, then

$$\int_\Omega u^T(0)(m_0 - \bar{m})dx = \int_\Omega (u^T(0) - \langle u^T(0)\rangle)(m_0 - \bar{m})dx \le c\|Du^T(0)\|_{L^2}$$

$\Rightarrow$ it is enough to bound $\|Du^T(0)\|$ independently of T.

In order to get estimates, we use different features according to local or nonlocal coupling.

(i) Nonlocal smoothing coupling $F(x, m)$: we use a semiconcavity estimate

$$D^2 u \leq K \quad \text{(uniformly with respect to time } T\text{)}$$

hence (since u is periodic) we deduce a Lipschitz bound $\|Du\|_\infty \leq K$.

(ii) Local coupling $F(x, m)$: we use the fact that in this case the system has an Hamiltonian structure $\Rightarrow$ there exists an invariant (constant in time).
Thanks to this fact we obtain a bound on $\|Du^T(0)\|_{L^2}$.

If $\mathcal{F}(x,m) = \int_0^m F(x,r)dr$ and

$$\mathcal{E}(u,m) = \int_\Omega [\frac{1}{2} m|\nabla u|^2 + \nabla u \cdot \nabla m - \mathcal{F}(x,m)]dx$$

then (u,m) is a (classical) solution of (MFG) if and only if (u,m) satisfies

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial \mathcal{E}}{\partial m} \\ \dfrac{\partial m}{\partial t} = -\dfrac{\partial \mathcal{E}}{\partial u} \end{cases}$$

In particular, the quantity $\mathcal{E}(u(t), m(t))$ is constant along the flow.

If $\mathcal{F}(x,m) = \int_0^m F(x,r)dr$ and

$$\mathcal{E}(u,m) = \int_\Omega [\frac{1}{2} m|\nabla u|^2 + \nabla u \cdot \nabla m - \mathcal{F}(x,m)]dx$$

then (u,m) is a (classical) solution of (MFG) if and only if (u,m) satisfies

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial \mathcal{E}}{\partial m} \\ \dfrac{\partial m}{\partial t} = -\dfrac{\partial \mathcal{E}}{\partial u} \end{cases}$$

In particular, the quantity $\mathcal{E}(u(t), m(t))$ is constant along the flow.

Consequence: it is enough to bound the average of $\mathcal{E}$, and one deduces

$$|\mathcal{E}(u(0), m(0))| \leq C$$

If m_0 is Lipschitz, $m_0 > 0$, then

$$\mathcal{E}(u(0), m(0)) = \int_\Omega [\tfrac{1}{2} m_0 |\nabla u(0)|^2 + \nabla u(0) \cdot \nabla m_0 - \mathcal{F}(x, m_0)]dx$$
$$\geq c_0 \int_\Omega |\nabla u(0)|^2 dx - C$$

and one concludes a bound on $|\nabla u(0)|$ in L^2.

Convergence at exponential rate

A very natural question concerns the rate of convergence:
is the ergodic limit exponentially stable ?

Convergence at exponential rate

A very natural question concerns the rate of convergence:
is the ergodic limit exponentially stable ?

1. Local case.

We strengthen the monotonicity condition

$$(F(x,m_1) - F(x,m_2))(m_1 - m_2) \geq \gamma (m_1 - m_2)^2 \qquad (1)$$

for some $\gamma > 0$. Then we prove an exponential convergence.

Convergence at exponential rate

A very natural question concerns the rate of convergence:
is the ergodic limit exponentially stable ?

1. Local case.

We strengthen the monotonicity condition

$$(F(x,m_1) - F(x,m_2))(m_1 - m_2) \geq \gamma (m_1 - m_2)^2 \qquad (1)$$

for some $\gamma > 0$. Then we prove an exponential convergence.

Theorem

Under assumption (1), there is some $\kappa > 0$ (independent of T) such that (we denote $\tilde{u} = u - \int_\Omega u dy$)

$$\|\tilde{u}(t) - \bar{u}\|_{L^1} \leq C \left(e^{-\kappa(T-t)} + e^{-\kappa t} \right) \qquad \forall t \in (1, T-1) \,,$$

$$\|m(t) - \bar{m}\|_{L^1} \leq C \left(e^{-\kappa(T-t)} + e^{-\kappa t} \right) \qquad \forall t \in (1, T-1)$$

In particular, the convergence of the rescaled functions $v^T = u(tT)$ and $\mu^T = m(tT)$ holds exponentially (on compact subsets of $(0,1)$).

Proof: back to the energy estimate. Recall that

$$-\frac{d}{dt}\int_\Omega(\tilde{u}-\bar{u})(m-\bar{m})dx =$$

$$\int_\Omega \frac{(m+\bar{m})}{2}|Du - D\bar{u}|^2 + (F(x,m)-F(x,\bar{m}))(m-\bar{m})\ dx$$

where $\tilde{u} = u - \int_\Omega u(y)dy$.

Proof: back to the energy estimate. Recall that

$$-\frac{d}{dt}\int_\Omega (\tilde{u}-\bar{u})(m-\bar{m})dx =$$

$$\int_\Omega \frac{(m+\bar{m})}{2}|Du-D\bar{u}|^2 + (F(x,m)-F(x,\bar{m}))(m-\bar{m})\ dx$$

where $\tilde{u} = u - \int_\Omega u(y)dy$.

Using the strong monotonicity of F and Poincaré-Wirtinger inequality

$$\int_\Omega |Du-D\bar{u}|^2 + (F(x,m)-F(x,\bar{m}))(m-\bar{m})\ dx$$
$$\geq c\left\{\|\tilde{u}-\bar{u}\|_{L^2}^2 + \frac{1}{2}\|m-\bar{m}\|_{L^2}^2\right\}$$

Proof: back to the energy estimate. Recall that

$$-\frac{d}{dt}\int_\Omega(\tilde{u}-\bar{u})(m-\bar{m})dx =$$
$$\int_\Omega \frac{(m+\bar{m})}{2}|Du-D\bar{u}|^2+(F(x,m)-F(x,\bar{m}))(m-\bar{m})\ dx$$

where $\tilde{u}=u-\int_\Omega u(y)dy$.

Using the strong monotonicity of F and Poincaré-Wirtinger inequality

$$\int_\Omega |Du-D\bar{u}|^2+(F(x,m)-F(x,\bar{m}))(m-\bar{m})\ dx$$
$$\geq c\left\{\|\tilde{u}-\bar{u}\|_{L^2}^2+\frac{1}{2}\|m-\bar{m}\|_{L^2}^2\right\}\geq c\left|\int_\Omega(\tilde{u}-\bar{u})(m-\bar{m})dx\right|$$

Proof: back to the energy estimate. Recall that

$$-\frac{d}{dt}\int_\Omega (\tilde u - \bar u)(m - \bar m)dx =$$
$$\int_\Omega \frac{(m+\bar m)}{2}|Du - D\bar u|^2 + (F(x,m) - F(x,\bar m))(m - \bar m)\ dx$$

where $\tilde u = u - \int_\Omega u(y)dy$.

Using the strong monotonicity of F and Poincaré-Wirtinger inequality

$$\int_\Omega |Du - D\bar u|^2 + (F(x,m) - F(x,\bar m))(m - \bar m)\ dx$$
$$\geq c\left\{\|\tilde u - \bar u\|_{L^2}^2 + \frac{1}{2}\|m - \bar m\|_{L^2}^2\right\} \geq c\left|\int_\Omega (\tilde u - \bar u)(m - \bar m)dx\right|$$

Therefore we have, for some $\sigma > 0$

$$\varphi(t) := \int_\Omega (\tilde u - \bar u)(m - \bar m)dx \quad \text{satisfies} \quad \varphi'(t) \leq -\sigma|\varphi(t)|$$

hence

$$\varphi(T)\, e^{-\sigma(T-t)} \leq \varphi(t) \leq \varphi(0)\, e^{-\sigma t}$$

Again, bounds only at T and at 0 allow to conclude the exponential decay of φ, then, by integration, of the energy.

2. Nonlocal case.

In the nonlocal case, we strengthen the regularizing property of the coupling term

$$\|F(x,m_1) - F(x,m_2)\|_{\mathcal{C}^{1+\alpha}} \le \bar{C}\|m_1 - m_2\|_{H^{-1}} \qquad \forall m_1, m_2 \qquad (2)$$

for some $\alpha > 0$.

Theorem

Under assumption (2), there exists $\kappa > 0$ (independent of T) such that

$$\|\tilde{u}(t) - \bar{u}\|_{C^{3,\alpha}} \le C\left(e^{-\kappa(T-t)} + e^{-\kappa t}\right) \qquad \forall t \in (a, T-a)\,,$$

$$\|m(t) - \bar{m}\|_{C^{2,\alpha}} \le C\left(e^{-\kappa(T-t)} + e^{-\kappa t}\right) \qquad \forall t \in (a, T-a)$$

(C, a depend on initial-terminal conditions).

The proof is completely different from the previous case ! It relies on underlying properties of the linearized system and decay estimates.

- We look at the linearized system around $(\bar{m}, \bar{u})$:

$$\begin{cases} -v_t \overbrace{-\Delta v + D\bar{u}Dv}^{A(v)} = F'(\bar{m})\mu \\ \mu_t \underbrace{-\Delta\mu - \operatorname{div}(\mu\, D\bar{u})}_{A^*(\mu)} \underbrace{- \operatorname{div}(\bar{m}\, Dv)}_{K(v)} = 0 \end{cases}$$

The proof is completely different from the previous case ! It relies on underlying properties of the linearized system and decay estimates.

- We look at the linearized system around $(\bar{m}, \bar{u})$:

$$\begin{cases} -v_t \overbrace{-\Delta v + D\bar{u}Dv}^{A(v)} = F'(\bar{m})\mu \\ \mu_t \underbrace{-\Delta\mu - \text{ div }(\mu\, D\bar{u})}_{A^*(\mu)} \underbrace{- \text{ div }(\bar{m}\, Dv)}_{K(v)} = 0 \end{cases}$$

and we show that $w := K^{-\frac{1}{2}}\mu$ satisfies

$$\frac{d^2}{dt^2}\|w(t)\|_2^2 \geq \omega_0^2 \ \|w(t)\|_2^2$$

for some $\omega_0 > 0$.

The proof is completely different from the previous case ! It relies on underlying properties of the linearized system and decay estimates.

- We look at the linearized system around $(\bar{m}, \bar{u})$:

$$\begin{cases} -v_t \overbrace{-\Delta v + D\bar{u}Dv}^{A(v)} = F'(\bar{m})\mu \\ \mu_t \underbrace{-\Delta\mu - \text{ div } (\mu\, D\bar{u})}_{A^*(\mu)} \underbrace{- \text{ div } (\bar{m}\, Dv)}_{K(v)} = 0 \end{cases}$$

and we show that $w := K^{-\frac{1}{2}}\mu$ satisfies

$$\frac{d^2}{dt^2}\|w(t)\|_2^2 \geq \omega_0^2\ \|w(t)\|_2^2$$

for some $\omega_0 > 0$. Hence there is an exponential decay

$$\|w(t)\|_2^2 \leq \max\{\|w(0)\|_2^2, \|w(T)\|_2^2\} \left(e^{-\omega_0 t} + e^{-\omega_0(T-t)}\right).$$

The proof is completely different from the previous case ! It relies on underlying properties of the linearized system and decay estimates.

- We look at the linearized system around $(\bar{m}, \bar{u})$:

$$\begin{cases} -v_t \overbrace{-\Delta v + D\bar{u} Dv}^{A(v)} = F'(\bar{m})\mu \\ \mu_t \underbrace{-\Delta\mu - \operatorname{div}(\mu\, D\bar{u})}_{A^*(\mu)} \underbrace{- \operatorname{div}(\bar{m}\, Dv)}_{K(v)} = 0 \end{cases}$$

and we show that $w := K^{-\frac{1}{2}}\mu$ satisfies

$$\frac{d^2}{dt^2}\|w(t)\|_2^2 \geq \omega_0^2\ \|w(t)\|_2^2$$

for some $\omega_0 > 0$. Hence there is an exponential decay

$$\|w(t)\|_2^2 \leq \max\{\|w(0)\|_2^2, \|w(T)\|_2^2\} \left(e^{-\omega_0 t} + e^{-\omega_0 (T-t)}\right) .$$

$$\Rightarrow \|\mu(t)\|_{H^{-1}}^2 \lesssim \|w(t)\|_2^2 \leq \max\{\|m_0\|_2^2, \|m(T)\|_2^2\} \left(e^{-\omega_0 t} + e^{-\omega_0 (T-t)}\right)$$

- Through a fixed point argument, we can preserve such property for the nonlinear problem:

$$\|m(t) - \bar{m}\|^2_{H^{-1}} \lesssim C \left(e^{-\omega_0 t} + e^{-\omega_0 (T-t)}\right)$$

- Using

$$\|F(x, m(t)) - F(x, \bar{m})\|_{\mathcal{C}^{1+\alpha}} \leq \bar{C}\|m(t) - \bar{m}\|_{H^{-1}}$$

we bootstrap the estimates between the two equations, using the exponential decay properties of the operators

$$A(v) = -\Delta v + D\bar{u}\, Dv\,, \qquad A^*(\mu) = -\Delta\mu - \text{ div } (\mu\, D\bar{u})$$

- Through a fixed point argument, we can preserve such property for the nonlinear problem:

$$\|m(t)-\bar{m}\|_{H^{-1}}^2 \lesssim C\left(e^{-\omega_0 t}+e^{-\omega_0(T-t)}\right)$$

- Using

$$\|F(x,m(t))-F(x,\bar{m})\|_{\mathcal{C}^{1+\alpha}} \leq \bar{C}\|m(t)-\bar{m}\|_{H^{-1}}$$

we bootstrap the estimates between the two equations, using the exponential decay properties of the operators

$$A(v)=-\Delta v+D\bar{u}\,Dv\,, \qquad A^*(\mu)=-\Delta\mu-\text{ div }(\mu\,D\bar{u})$$

Rmk: This proof follows the algebraic hint of the 1-d case; if $\lambda>0$, $\alpha,\beta\geq 0$, the linear system

$$\begin{cases} -v_t-\lambda v=\beta\mu \\ \mu_t+\lambda\mu=-\alpha\,v \\ \mu(0)=\mu_0,\ v(T)=v_T \end{cases}$$

decays exponentially inside $|\mu(t)|,|v|\leq c(e^{-\omega_0 t}+e^{-\omega_0(T-t)})$.

Links with optimal control problems

From the optimal control viewpoint, MFG system is an optimality system in $[0, T]$ (for a bilinear control problem).

Ex: Optimize in terms of the field α

$$\inf_{\alpha} \int_0^T [\int_\Omega \frac{1}{2}\, m\,|\alpha|^2 + \mathcal{F}(m(s))]ds$$

where $\mathcal{F}'(m) = F(m)$ and where

$$m_t = \Delta m + \text{ div } (\alpha\, m)\,, \qquad m(0) = m_0$$

Optimality gives $\alpha_{opt} = \nabla u(t, x)$ solution of the adjoint equation.

Links with optimal control problems

From the optimal control viewpoint, MFG system is an optimality system in $[0, T]$ (for a bilinear control problem).

Ex: Optimize in terms of the field α

$$\inf_{\alpha} \int_0^T [\int_\Omega \frac{1}{2} m |\alpha|^2 + \mathcal{F}(m(s))] ds$$

where $\mathcal{F}'(m) = F(m)$ and where

$$m_t = \Delta m + \text{ div } (\alpha\, m) , \qquad m(0) = m_0$$

Optimality gives $\alpha_{opt} = \nabla u(t, x)$ solution of the adjoint equation.

What we prove is the convergence of optimal controls $[\nabla u^T]$ and trajectories $[m^T]$ which are optimal in $[0, T]$ towards the corresponding steady-state ones.

The convergence holds in average and exponentially in the transient time.

Is this a general issue of optimality systems?

It turns out that similar exponential estimates hold for a wide class of optimal control problems in the long horizon
(work in progress with E. Zuazua).

Ex: back to the linear case $\iff$ minimize a quadratic cost

$$J(u) = \frac{1}{2}\int_0^T \left[\|Cx - z\|^2 + \|u\|^2\right] dt$$

over the dynamics

$$\begin{cases} x_t + Ax = Bu \\ x(0) = x_0. \end{cases}$$

where $A, B, C \in \mathcal{M}_N$, $z \in \mathbb{R}^N$ is some target observation.

The optimality system reads as

$$\begin{cases} -p_t - A^* p = C^* Cx - C^* z \\ x_t + Ax = -BB^* p \\ x(0) = x_0, \ p(T) = 0 \end{cases}$$

Theorem (Finite dimensional case)

If (A, B) is controllable [i.e. the Kalman rank condition $\text{Rank}\big[B\ AB\ A^2B \ldots\ A^{N-1}B\big] = N$ holds] and (A, C) is observable, then there exist $\kappa > 0$ and K:

$$|u^T(t) - \bar{u}| + |x^T(t) - \bar{x}| \leq K(e^{-\kappa t} + e^{-\kappa(T-t)}) \qquad \forall t \in [0, T],$$

where (u^T, x^T) and $(\bar{u}, \bar{x})$ are the evolution and the stationary optimal control and state.

Theorem (Finite dimensional case)

If (A, B) is controllable [i.e. the Kalman rank condition $\mathrm{Rank}\big[B\ AB\ A^2B\ldots\ A^{N-1}B\big] = N$ holds] and (A, C) is observable, then there exist $\kappa > 0$ and K:

$$|u^T(t) - \bar{u}| + |x^T(t) - \bar{x}| \leq K(e^{-\kappa t} + e^{-\kappa(T-t)}) \qquad \forall t \in [0, T],$$

where (u^T, x^T) and $(\bar{u}, \bar{x})$ are the evolution and the stationary optimal control and state.

- Actually, κ is characterized as the exponential rate of the dynamics stabilized through the solution of algebraic Riccati equation.
- The same approach extends to infinite dimensional setting (at least for a large class of examples).
  The linearized MFG system (around the ergodic solution) is an example of this kind.

Conclusions

- Under mild monotonicity conditions, we have shown, as $T \to \infty$

  (i) the convergence of $\frac{u(t)}{T}$ to $\bar{\lambda}(T-t)$

  (ii) the convergence of $u(t) - \int_\Omega u(t,y)dy$ to $\bar{u}$

  (iii) the convergence of $m(t)$ to $\bar{m}$

  expressed in different norms or scales.

Conclusions

- Under mild monotonicity conditions, we have shown, as $T \to \infty$

  (i) the convergence of $\frac{u(t)}{T}$ to $\bar{\lambda}(T-t)$

  (ii) the convergence of $u(t) - \int_\Omega u(t,y)dy$ to $\bar{u}$

  (iii) the convergence of $m(t)$ to $\bar{m}$

  expressed in different norms or scales.
- Under either stronger monotonicity in the local case or stronger continuity in the nonlocal case we have shown that the convergence has exponential rate in the transient time.

Conclusions

- Under mild monotonicity conditions, we have shown, as $T \to \infty$

  (i) the convergence of $\frac{u(t)}{T}$ to $\bar{\lambda}(T-t)$

  (ii) the convergence of $u(t) - \int_\Omega u(t,y)dy$ to $\bar{u}$

  (iii) the convergence of $m(t)$ to $\bar{m}$

  expressed in different norms or scales.
- Under either stronger monotonicity in the local case or stronger continuity in the nonlocal case we have shown that the convergence has exponential rate in the transient time.
- The behavior of $u(t) - \lambda(T-t)$ is not well understood yet (comparing with the case of a single equation).

Conclusions

- Under mild monotonicity conditions, we have shown, as $T \to \infty$

  (i) the convergence of $\frac{u(t)}{T}$ to $\bar{\lambda}(T-t)$

  (ii) the convergence of $u(t) - \int_\Omega u(t,y)dy$ to $\bar{u}$

  (iii) the convergence of $m(t)$ to $\bar{m}$

  expressed in different norms or scales.
- Under either stronger monotonicity in the local case or stronger continuity in the nonlocal case we have shown that the convergence has exponential rate in the transient time.
- The behavior of $u(t) - \lambda(T-t)$ is not well understood yet (comparing with the case of a single equation).
- The results obtained are consistent with a general behavior of optimality systems in the long horizon. The structure of the linearized system explains the exponential stability and suggests possibly new approaches
- Long time behavior for finite state space, discrete time models is analyzed in [Gomes-Mohr-Souza]