# Balanced and unbalanced collections

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Balanced collections were introduced 50 years ago by Lloyd Shapley (Nobel Prize in Economics, 2012) in his study of economic equilibria (the "core of a cooperative game").

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We are interested in enumerating these collections.

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| 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|
| 2 | 6 | 32 | 370 | 11,292 | 1,066,044 | 347,326,352 | 419,172,756,930 |

# A few examples

For $n = 3$, the 6 maximal unbalanced collections are

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- they always have $2^{n-1} - 1$ sets and rank $n - 2$ with $(n-1)!$ maximal chains.
- their order complexes are always shellable balls with a single interior vertex
- their $f$-vectors are all the same; in fact, $h_i(\Delta(\mathcal{F}))$ is the number of permutations in $S_{n-1}$ with $i$ descents (classical Eulerian numbers).

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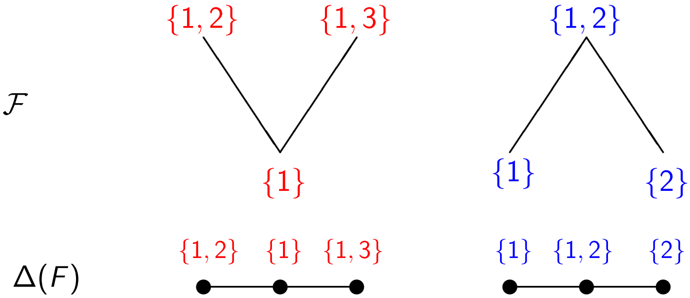

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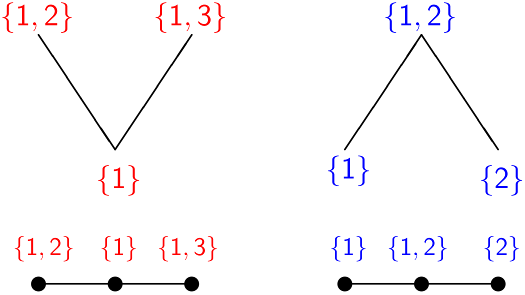


Note: both have $f(\Delta) = (3, 2)$ and a unique interior vertex

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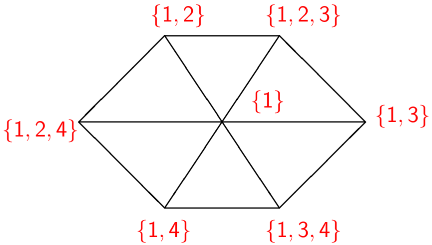


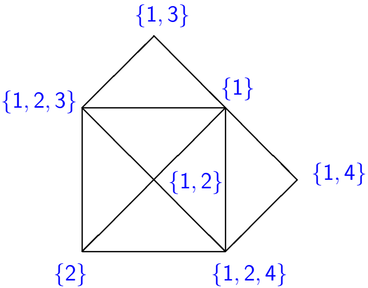

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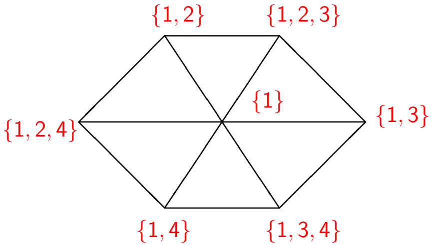


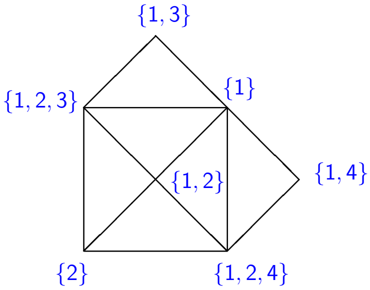


Here both have $f(\Delta) = (7, 12, 6)$ and a single interior vertex.

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Recall: $\mathcal{F} \subset 2^{[n]}$ is unbalanced $\iff$
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# All-subset arrangement in $\mathbb{R}^{n-1}$

Combinatorially equivalent to the restricted all-subset arrangement in $\mathbb{R}^n$ is the all-subset arrangement $\mathcal{A}_{n-1}$ in $\mathbb{R}^{n-1}$, consisting of all hyperplanes with normals $e_S, S \subseteq [n-1], S \neq \emptyset$.

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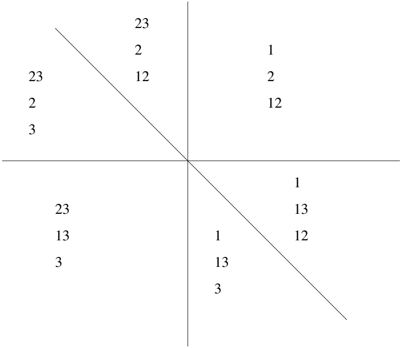
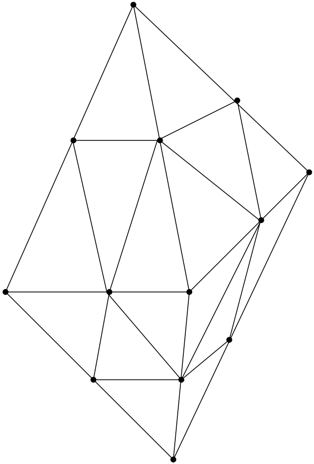


# $\mathcal{A}_3$ has 7 planes and 32 regions



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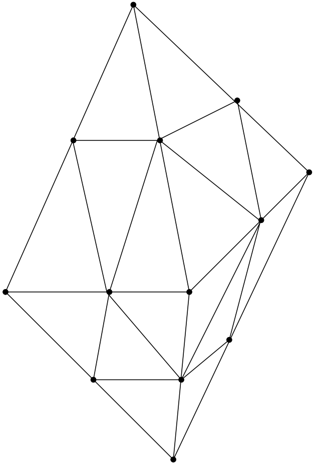

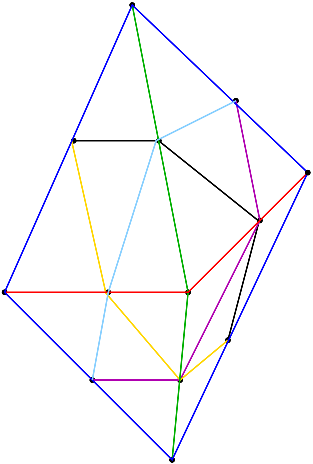

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Recall the characteristic polynomial of $\mathcal{A}_n$ is defined by

$$\chi(\mathcal{A}_n, t) = \sum_{x \in L_n} \mu(0, x)\ t^{\mathrm{rank}(L_n) - \mathrm{rank}(x)} = \sum_{k=0}^{n} w_k(L_n)\ t^{n-k}$$

($L_n$ = lattice of flats of $\mathcal{A}_n$)

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Unfortunately, we don't know $\chi(\mathcal{A}_n, t)$.

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Theorem: The number of maximal unbalanced families in $[n]$, equivalently, the number of chambers of the arrangement $\mathcal{A}_{n-1}$, is at least $\prod_{i=0}^{n-2}(2^i+1)$. Thus the number of maximal unbalanced collections is more than

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This answers a question raised by the physicist T.S. Evans, who asked if the number of such collections exceeded $n!$.

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Theorem: There are fewer than $2^{(n-1)^2}$ maximal unbalanced families in $[n]$.

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# Threshold collections and threshold functions

- A collection of subsets $\mathcal{T} \subset 2^{[n]}$ is a threshold collection if there is a weight vector $w \in \mathbb{R}^n$ and $q \in \mathbb{R}$ so that

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Thus $\{$ unbalanced $\mathcal{T}\} \subset \{$ 0-threshold $\mathcal{T}\} \subset \{$ threshold $\mathcal{T}\}$

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The argument uses a theorem of Odlyzko on random $\pm 1$ vectors.

# Open questions

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- The signature, and more generally, the degree sequence of graphs and threshold complexes, behaves like the coordinates for secondary polytopes given by Gel'fand, Kapranov and Zelevinski. Is there some relation here?

# Some references

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