The most and the least avoided consecutive patterns

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A new Catalan number

2,305,290

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2,305,290: number of Catalans who voted yesterday in the informal poll for Catalan independence

Consecutive patterns

$\pi = \pi_1\pi_2\dots\pi_n \in \mathcal{S}_n, \quad \sigma \in \mathcal{S}_m.$

Definition. π *contains* σ *as a consecutive pattern* if it has a subsequence of adjacent entries order-isomorphic to σ.

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Examples:
25134 avoids 132
42531 contains 132
15243 contains two occurrences of 132

In this talk, containment and avoidance will always refer to consecutive patterns.

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- Occurrences of 21 are *descents*.
- Occurrences of 132 and 231 are *peaks*.
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Consecutive patterns arise naturally in dynamical systems, and play a role in distinguishing deterministic from random sequences.

Notation

For a fixed pattern σ, let

$$P_\sigma(u,z) = \sum_{n\geq 0} \sum_{\pi\in\mathcal{S}_n} u^{\#\{\text{occurrences of } \sigma \text{ in } \pi\}} \frac{z^n}{n!},$$

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$$P_\sigma(0,z) = \sum_{n\geq 0} \alpha_n(\sigma) \frac{z^n}{n!},$$

where $\alpha_n(\sigma) = \#\{\pi \in \mathcal{S}_n : \pi \text{ avoids } \sigma\}$.

Some questions being studied

- Exact enumeration: find $P_\sigma(u,z)$ or $P_\sigma(0,z)$.

  In this talk: Formulas for $P_\sigma(u,z)$ for σ of certain shapes.

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- Classification of patterns according to *c-Wilf-equivalence*.
  We write $\sigma \sim \tau$ if $P_\sigma(u,z) = P_\tau(u,z)$.

  Example: $1342 \sim 1432$.

  In this talk: Classification of patterns of length up to 6.

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  In this talk: Classification of patterns of length up to 6.

- Comparison of $\alpha_n(\sigma)$ for different patterns.

  Example: $\alpha_n(132) < \alpha_n(123)$ for $n \geq 4$.

  In this talk: For which pattern $\sigma \in \mathcal{S}_m$ is $\alpha_n(\sigma)$ largest.

Patterns of small length

Length 3: **2** c-Wilf classes (compare: 1 Wilf class in classical case)

$123 \sim 321$

$132 \sim 231 \sim 312 \sim 213$

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Length 4: **7** c-Wilf classes (compare: 3 Wilf classes in classical case)

$1234 \sim 4321$

$2413 \sim 3142$

$2143 \sim 3412$

$1324 \sim 4231$

$1423 \sim 3241 \sim 4132 \sim 2314$

$1342 \sim 2431 \sim 4213 \sim 3124 \overset{*}{\sim} 1432 \sim 2341 \sim 4123 \sim 3214$

$1243 \sim 3421 \sim 4312 \sim 2134$

All $\sim$ follow from reversal and complementation except for $\overset{*}{\sim}$.

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$1234 \sim 4321$ enumeration solved

$2413 \sim 3142$ enumeration unsolved

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Clusters

We use an adaptation of the cluster method of Goulden and Jackson, based on inclusion-exclusion.

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Example: $\underline{14\overline{25}}\overline{3}\underline{\overline{6}879}$ is a 3-cluster w.r.t. 1324.

The cluster method

Let the EGF for clusters be

$$C_\sigma(u,z) = \sum_{n,k} c^\sigma_{n,k} u^k \frac{z^n}{n!},$$

where $c^\sigma_{n,k}$:= number of k-clusters of length n w.r.t. σ.

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Theorem (Goulden-Jackson '79, adapted)

$$P_\sigma(u,z) = \frac{1}{1 - z - C_\sigma(u-1,z)} \stackrel{\text{def}}{=} \frac{1}{\omega_\sigma(u,z)}.$$

This reduces the computation of $P_\sigma(u,z)$ to the enumeration of clusters.

Clusters as linear extensions of posets

$\underline{\pi_1\pi_2\overline{\pi_3\pi_4\pi_5}}\overline{\pi_6\underline{\pi_7}}\underline{\pi_8\pi_9\pi_{10}\pi_{11}}$ is a cluster w.r.t. $\sigma = 14253$

$$\Updownarrow$$

$$\pi_1 < \pi_3 < \pi_5 < \pi_2 < \pi_4$$

$$\pi_3 < \pi_5 < \pi_7 < \pi_4 < \pi_6$$

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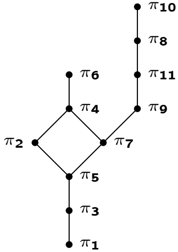
π is a linear extension of the poset given by these relations (called a cluster poset)



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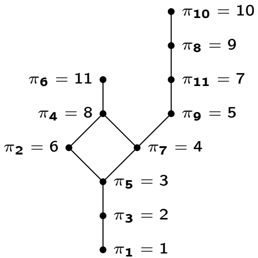

Ex: $\underline{1\ 6\ \overline{2\ 8\ 3}}\ \overline{11\ \underline{4}}\ \underline{9\ 5\ 10\ 7}$

The pattern $\sigma = 12\ldots m$ and generalizations

Theorem (Goulden-Jackson '83, E.-Noy '01)

For $\sigma = 12\ldots m$, $\omega_\sigma(u,z)$ *is the solution of*

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More generally, we get a similar differential equation for any σ such that all its cluster posets are chains. For example,

$$\sigma = 12\dots(s-1)(s+1)s(s+2)(s+3)\dots m.$$

Non-overlapping patterns

$\sigma \in \mathcal{S}_m$ is non-overlapping if two occurrences of σ can't overlap in more than one position.

Example: 132, 1243, 1342, 21534, 34671285 are non-overlapping.

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Similar arguments give differential equations for $\sigma = 12534$ and $\sigma = 13254$, which aren't non-overlapping.

The pattern $134\ldots(s+1)2(s+2)(s+3)\ldots m$

Theorem (E.-Noy, Liese-Remmel, Dotsenko-Khoroshkin)

For $\sigma = 1324$, $\omega_\sigma(u,z)$ *is the solution of*

$$z\omega^{(5)}-((u-1)z-3)\omega^{(4)}-3(u-1)(2z+1)\omega^{(3)}+(u-1)((4u-5)z-6)\omega'' + (u-1)(8(u-1)z-3)\omega' + 4(u-1)^2 z\omega = 0$$

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The construction generalizes to patterns of the form

$$\sigma = 134\ldots(s+1)2(s+2)(s+3)\ldots m.$$

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For the remaining cases, 1423, 2143 and 2413, we do not know of similar differential equations for $\omega_\sigma(u, z)$.

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"Conjecture" (Noonan-Zeilberger '96)

For every classical pattern σ *(i.e., where occurrences are not constrained to consecutive positions), the generating function for* σ*-avoiding permutations is D-finite.*

Consecutive Wilf-equivalence

One can classify patterns of length up to 6 into c-Wilf-equivalence classes, proving four conjectures of Nakamura:

n	# of classes
3	2
4	7
5	25
6	92

Theorem (E.-Noy)

- $123546 \sim 124536 \rightarrow$ *solution of* $\omega^{(5)} + (1-u)(\omega' + \omega) = 0$.
- $123645 \sim 124635 \rightarrow$ *solution of* $\omega^{(5)} + (1-u)z(\omega'' + \omega') = 0$.
- $132465 \sim 142365 \rightarrow$ *solution of* $\omega^{(5)} + (1-u)(\omega'' + z\omega') = 0$.
- $154263 \sim 165243$.

Asymptotic behavior

Theorem (E. '05)

For every σ, the limit

$$\rho_\sigma := \lim_{n\to\infty} \left(\frac{\alpha_n(\sigma)}{n!}\right)^{1/n} \qquad \textit{exists.}$$

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Theorem (Ehrenborg-Kitaev-Perry '11)

For every σ,

$$\frac{\alpha_n(\sigma)}{n!} = \gamma_\sigma \rho_\sigma^n + O(\delta^n),$$

for some constants γ_σ and $\delta < \rho_\sigma$.

The proof uses methods from spectral theory.

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Theorem (E.)

For every $\sigma \in \mathcal{S}_m$ there exists n_0 such that

$$\alpha_n(\sigma) \leq \alpha_n(12\ldots m)$$

for all $n \geq n_0$.

Interestingly, the analogous result for classical patterns (i.e., without the adjacency requirement) is false.

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The theorem is equivalent to ρ_σ being largest for $\sigma = 12 \dots m$.

Proof idea — 1. Singularity analysis

Let $\sigma \in \mathcal{S}_m \setminus \{12\dots m, m\dots 21\}$. Want to show: $\rho_\sigma < \rho_{12\dots m}$.

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$$P_\sigma(0,z) = \frac{1}{\omega_\sigma(0,z)} = \sum_{n\geq 0} \alpha_n(\sigma)\frac{z^n}{n!},$$

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One can show that $\omega_\sigma(z) := \omega_\sigma(0,z)$ is analytic near the origin, so

- ρ_σ^{-1} is the smallest zero of $\omega_\sigma(z)$,
- $\rho_{12\dots m}^{-1}$ is the smallest zero of $\omega_{12\dots m}(z)$.

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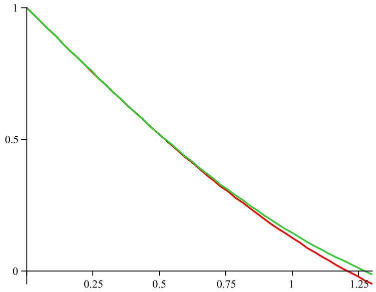
To show that $\rho_\sigma < \rho_{12\dots m}$, it is enough to show that

$$\omega_{12\dots m}(z) < \omega_\sigma(z)$$

for $0 < z < 1.276$.



Proof idea — 2. Comparing cluster numbers

We show that $\omega_{12\dots m}(z) < \omega_\sigma(z)$ for $0 < z < 1.276$:

$$\omega_{12\dots m}(z) = \sum_{j\geq 0}\left(\frac{z^{jm}}{(jm)!} - \frac{z^{jm+1}}{(jm+1)!}\right) < 1 - z + \frac{z^m}{m!} - \frac{z^{m+1}}{(m+1)!} + \frac{z^{2m}}{(2m)!},$$

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$$\omega_\sigma(z) = 1 - z - \sum_{k\geq 1}(-1)^k \underbrace{\sum_n r^\sigma_{n,k}\frac{z^n}{n!}}_{s^\sigma_k(z)}$$

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Key fact #1: The sequence $\{s^\sigma_k(z)\}_{k\geq 1}$ is decreasing.

Key fact #2: $s^\sigma_2(z) < \frac{z^{m+1}}{(m+1)!} - \frac{z^{2m}}{(2m)!}$.

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Theorem (E., conjectured by Nakamura)

For every $\sigma \in \mathcal{S}_m$ there exists n_0 such that

$$\alpha_n(123\ldots(m-2)m(m-1)) \leq \alpha_n(\sigma)$$

for all $n \geq n_0$.

Thank you