Zeros of Generalized Eulerian Polynomials

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Outline

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Polynomials with (only) real zeros

Combinatorics, algebra, geometry, analysis, . . .

Surveys by: Stanley ('86), Brenti ('94), Brändén (2014+).

Combinatorial significance

Consider a generating polynomial

$$\sum_{k=0}^{n} a_k x^k ,$$

if it has only real zeros then the coefficients are known to be

strongly log-concave: $\frac{a_k^2}{\binom{n}{k}\binom{n}{k}} \geqslant \frac{a_{k-1}}{\binom{n}{k-1}} \frac{a_{k+1}}{\binom{n}{k+1}}$

log-concave: $a_k^2 \geqslant a_{k-1} a_{k+1}$

unimodal: $a_0 \leqslant \cdots \leqslant a_m \geqslant \cdots \geqslant a_n$ (if $a_k > 0$)

Other, geometrically inspired notions: γ-*nonnegativity*.

Outline

Eulerian polynomials

as generating polynomials

For a permutation $\pi = \pi_1 \dots \pi_n$ in $\mathfrak{S}_n$, let

$$\mathrm{des}(\pi) = |\{i \mid \pi_i > \pi_{i+1}\}|$$

denote the number of *descents* in π.

Definition

The Eulerian polynomial is

$$\mathfrak{S}_n(x) := \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{des}(\pi)} = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle x^k,$$

where $\left\langle {n \atop k} \right\rangle = |\{\pi \in \mathfrak{S}_n | \, \mathrm{des}(\pi) = k\}|$.

Eulerian numbers: $\left\langle {n \atop k} \right\rangle$

Euler's triangle

		k:					
		0	1	2	3	4	5
n:	1	1					
	2	1	1				
	3	1	4	1			
	4	1	11	11	1		
	5	1	26	66	26	1	
	6	1	57	302	302	57	1

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- $\mathfrak{S}_1(x) = 1$,
- $\mathfrak{S}_2(x) = 1 + x$,
- $\mathfrak{S}_3(x) = 1 + 4x + x^2$,
- $\mathfrak{S}_4(x) = 1 + 11x + 11x^2 + x^3, \ldots$

The zeros of $\mathfrak{S}_n(x)$

Theorem (Frobenius)

$\mathfrak{S}_n(x)$ *has only (negative and simple) real zeros.*

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Corollary

For all $n \geqslant 1$*, the Eulerian numbers*

$$\left\langle {n \atop 0} \right\rangle, \left\langle {n \atop 1} \right\rangle, \ldots, \left\langle {n \atop n-1} \right\rangle$$

form a (strongly) log-concave, and hence unimodal sequence.

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Most proofs of the theorem rely on the recurrence:

$$\mathfrak{S}_{n+1}(x) = (1+nx)\mathfrak{S}_n(x) + x(1-x)\mathfrak{S}_n'(x).$$

Frobenius' proof (via interlacing)

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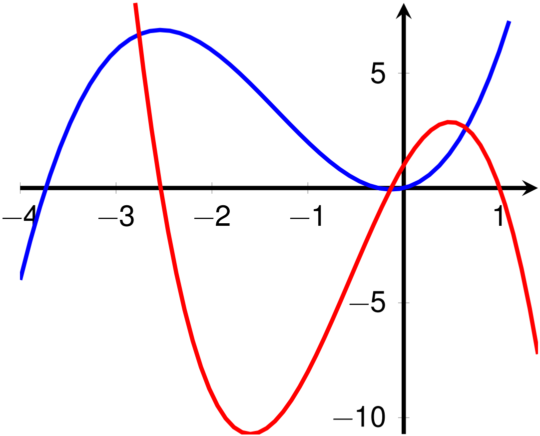
$x(1+4x+x^2)$



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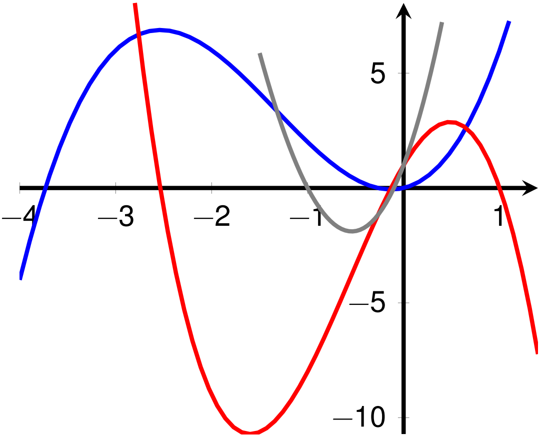

Interlacing polynomials

Theorem (Obreschkoff)

- $f(x)$ *and* $g(x)$ *have interlacing zeros*
- $\lambda f + \mu g$ *has only real zeros for any* $\lambda, \mu \in \mathbb{R}$.

Problem with this method:

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Problem with this method: Does not scale.

$$\begin{aligned} D_{n+2}(x) \quad = \quad & (n(1+5x)+4x)D_{n+1}(x)+4x(1-x)D'_{n+1}(x) \\ & +((1-x)^2-n(1+3x)^2-4n(n-1)x(1+2x))D_n(x) \\ & -(4nx(1-x)(1+3x)+4x(1-x)^2)D'_n(x)-4x^2(1-x)^2D''_n(x) \\ & +(2n(n-1)x(3+2x+3x^2)+4n(n-1)(n-2)x^2(1+x))D_{n-1}(x) \\ & +(2nx(1-x)^2(3+x)+8n(n-1)x^2(1-x)(1+x))D'_{n-1}(x) \\ & +4nx^2(1-x)^2(1+x)D''_{n-1}(x). \end{aligned}$$

Outline

Compatible polynomials

Definition

The polynomials $f_1(x), \ldots, f_m(x)$ over $\mathbb{R}$ are *compatible*, if all their conic combinations, i.e., the polynomials

$$\sum_{i=1}^{m} c_i f_i(x) \quad \text{for all } \; c_1, \ldots, c_m \geqslant 0$$

have only real zeros.

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have only real zeros.

Remark (Chudnovsky–Seymour)

- $f_1(x), \ldots, f_m(x)$ *are compatible if and only if*
- $f_1(x), \ldots, f_m(x)$ *have a common interleaver* $g(x)$.

Compatible polynomials

Definition

The polynomials $f_1(x), \ldots, f_m(x)$ are *pairwise compatible* if

$$f_i(x) \text{ and } f_j(x) \text{ are compatible}$$

for all $1 \leqslant i < j \leqslant m$.

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Lemma (Chudnovsky–Seymour)

The polynomials $f_1(x), \ldots, f_m(x)$ are compatible if and only if they are pairwise compatible.

Advantage of compatible polynomials

- Can handle nonnegative sum of *many* polynomials.
- Enough to prove *pairwise* compatibility.

Outline

Intro

- Polynomials with (only) real zeros
- Eulerian polynomials

Toolbox

- Compatible polynomials
- Inversion sequences

s-Eulerian polynomials

- Generalized inversion sequences
- Proving real zeros via compatible polynomials
- Consequences

Summary

Inversion sequences

an alternative way to represent $\mathfrak{S}_n$

Definition

The inversion sequence of a permutation $\pi = \pi_1 \cdots \pi_n$ is an n-tuple

$$\mathbf{e} = (e_1, \ldots, e_n),$$

where

$$e_j = \left| \left\{ i \in \{1, 2, \ldots, j-1\} \mid \pi_i > \pi_j \right\} \right|$$

counts the number of inversions "ending" in the jth position.

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Example ($n = 3$)

$\pi_1\pi_2\pi_3$	1 2 3	1 3 2	2 1 3	2 3 1	3 1 2	3 2 1
$e_1e_2e_3$	0 0 0	0 0 1	0 1 0	0 0 2	0 1 1	0 1 2

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$e_1 e_2 e_3$	0 0 0	0 0 1	0 1 0	0 0 2	0 1 1	0 1 2

Variants known under different names: *Lehmer code, inversion code, inversion table, etc.*

Inversion sequences

ascent statistic

Definition

For an inversion sequence $e = (e_1, \ldots, e_n) \in I_n$, let

$$\mathrm{asc}_I(e) = |\{i \in \{1, \ldots, n-1\} : e_i < e_{i+1}\}| \ ,$$

denote the number of ascents in e.

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$e_1 e_2 e_3$	$\mathrm{asc}_{\mathrm{I}}(\boldsymbol{e})$
0 0 0	0
0 0 1	1
0 0 2	1
0 1 0	1
0 1 1	1
0 1 2	2

Observation

The ascent statistics over inversion sequences is Eulerian.

Theorem (Savage–Schuster)

$$\sum_{e \in \mathrm{I}_n} x^{\mathsf{asc}_{\mathrm{I}}(e)} = \sum_{\pi \in \mathfrak{S}_n} x^{\mathsf{des}(\pi)}.$$

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0 0 0	0	1 2 3	0
0 0 1	1	1 3 2	1
0 0 2	1	2 3 1	1
0 1 0	1	2 1 3	1
0 1 1	1	3 1 2	1
0 1 2	2	3 2 1	2

Advantage of inversion sequences

- Easy recurrence, the *change* in the ascent statistic

$$\text{asc}_I(e) = |\{i \in \{1, \ldots, n-1\} : e_i < e_{i+1}\}| \,,$$

only depends on the last entry.

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  only depends on the last entry.

- Lend themselves to generalizations.

Outline

Generalized inversion sequences

Recall some facts about the inversion sequences:

$$\begin{aligned} I_n &= \{(e_1, \ldots, e_n) \in \mathbb{Z}^n \mid 0 \leqslant e_i < i\} \\ &= \{0\} \times \{0, 1\} \times \cdots \times \{0, 1, \ldots, n-1\}. \end{aligned}$$

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Definition

For a given sequence $s = (s_1, \ldots, s_n) \in \mathbb{N}^n$, let $I_n^{(s)}$ denote the set of s-*inversion sequences* by

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$$I_n^{(s)} = \{0, \ldots, s_1 - 1\} \times \{0, \ldots, s_2 - 1\} \times \cdots \times \{0, \ldots, s_n - 1\}.$$

The *ascent* statistic on s-inversion sequences

Savage and Schuster extended the definition of the *ascent* statistic to s-inversion sequences.

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For $\boldsymbol{e} = (e_1, \ldots, e_n) \in I_n^{(s)}$, let

$$\text{asc}_I(\boldsymbol{e}) = \left| \left\{ i \in \{0, \ldots, n-1\} : \frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}} \right\} \right| ,$$

where we use the convention $e_0 = 0$ (and $s_0 = 1$).

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where we use the convention $e_0 = 0$ (and $s_0 = 1$).

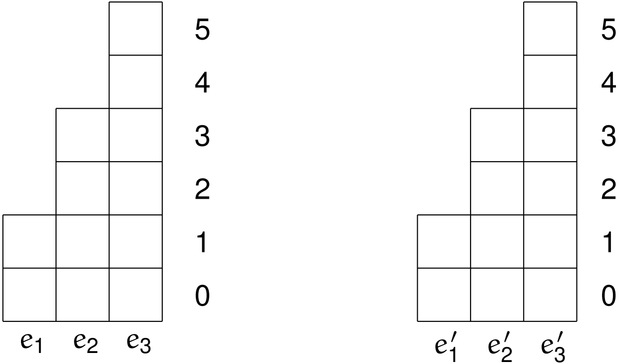
Fact: The case $s_i = i$ agrees with usual inversion sequences.

$$e_i < e_{i+1} \iff \frac{e_i}{i} < \frac{e_{i+1}}{i+1} ,$$

whenever $0 \leqslant e_k < k$, for all k.

Examples

The ascent statistic on s-inversion sequences

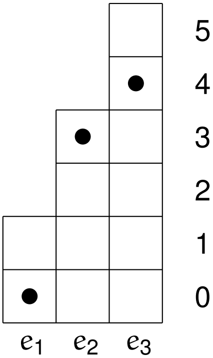


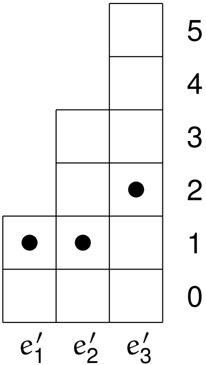
Two examples for the sequence $s = (2, 4, 6)$

Examples

The ascent statistic on s-inversion sequences

$\mathbf{e} = (0, 3, 4)$ $\mathbf{e}' = (1, 1, 2)$





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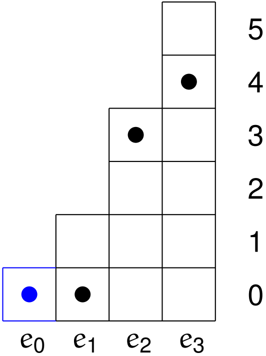

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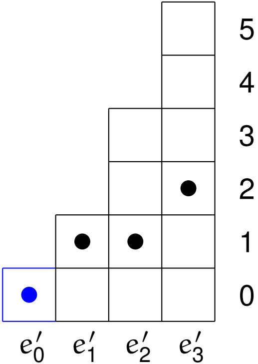

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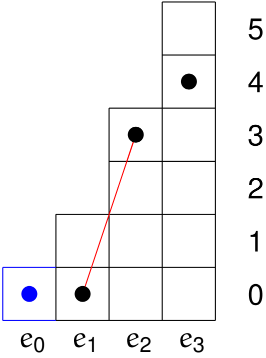
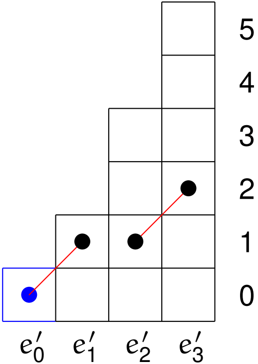
$e' = (1,1,2)$ with

- $\mathrm{asc}_I(e') = 2$.





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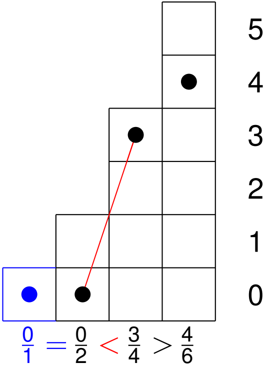

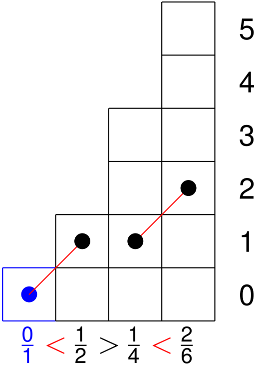

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s-Eulerian polynomials

Recall that

$$\mathfrak{S}_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{des}(\pi)} = \sum_{e \in I_n^{(s)}} x^{\mathrm{asc}_I(e)},$$

when $s = 1, 2, \ldots, n$.

s-Eulerian polynomials

Definition (s-Eulerian polynomials)

For an arbitrary sequence $s = s_1, s_2, \ldots,$ let

$$\mathcal{E}_n^{(s)}(x) := \sum_{e \in I_n^{(s)}} x^{\mathrm{asc}_I(e)}.$$

Recall that

$$\begin{aligned}\mathfrak{S}_n(x) &= \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{des}(\pi)} \\ &= \sum_{e \in I_n^{(s)}} x^{\mathrm{asc}_I(e)},\end{aligned}$$

when $s = 1, 2, \ldots, n$.

s-Eulerian polynomials

and why do we care

Special cases of s-Eulerian polynomials, $\mathcal{E}_n^{(\mathbf{s})}(x)$:

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s-Eulerian polynomials

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Special cases of s-Eulerian polynomials, $\mathcal{E}_n^{(s)}(x)$:

- $s = (1, 2, \ldots, n)$: the Eulerian polynomial, $\mathfrak{S}_n(x)$,
- $s = (2, 4, \ldots, 2n)$: the type B Eulerian polynomial, $B_n(x)$,

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- $s = (k, 2k, \ldots, nk)$: the descent polynomial for the wreath products, $G_{n,r}(x)$,

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- $s = (k+1, 2k+1, \ldots, (n-1)k+1)$: the $1/k$-Eulerian polynomial, $x^{\text{exc}\,\pi}(1/k)^{\text{cyc}\,\pi}$,

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- $s = (1, 1, 3, 2, 5, 3, 7, 4, \ldots, 2n-1, n)$: the descent polynomial for the multiset $\{1, 1, 2, 2, \ldots, n, n\}$.

Outline

On the zeros of s-Eulerian polynomials

The theorem of Frobenius can be generalized to the following.

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The theorem of Frobenius can be generalized to the following.

Theorem (Savage, V.)

For any sequence $\mathbf{s}$ of nonnegative integers, the $\mathbf{s}$-Eulerian polynomials

$$\mathcal{E}_n^{(s)}(x) = \sum_{e \in I_n^{(s)}} x^{\mathrm{asc}_I(e)}$$

have only real zeros.

Proving more is sometimes easier...

Instead of working with $\mathcal{E}_n^{(s)}(x) = \sum_{e \in I_n^{(s)}} x^{\mathrm{asc}_I(e)}$ we will be working with the partial sums

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IDEA: $P_{n,i}^{(s)}(x)$ are compatible $\Longrightarrow$ $\mathcal{E}_n^{(s)}(x)$ has only real zeros.

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$$P^{(s)}_{n+1,i}(x) = \sum_{j=0}^{\ell-1} x P^{(s)}_{n,j}(x) + \sum_{j=\ell}^{s_n-1} P^{(s)}_{n,j}(x)$$

A simple recurrence

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Back to the proof of the main result

Again, prove something stronger

Theorem (Savage, V.)

Given a sequence $s = \{s_i\}_{i \geqslant 1}$, *for all* $0 \leqslant i \leqslant j < s_n$,

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$P^{(s)}_{n,0}(x), P^{(s)}_{n,1}(x), \ldots, P^{(s)}_{n,s_n-1}(x)$ *are compatible.*

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Proof of compatibility

Use induction. Base case: $(x, 1)$ or (x, x) or (x^2, x).

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Now

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Q.E.D.

Outline

The Eulerian polynomials have only real zeros

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have only real zeros.

Combinatorics of Coxeter groups

The descents can be defined in a more general setting.

Definition (Björner–Brenti)

Let S be a set of Coxeter generators, m be a Coxeter matrix, and

$$W = \langle S : (ss')^{m(s,s')} = \mathrm{id}, \text{ for } s, s' \in S, m(s,s') < \infty \rangle$$

be the corresponding Coxeter group. Given a pair (W, S) and $\sigma \in W$, let $\ell_W(\sigma)$ be the length of σ in W with respect to S.

Eulerian polynomials for Coxeter groups

Definition

For W a finite Coxeter group, with generator set $S = \{s_1, \ldots, s_n\}$ the descent set of $\sigma \in W$ is

$$\mathcal{D}_W(\sigma) = \{s_i \in S : \ell_W(\sigma s_i) < \ell_W(\sigma s_i)\}.$$

$$W(x) = \sum_{\sigma \in W} x^{|\mathcal{D}_W(\sigma)|}.$$

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Conjecture (Brenti)

Eulerian polynomials $W(x)$ for all Coxeter groups W have only real zeros.

Eulerian polynomials for Coxeter groups

Theorem (Brenti)

The Eulerian polynomial for type B_n *and for all the exceptional Coxeter groups has only real zeros.*

Eulerian polynomials for Coxeter groups

Theorem (Brenti)

The Eulerian polynomial for type B_n and for all the exceptional Coxeter groups has only real zeros.

Observation: Eulerian polynomials are "multiplicative". Enough to consider irreducible groups. Type D_n is the last remaining piece of the puzzle. (Verified up to $n = 100$.)

Eulerian polynomials for type D_n

Conjecture (Brenti)

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Eulerian polynomials for type D_n

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1. Why did type D_n resist so far?
2. Motivating question raised by Krattenthaler at SLC (Strobl): Why don't you apply your method to type D_n?

Answer to the first question

Why did type D_n resist so far?

Combinatorial definition is not as "pretty." Think of elements of B_n (resp. D_n) as signed (resp. even-signed) permutations. The definition of descents:

$$\mathrm{des}_B(\sigma) = |\{i \mid \sigma_i > \sigma_{i+1}\} \cup \{0 \mid \sigma_1 > 0\}|$$

$$\mathrm{des}_D(\sigma) = |\{i \mid \sigma_i > \sigma_{i+1}\} \cup \{0 \mid \sigma_1 + \sigma_2 > 0\}|$$

No "nice" recurrence. The only recurrence (due to Chow) is rather complicated.

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$$\begin{aligned}
D_{n+2}(x) \quad = \quad & (n(1+5x)+4x)D_{n+1}(x) + 4x(1-x)D'_{n+1}(x) \\
& +((1-x)^2 - n(1+3x)^2 - 4n(n-1)x(1+2x))D_n(x) \\
& -(4nx(1-x)(1+3x)+4x(1-x)^2)D'_n(x) - 4x^2(1-x)^2D''_n(x) \\
& +(2n(n-1)x(3+2x+3x^2)+4n(n-1)(n-2)x^2(1+x))D_{n-1}(x) \\
& +(2nx(1-x)^2(3+x)+8n(n-1)x^2(1-x)(1+x))D'_{n-1}(x) \\
& +4nx^2(1-x)^2(1+x)D''_{n-1}(x).
\end{aligned}$$

Answer to the second question

Why don't you apply your method to type D_n?

Short answer: Does not work.

Answer to the second question

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Long answer: Does not work *out of the box*.

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- Trick 2 Find an ascent statistic for $2D_n(x)$.
- Trick 3 Believe in your method!

Trick 1

Getting rid of parity

Recall,

$$\mathrm{des}_D(\sigma) = |\{i \mid \sigma_i > \sigma_{i+1}\} \cup \{0 \mid \sigma_1 + \sigma_2 > 0\}|.$$

Proposition

For $n \geqslant 2$,

$$\sum_{\sigma \in B_n} x^{\mathrm{des}_D \sigma} = 2 \sum_{\sigma \in D_n} x^{\mathrm{des}_D \sigma}.$$

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$$\bar{2}\bar{1}56\bar{3}4 \Longleftrightarrow \bar{2}156\bar{3}4$$

Trick 2

A type D_n ascent statistic

$$\mathrm{Asc}_A(e) = \left\{i \mid \frac{e_i}{i} < \frac{e_{i+1}}{i+1}\right\}$$

$$\mathrm{Asc}_B(e) = \left\{i \mid \frac{e_i}{i} < \frac{e_{i+1}}{i+1}\right\} \cup \{0 \mid e_1 > 0\}$$

$$\mathrm{Asc}_D(e) = \left\{i \mid \frac{e_i}{i} < \frac{e_{i+1}}{i+1}\right\} \cup \left\{0 \mid e_1 + \frac{e_2}{2} \geqslant \frac{3}{2}\right\}$$

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Trick 2

A type D_n ascent statistic

$$2D_n(x) = \sum_{e \in I_n^{2,4,6,\ldots}} x^{\mathrm{Asc}_D(e)}.$$

$$\mathrm{Asc}_A(e) = \left\{ i \mid \frac{e_i}{i} < \frac{e_{i+1}}{i+1} \right\}$$

$$\mathrm{Asc}_B(e) = \left\{ i \mid \frac{e_i}{i} < \frac{e_{i+1}}{i+1} \right\} \cup \{0 \mid e_1 > 0\}$$

$$\mathrm{Asc}_D(e) = \left\{ i \mid \frac{e_i}{i} < \frac{e_{i+1}}{i+1} \right\} \cup \left\{0 \mid e_1 + \frac{e_2}{2} \geqslant \frac{3}{2} \right\}$$

$$\mathrm{Des}_A(\sigma) = \{i \mid \sigma_i > \sigma_{i+1}\}$$

$$\mathrm{Des}_B(\sigma) = \{i \mid \sigma_i > \sigma_{i+1}\} \cup \{0 \mid \sigma_1 > 0\}$$

$$\mathrm{Des}_D(\sigma) = \{i \mid \sigma_i > \sigma_{i+1}\} \cup \{0 \mid \sigma_1 + \sigma_2 > 0\}$$

Putting all together

A recursive proof for type D_n

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Also, $D_{3,0}(x), \ldots, D_{3,5}(x)$ are not compatible.

Trick 3

Leap of faith

$D_{4,0}(x), \ldots, D_{4,7}(x)$ are compatible. ✓

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By induction,

$$D_{n,0}(x), \ldots, D_{n,2n-1}(x)$$

are compatible for all $n \geqslant 4$.

Corollary

$D_n(x) = \sum_{i=0}^{2n-1} D_{n,i}(x)$ *has only real zeros.*

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- Unified proof of existing results, but also can be used to solve new problems (Brenti's type D conjecture).
- The method of compatible polynomials is a simple yet powerful method to prove real zeros.
- A reformulation, or even generalization (s-inversion sequences) often makes the problem easier.