# Many Looks on the Fibonacci Polynomials

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IMA Workshop

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# Dialog through the times ...

- "What's enumeration?" - Socrates (400 BC)
- "To count, or not to count." - Shakespeare (1600's AD)
- "Oh, rabbits!" - Fibonacci
- "Huh?" - Anonymous.

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# "partners in crime"

Mahir Can

Xi Chen

Melanie. Jensen

Victor Moll

Bruce Sagan

# Zero, One, One, Two, Three, Five, Eight, ...

$F_0 = 0, F_1 = 1$ and

$$F_n = F_{n-1} + F_{n-2}.$$

$\{0\}_{s,t} = 0, \{1\}_{s,t} = 1$ and

$$\{n\}_{s,t} = s\{n-1\}_{s,t} + t\{n-2\}_{s,t}.$$

Examples:

$$\{2\} = s, \qquad \{3\} = s^2 + t,$$

$$\{4\} = s^3 + 2st, \qquad \{5\} = s^4 + 3s^2t + t^2.$$

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# analogue of Binet

Let $X$ and $Y$ be roots of

$$z^2 - sz - t = 0.$$

Then

$$\{n\} = \frac{X^n - Y^n}{X - Y},$$

$$\langle n \rangle = X^n + Y^n.$$

# Tilings and explicit forms

**Proposition.** Let
$L_n = \{T : T \text{ a linear tiling of a row of } n \text{ squares}\}$. Then

$$\{n+1\} = \sum_{T \in L_n} \mathrm{wt}\ T.$$

**Proposition.**

$$\{n\} = \sum_{k \geq 0} \binom{n-k-1}{k} s^{n-2k-1} t^k.$$

# Fibotorials and Fibonomials

$$\{n\}_{s,t}! = \prod_{i=1}^{n} \{i\}_{s,t},$$

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{s,t} = \frac{\{n\}_{s,t}!}{\{k\}_{s,t}!\{n-k\}_{s,t}!}.$$

**Theorem (Sagan-Savage).** Combinatorial interpretation of Fibonomials.

**Proof.** Tilings of a $k \times (n-k)$ rectangle containing a partition.
□

# Very useful

**Theorem.**

$$\{m+n\} = \{m\}\{n+1\} + t\{m-1\}\{n\}.$$

**Theorem (Hoggart-Long)**

$$\gcd(\{m\},\{n\}) = \{\gcd(m,n)\}.$$

# a cute identity

**Theorem.** For $s, t \in \mathbb{P}$, we have

$$\sum_{n=0}^{\infty} \frac{t(s+t-1)\{n\}_{s,t}}{(s+t)^{n+1}} = 1.$$

**Proof.** Generating function

$$\sum_{n \geq 0} \{n\} z^n = \frac{z}{1 - sz - tz^2}.$$

# enter arithmetic

*The $d$-adic valuation*

$$\nu_d(n) = \begin{cases} \text{the highest power of } d \text{ dividing } n, & n \neq 0 \\ \infty & n = 0. \end{cases}.$$

**Theorem.** Let $s, t$ be odd. Then

$$\nu_2(\{3k\}) = \begin{cases} 1 + \delta_E(k) \cdot (\nu_2(\{6\}) - 2), & t = 1 \mod 4 \\ \nu_2(k\{3\}) & t = 3 \mod 4. \end{cases}$$

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# ... more generally

**Theorem.** Let $d \geq 2$ be an integer. Then,

$$\nu_d(\{n\}_{d,-1}) = \delta_E(n) \cdot \nu_d(dn/2).$$

**Conjecture.** Let $s \geq 2$ and $d \geq 3$ odd. Then $s^*, d^*$ such that

$$\nu_d(\{n\}_{s,-1}) = \delta_{d^*\mathbb{Z}}(n) \cdot \nu_d(s^*n/d^*).$$

Recently proved by S. Park.

Note: $\{n\}_{\ell,-1}$ linked to *Lecture Hall Partitions*.

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# Generalized Lecture Hall Theorem ...

With $\{n\} = \{n\}_{\ell,-1}$,
the proof by **Bousquet-Mélou and Eriksson** utilizes

$$\prod_{j=1}^{n} \frac{\{n\} + \{n-1\} + \cdots + \{j\}}{\{j\}} \quad \in \quad \mathbb{N}.$$

**Conjecture.** Parity-splits

$$\prod_{j=1}^{n} \frac{\{2n\}^{2k-1} + \{2n-2\}^{2k-1} + \cdots + \{2j\}^{2k-1}}{\{2j\}} \quad \in \quad \mathbb{N}.$$

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# Euler-Cassini

*log concave* sequences

$$a_n^2 - a_{n-1}a_{n+1} \geq 0 \qquad \Leftrightarrow \qquad a_n a_{n+m-1} \geq a_{n-1}a_{n+m}.$$

See works by Brenti, Stanley and Wilf.

**Example.** For ordinary Fibonacci numbers,

$$F_{rn}^2 - F_{r(n+1)}F_{r(n-1)} = (-1)^r F_r^2.$$

Cigler's *q-extension*: $\{0\}(q) = 0, \{1\}(q) = 1$ and

$$\{n\}(q) = s\{n-1\}(q) + tq^{n-2}\{n-2\}(q).$$

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# Cassini a la Dodgson

**Theorem.**

$$\{n\}_{s,t}(q) \cdot \{n\}_{s,qt}(q) - \{n-1\}_{s,qt}(q) \cdot \{n+1\}_{s,t}(q) = (-t)^{n-1} q^{\binom{n}{2}}.$$

**Proof.** Write $\{n\}(q)$ as a determinant and use

$$\det M = \frac{NW \cdot SE - SW \cdot NE}{CTR}. \qquad \square$$

$$\det \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

$$= \frac{\det \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \det \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix} - \det \begin{pmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{pmatrix} \det \begin{pmatrix} A_{12} & A_{13} \\ A_{22} & A_{23} \end{pmatrix}}{\det \left( A_{22} \right)}$$

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# tails of Riemann a la Cassini

Fibonacci analogue of the Riemann zeta

$$\zeta_F(z) = \sum_{k=1}^{\infty} \frac{1}{F_k^z}.$$

**Theorem.** Let $s \geq t \geq 1$ and $n, r \in \mathbb{P}$. Then,

$$\left\lfloor \left( \sum_{k=n}^{\infty} \frac{1}{\{rk\}_{s,t}} \right)^{-1} \right\rfloor = \{rn\}_{s,t} - \{r(n-1)\}_{s,t} - \delta_E(r(n-1)),$$

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# enter Catalanomials

Notations. Ask Stanley what $C_n$ is.

$$d_{\boldsymbol{b}}(n) = \#\text{ of non-zero digits of } n \text{ is base } \boldsymbol{b}.$$

**Well-known.** $\nu_2(C_n) = d_{\mathbf{2}}(n+1) - 1$. Here $\mathbf{2} = (1, 2, 2^2, \ldots)$.

By analogy, define

$$C_{\{n\}} = \frac{1}{\{n+1\}} \left\{ \begin{matrix} 2n \\ n \end{matrix} \right\}.$$

Ekhad noted

$$C_{\{n\}} = \left\{ \begin{matrix} 2n-1 \\ n-1 \end{matrix} \right\} + t \left\{ \begin{matrix} 2n-1 \\ n-2 \end{matrix} \right\}.$$

"Is there a combinatorial interpretation?" - Shapiro.

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# ...what power of 2 divides ...

**Theorem.** Let $s, t$ be of opposite parity. Then,

$$\nu_2(C_{\{n\}}) \begin{cases} d_{\mathbf{2}}(n+1)-1, & t \text{ odd} \\ d_{\boldsymbol{F}}(n+1)-1, & t \text{ even.} \end{cases}$$

**Theorem.** Let $\boldsymbol{F} = (1, 3, 3\cdot 2, 3\cdot 2^2, 3\cdot 3^3, \dots)$ and $s, t$ odd.

Then,

$$\nu_2(C_{\{n\}}) \begin{cases} d_{\boldsymbol{F}}(n+1) - \nu_2(\{6\}) - 3 \\ d_{\boldsymbol{F}}(n+1) - 1. \end{cases}$$

# … square-freeness

**Theorem.** Let $1 \neq m$ divide $n$. Viewed as polynomials:

$$\{m\}^2 \text{ does not divide } \{n\}.$$

For example, $\{n\}^2$ does not divide $\{n^2\}$.

Introduce flat and sharp analogues

$$\{n\}^\flat = \{p_1\} \cdots \{p_r\}, \qquad \{n\}^\sharp = \frac{\{n\}}{\{n\}^\flat}.$$

**Theorem.** The following are polynomials in $\mathbb{N}[s, t]$:

$$\left\{ {n \atop k} \right\}^\flat, \qquad \left\{ {n \atop k} \right\}^\sharp, \qquad C^\flat_{\{n\}}, \qquad C^\sharp_{\{n\}}.$$

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# SQUEEZE ... POUR ...

# … and ENJOY!