Hard tiling problems with triangles and rhombi

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(Joint work with Igor Pak)

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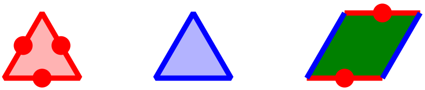
Tiling



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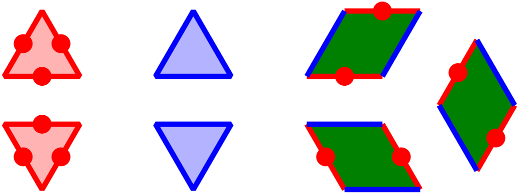
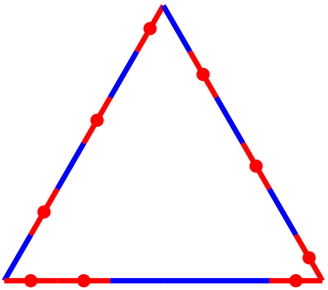
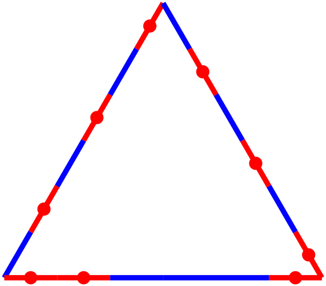
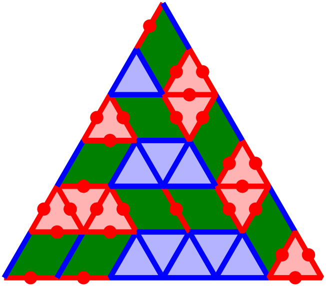


Puzzles



Puzzles





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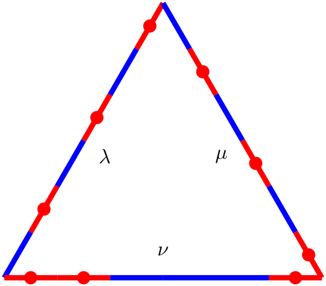

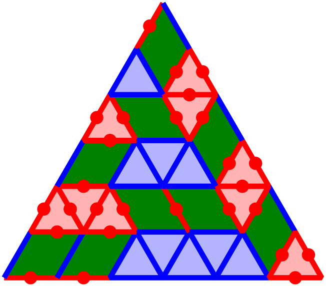

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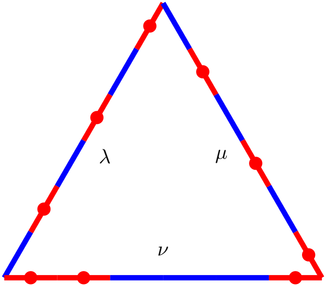
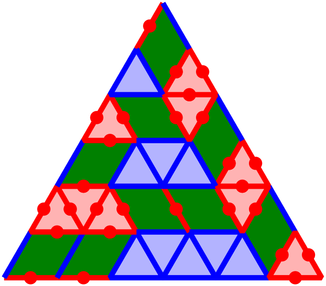
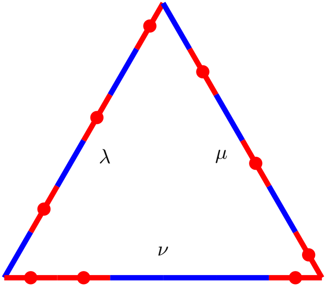
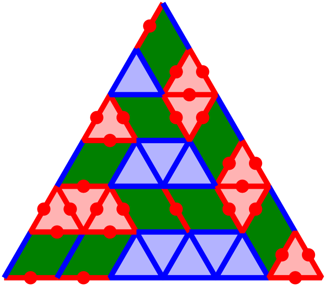




$r(\lambda) = \#$ of red edges in λ
$|\lambda| = \#$ of times red edges to the left of blue edges in λ

Puzzles



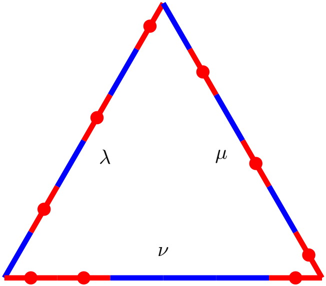


$r(\lambda) = \#$ of red edges in λ
$|\lambda| = \#$ of times red edges to the left of blue edges in λ

Tileable implies

- $r(\lambda) = r(\mu) = r(\nu)$,
- $|\lambda| + |\mu| = |\nu|$.

Partitions



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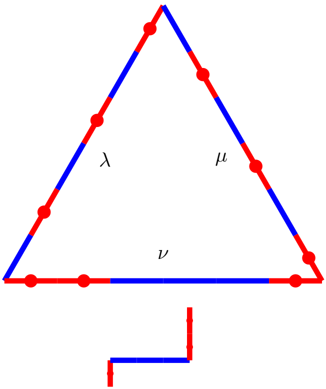

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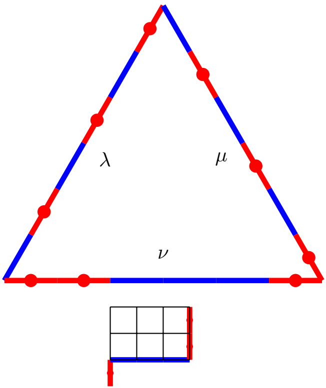

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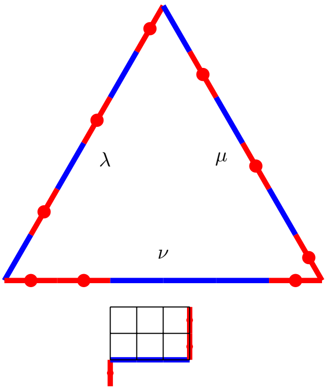
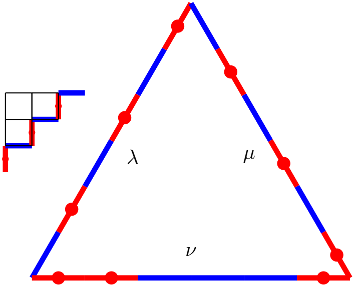
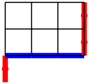


$\nu = 33$

Partitions





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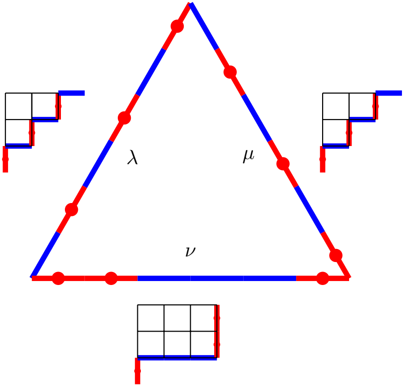

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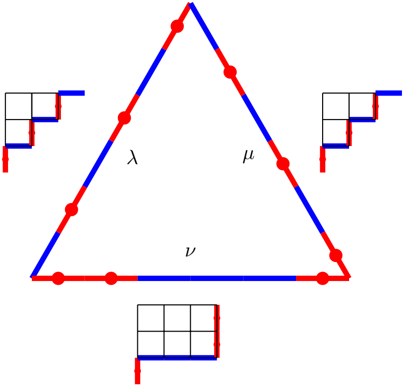

$\lambda = \mu = 21$, $\nu = 33$

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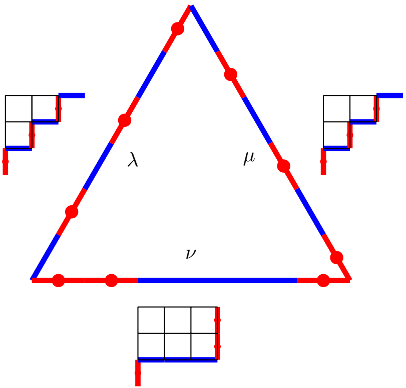

$\lambda = \mu = 21$, $\nu = 33$

Theorem (Knutson–Tao–Woodward 2001)

The number of tilings is the Littlewood–Richardson coefficient $c^{\nu}_{\lambda\mu}$.

Littlewood–Richardson coefficients

The Littlewood–Richardson coefficient $c^{\nu}_{\lambda\mu}$ appear as

1. multiplication of Schur functions: $s_\lambda s_\mu = \sum c^{\nu}_{\lambda\mu} s_\nu$
2. intersection numbers on a Grassmannian: $\sigma_\lambda \sigma_\mu = \sum c^{\nu}_{\lambda\mu} \sigma_\nu$
3. tensors of irreducible representations of $\mathrm{GL}_n(\mathbb{C})$: $V_\lambda \otimes V_\mu = \bigoplus c^{\nu}_{\lambda\mu} V_\nu$
4. induced representations of S_n: $\uparrow^{S_{|\nu|}}_{S_{|\lambda|} \times S_{|\mu|}} W_\lambda \otimes W_\mu = \bigoplus c^{\nu}_{\lambda\mu} W_\nu$

Complexity

Theorem (Narayanan 2006)

Computing $c^{\nu}_{\lambda\mu}$ *is* **#P***-complete, i.e., "hard."*

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Theorem (Knutson–Tao 2001)

Deciding $c^{\nu}_{\lambda\mu} > 0$ *is in* **P**, *i.e., "easy."*

Complexity

Theorem (Narayanan 2006)

*Computing $c_{\lambda\mu}^{\nu}$ is #**P**-complete, i.e., "hard."*

Theorem (Knutson–Tao 2001)

*Deciding $c_{\lambda\mu}^{\nu} > 0$ is in **P**, i.e., "easy."*

Example (Pak–Y. 2013)

Tileability of (contractible) regions in three dimensions by dominoes is in **P**, but counting is #**P**-complete.

Tileability

Tileability of

1. triangular regions is in **P**

Tileability

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1. triangular regions is in **P**
2. convex regions

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Example (Robson 1991; Kenyon–Kenyon 1992)

Tileability by straight trominoes is **NP**-complete in general

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but is in **P** when restricted to simply connected regions.

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Example (Robson 1991; Kenyon–Kenyon 1992)

Tileability by straight trominoes is **NP**-complete in general but is in **P** when restricted to simply connected regions.

Theorem (Pak–Y. 2014+)

Tileability of general regions by puzzle pieces is **NP***-complete.*

Planar 3SAT

Boolean expression in Conjunctive Normal Form

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$(x \vee \neg y \vee z) \wedge (x \vee \neg z \vee w)$

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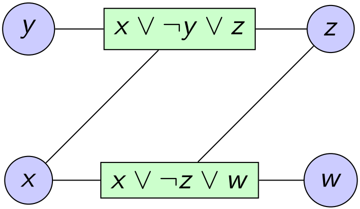

Planar 3SAT

Boolean expression in Conjunctive Normal Form
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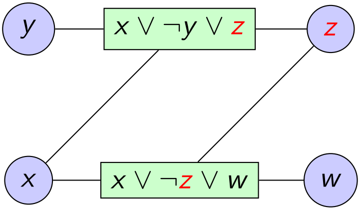

Planar 3SAT

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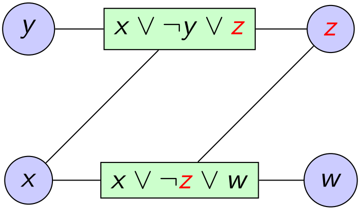

Theorem (Lichtenstein 1982)

PLANAR 3SAT *is* **NP**-*complete.*

Quasi-wire

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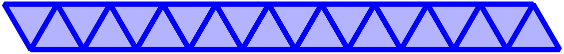
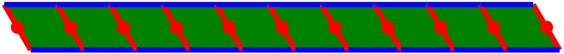
Can be tiled in two ways:





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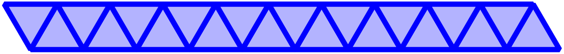
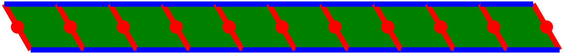
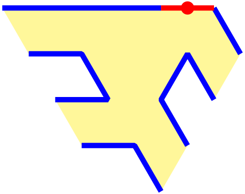
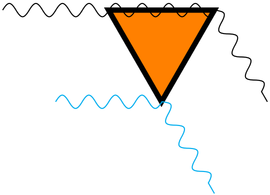




Schematic:

Bender





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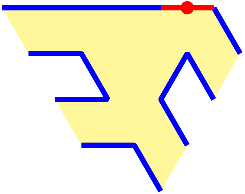

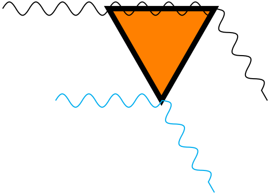

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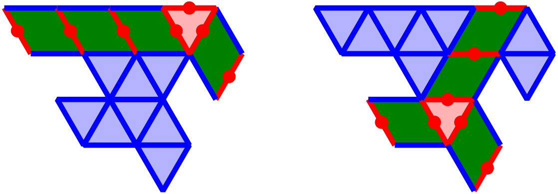
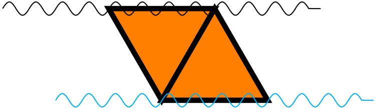
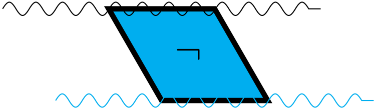
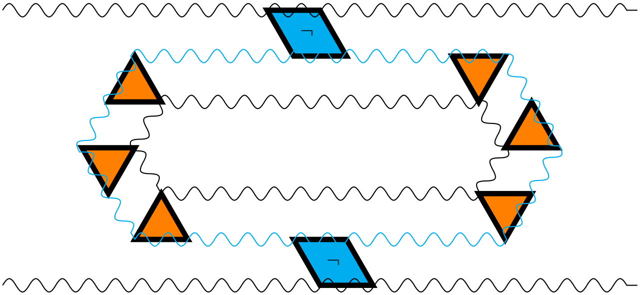


Negator

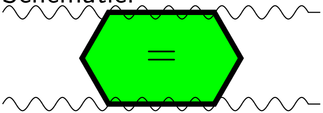


Schematic:

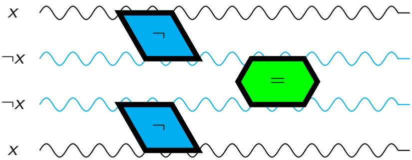




Schematic:



Wire



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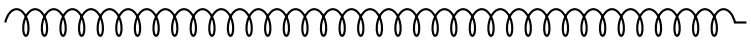
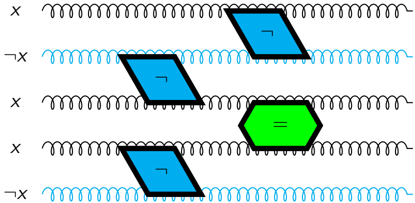
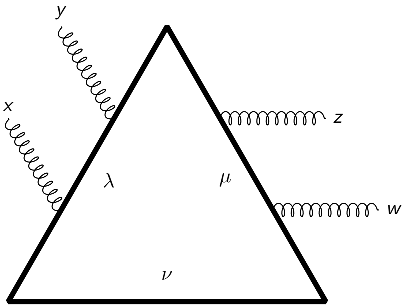


Note that wires can self-terminate.

Variable Gadget



Clause Gadget



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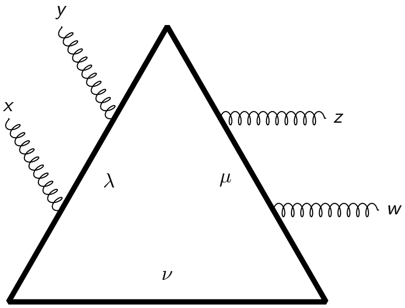
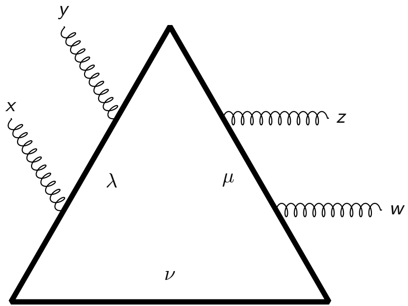


$\lambda = 11\mathrm{x}1\overline{\mathrm{x}}1\overline{\mathrm{x}}1\mathrm{x}1\mathrm{y}1\overline{\mathrm{y}}1\overline{\mathrm{y}}1\mathrm{y}1000$

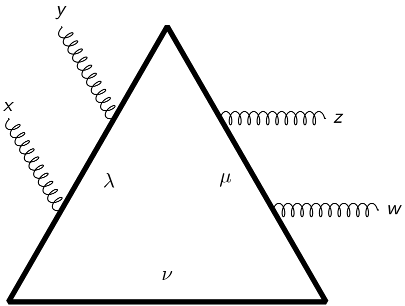
Clause Gadget



$\lambda = 11\mathtt{x}1\overline{\mathtt{x}}1\overline{\mathtt{x}}1\mathtt{x}1\mathtt{y}1\overline{\mathtt{y}}1\overline{\mathtt{y}}1\mathtt{y}1000$

$\mu = 11\mathtt{z}1\overline{\mathtt{z}}1\overline{\mathtt{z}}1\mathtt{z}1\mathtt{w}1\overline{\mathtt{w}}1\overline{\mathtt{w}}1\mathtt{w}1000$

Clause Gadget



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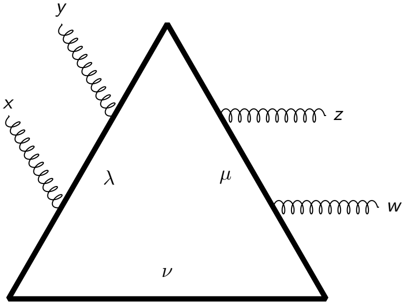
$\nu = 000111111111101100011$

Clause Gadget



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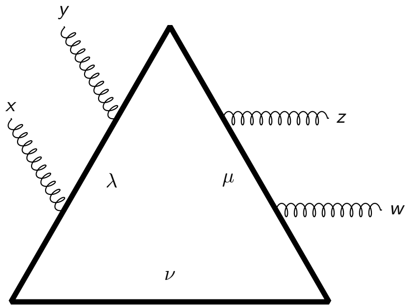
$\nu = 0001111111111101100011$

			$c^{\nu}_{\lambda\mu}$ with	
x	y	z	$w = 0$	$w = 1$
0	0	0	0	0
0	0	1	1	4
0	1	0	0	1
0	1	1	0	1
1	0	0	1	0
1	0	1	1	0
1	1	0	4	1
1	1	1	0	0

Clause Gadget



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			$c^{\nu}_{\lambda\mu}$ with	
x	y	z	$w = 0$	$w = 1$
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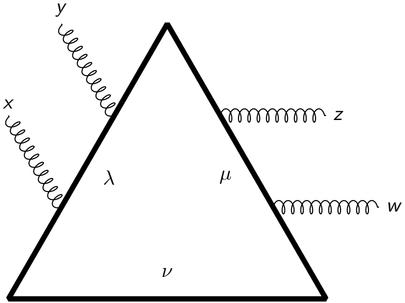
0	1	0	0	1
0	1	1	0	1
1	0	0	1	0
1	0	1	1	0
1	1	0	4	1
1	1	1	0	0

Tileable if and only if $\{x, y, z\} = \{1, 0\}$, *i.e.*, "not all equal."

Clause Gadget



$\lambda = \texttt{11x1}\overline{\texttt{x}}\texttt{1}\overline{\texttt{x}}\texttt{1x1y1}\overline{\texttt{y}}\texttt{1}\overline{\texttt{y}}\texttt{1y1000}$

$\mu = \texttt{11z1}\overline{\texttt{z}}\texttt{1}\overline{\texttt{z}}\texttt{1z1w1}\overline{\texttt{w}}\texttt{1}\overline{\texttt{w}}\texttt{1w1000}$

$\nu = \texttt{0001111111111101100011}$

			$c^{\nu}_{\lambda\mu}$ with	
x	y	z	$w = 0$	$w = 1$
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0	0	1	1	4
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1	0	1	1	0
1	1	0	4	1
1	1	1	0	0

Tileable if and only if $\{x, y, z\} = \{1, 0\}$, *i.e.*, "not all equal."

$x \vee y \vee z \iff NAE(x, y, t) \wedge NAE(\bar{t}, z, 0)$

Tileability

Tileability of

1. triangular regions is in **P**
2. convex regions
3. simply connected regions
4. general regions

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Conjecture (Pak–Tassy, work in progress)

Tileability of simply connected regions is in **P**.

Cohomology

Puzzles compute Schubert calculus:
as structure constants of $H^*(\mathbf{Gr}_k(\mathbb{C}^n))$.

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Puzzles compute Schubert calculus:
as structure constants of $H^*(\mathbf{Gr}_k(\mathbb{C}^n))$.

Puzzles with a reflection piece and extra rules:
as structure constants of $H^*_T(\mathbf{Gr}_k(\mathbb{C}^n))$.

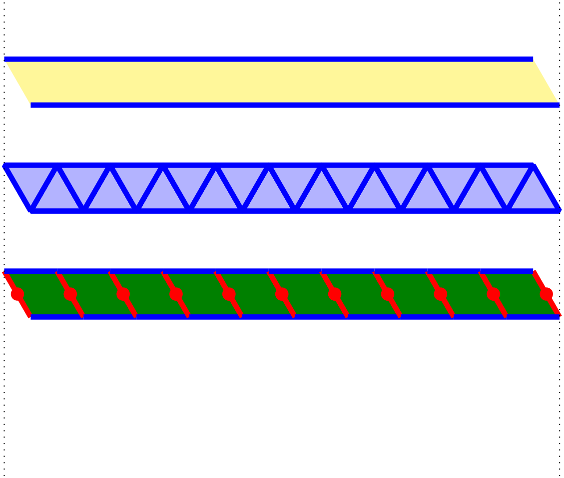
Reflections

Consider tileability by puzzle pieces **allowing reflections**.

Theorem (Pak–Y. 2014+)

Tileability of general regions is **NP***-complete.*

Wire revisited



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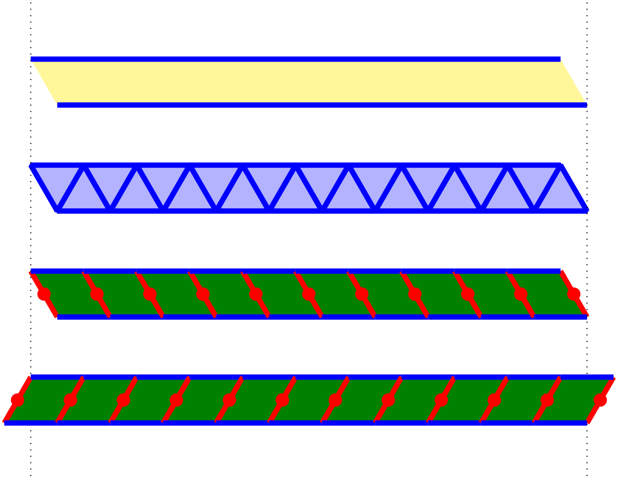
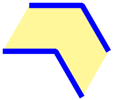
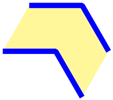


Bend



Bend



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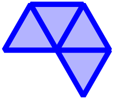

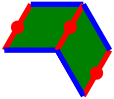

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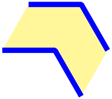

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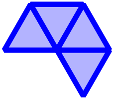
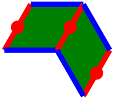
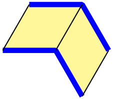




Schematic:



Wire

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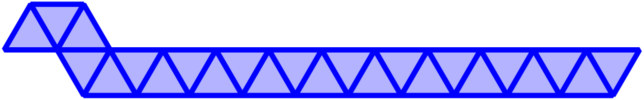

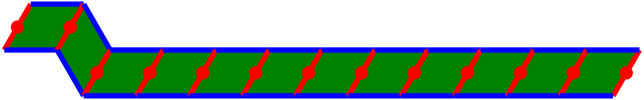

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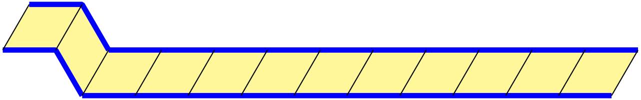

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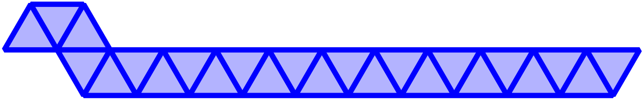

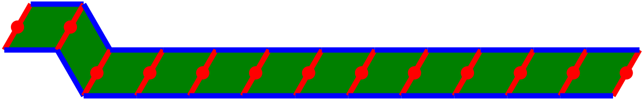

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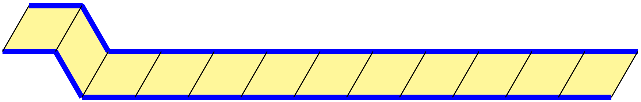

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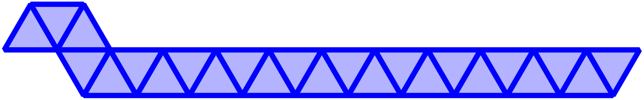

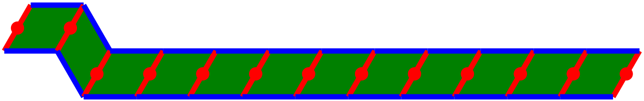

Schematic:

Reflections

Consider tileability by puzzle pieces **allowing reflections**.

Theorem (Pak–Y. 2014+)

Tileability of general regions is **NP***-complete.*

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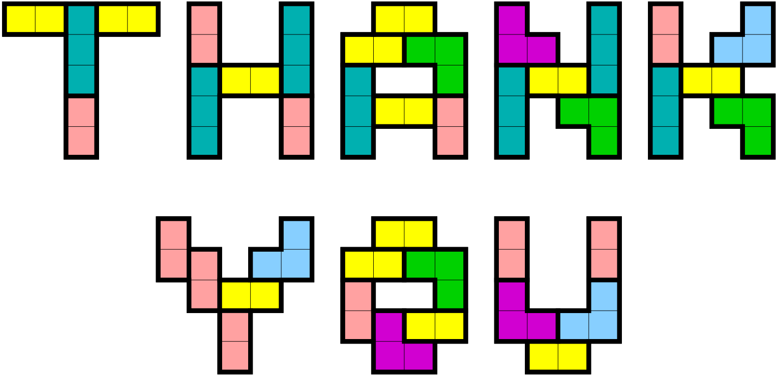
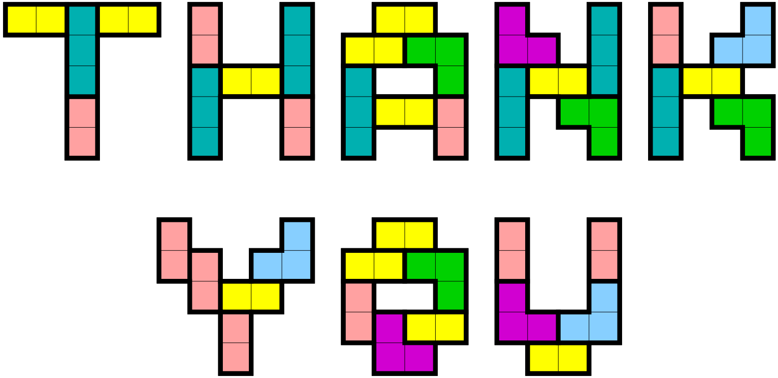
Main ingredient: counting lattice points in a polyhedron (Barvinok)

Conjecture (Pak–Tassy, work in progress)

Tileability of **simply connected regions** is in **P**.

The work is supported by the NSF GRFP grant NSF/DGE-0707424 and NSF RTG grant NSF/DMS-1148634.


THANK
YOU


THANK
YOU

24 Appendix

25 Equality Gadget

26 Crossover Gadget

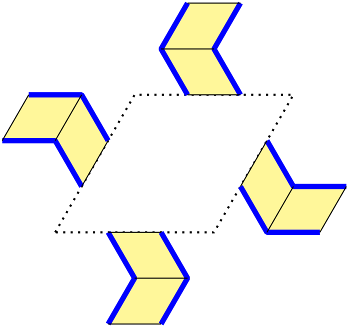
27 Cubic Monotone 1-in-3 SAT

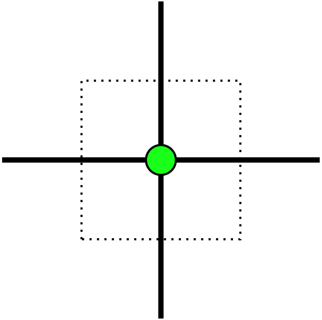
28 Inflation

29 Hives

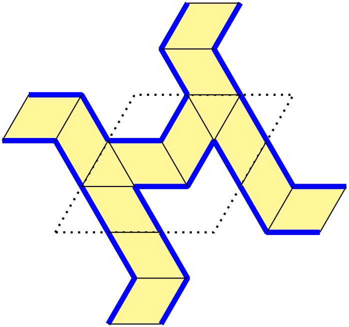
30 Decision

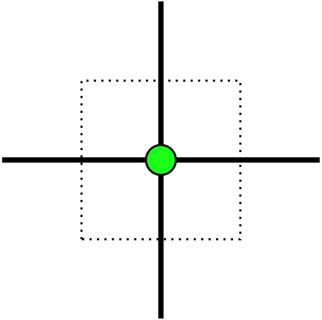
Equality Gadget



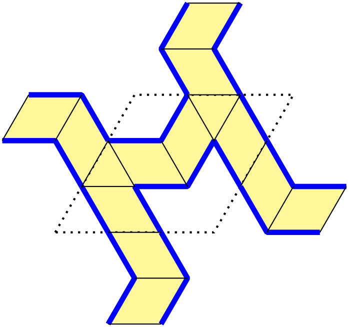


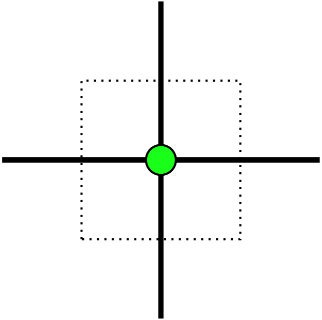
Equality Gadget





Equality Gadget





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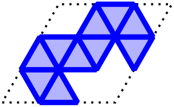
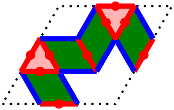
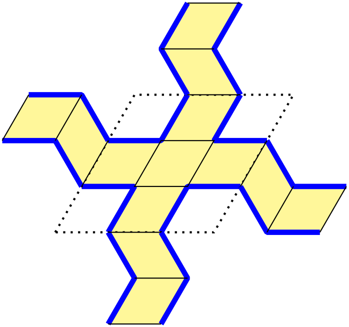
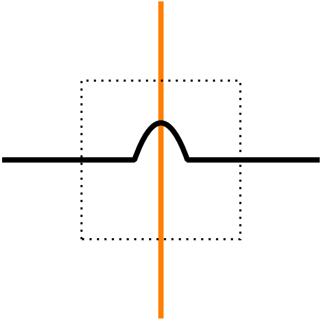


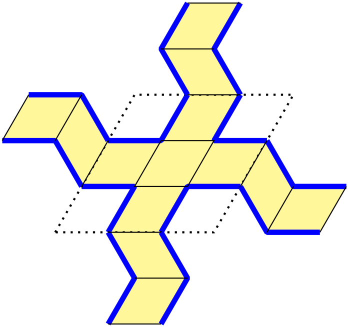


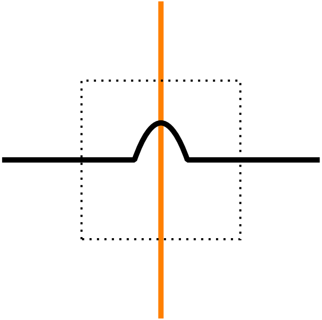
Crossover Gadget





Crossover Gadget





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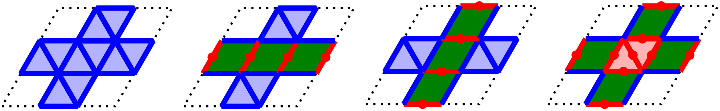
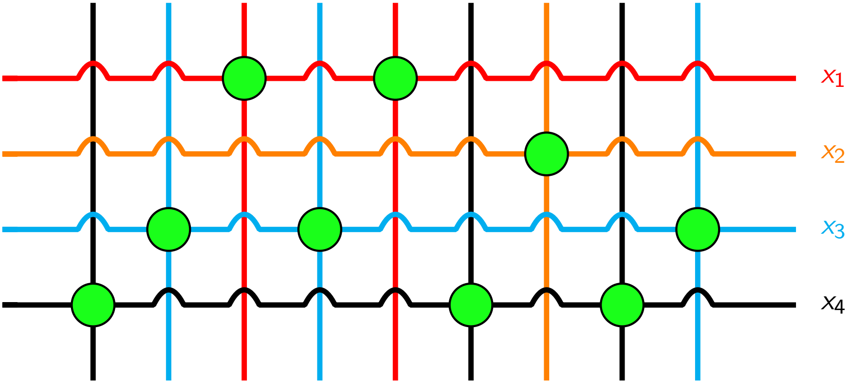
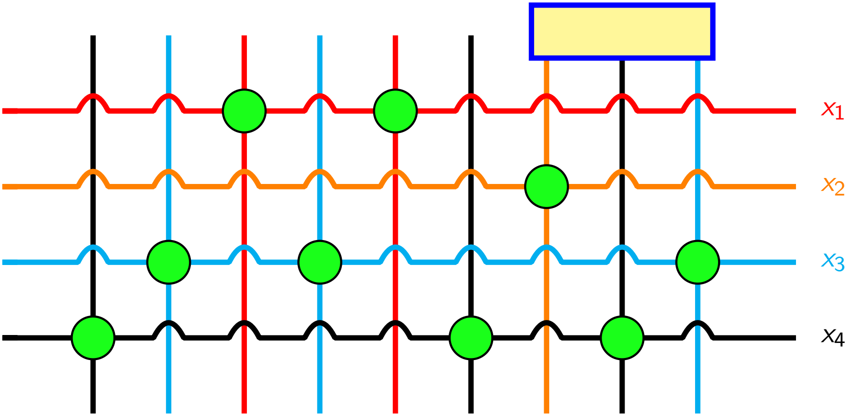


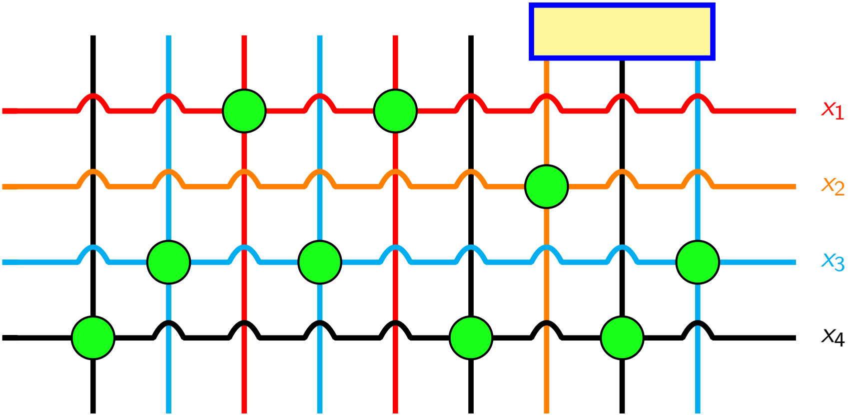
SAT



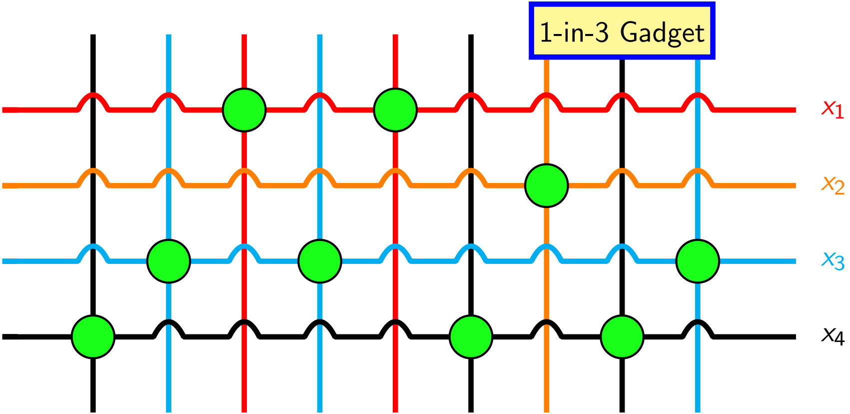
3 SAT



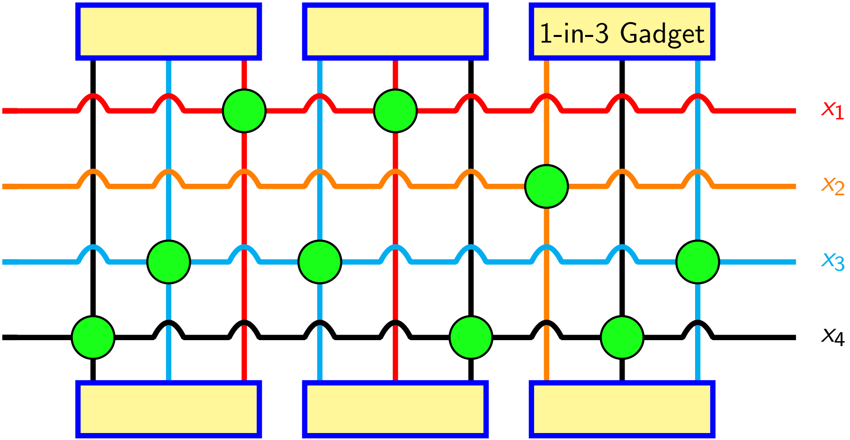
Monotone 3 SAT



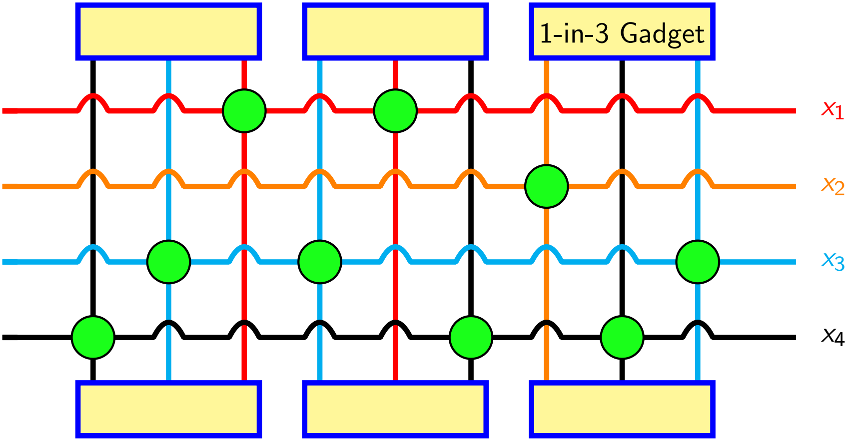
Monotone 1-in-3 SAT



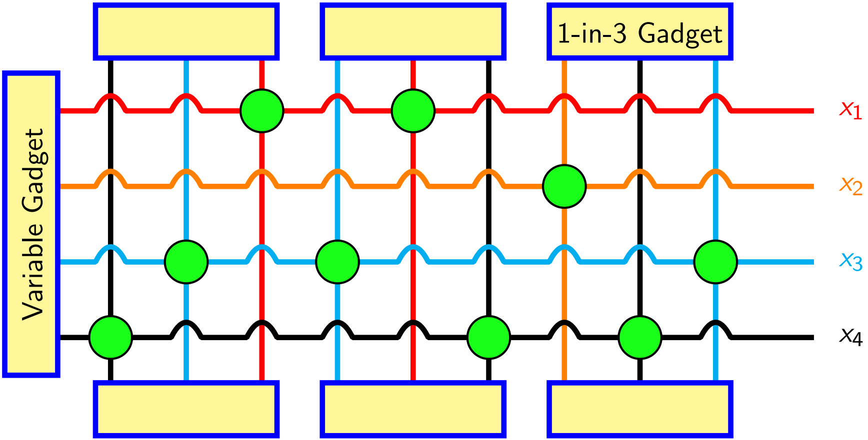
Monotone 1-in-3 SAT



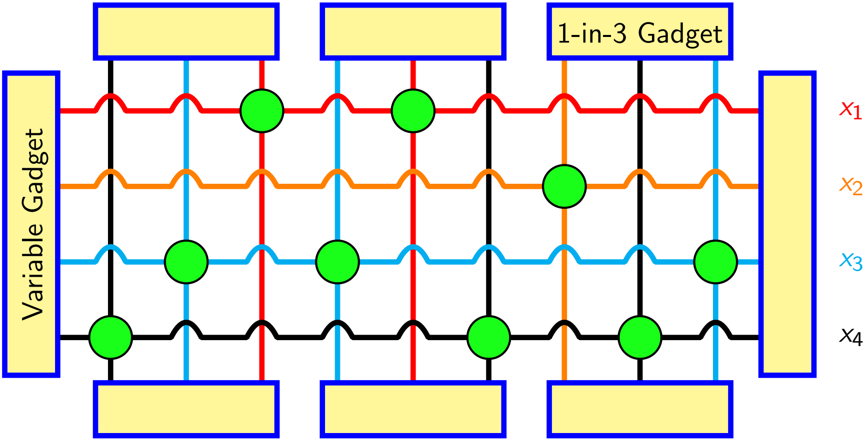
Cubic Monotone 1-in-3 SAT



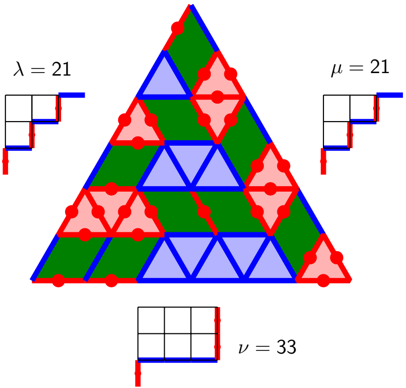
Cubic Monotone 1-in-3 SAT



Cubic Monotone 1-in-3 SAT



Inflation



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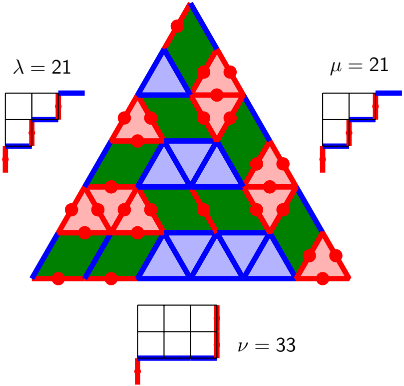

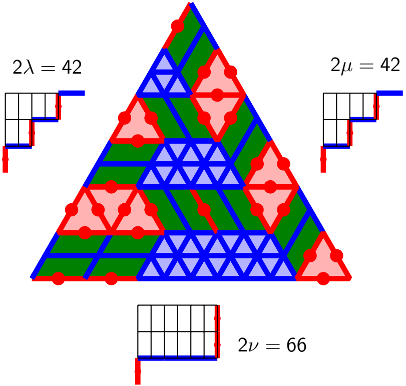

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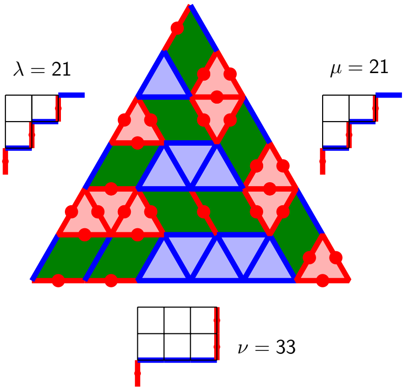
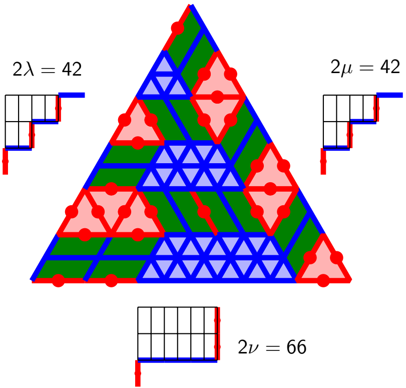
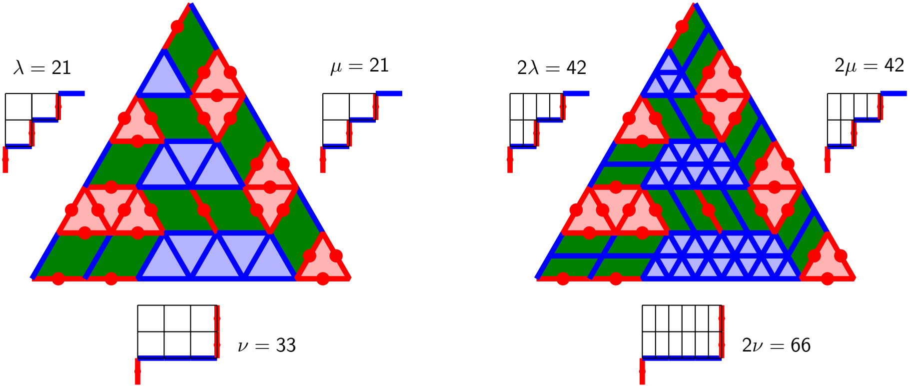




Theorem (Knutson–Tao 1998; formerly "Saturation Conjecture")

$c^{\nu}_{\lambda\mu} > 0 \Longrightarrow c^{N\nu}_{N\lambda,N\mu} > 0$ *for all* $N \geq 1$.

Inflation



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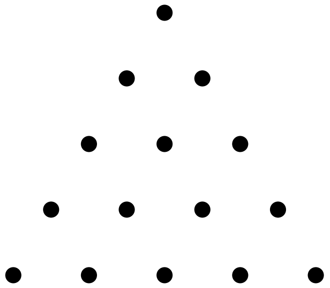
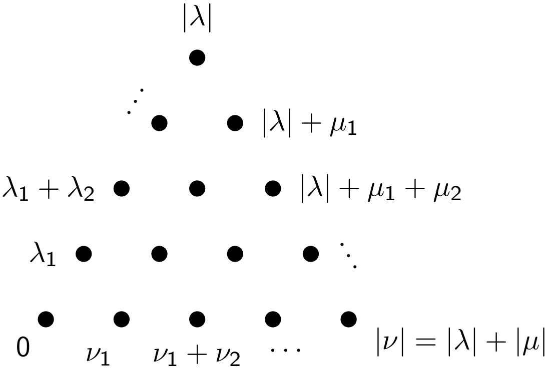
$c^{\nu}_{\lambda\mu} > 0 \Longleftarrow c^{N\nu}_{N\lambda,N\mu} > 0$ *for some* $N \geq 1$.

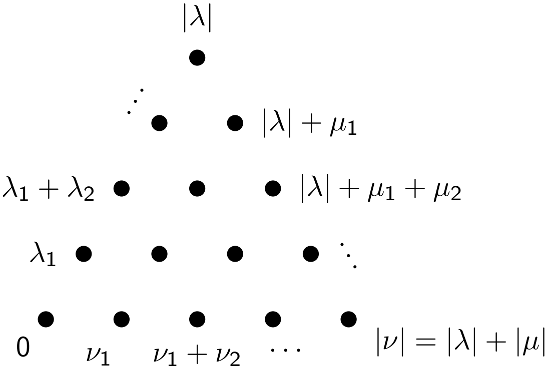
Hives



Hives


$|\lambda|$
$|\lambda| + \mu_1$
$\lambda_1 + \lambda_2$
$|\lambda| + \mu_1 + \mu_2$
λ_1
0
ν_1
$\nu_1 + \nu_2$
$\cdots$
$|\nu| = |\lambda| + |\mu|$

Hives


$|\lambda|$
$|\lambda| + \mu_1$
$\lambda_1 + \lambda_2$
$|\lambda| + \mu_1 + \mu_2$
λ_1
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$\nu_1 + \nu_2$
$\cdots$
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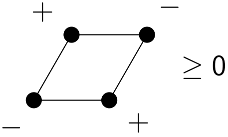
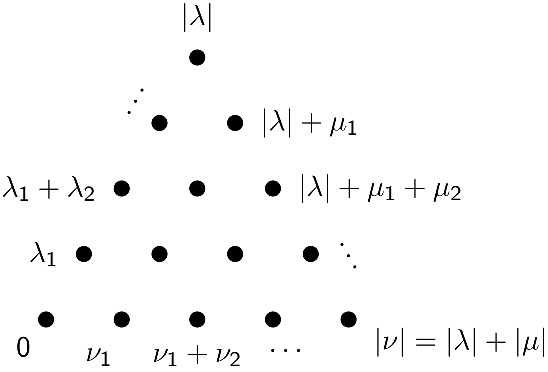
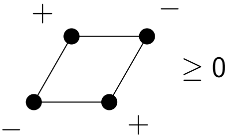

$+$
$-$
≥ 0
$-$
$+$

Hives

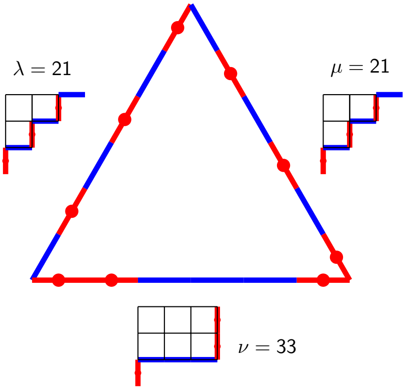




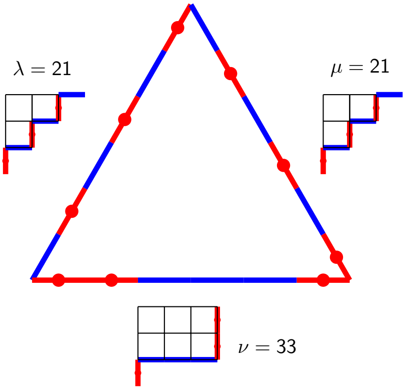
Theorem (Knutson–Tao 1998)

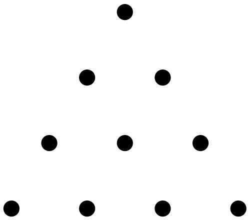
$c^{\nu}_{\lambda\mu}$ is the number of integral hives.

Decision

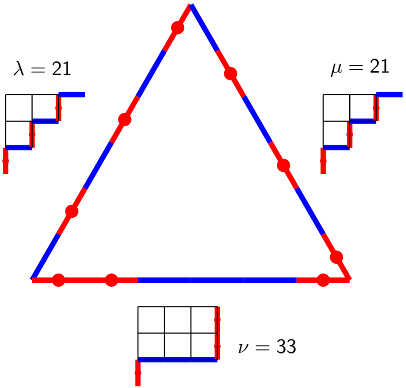


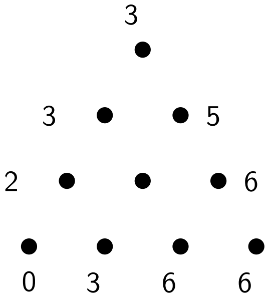
Decision



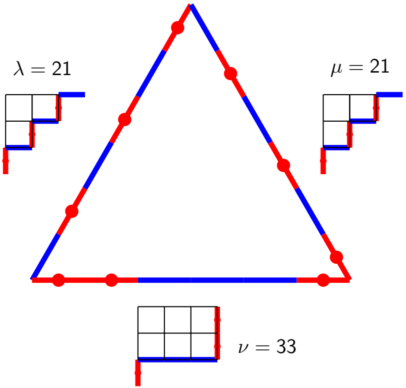


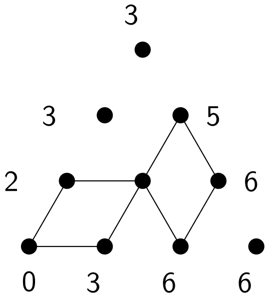
Decision





Decision





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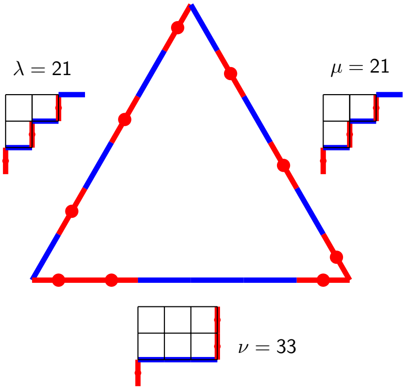
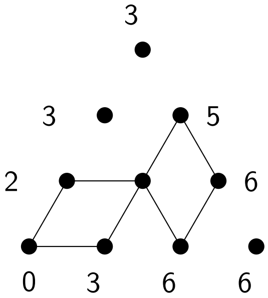
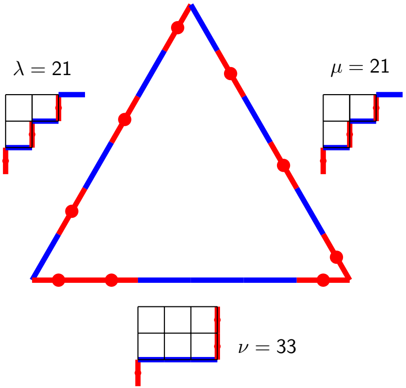
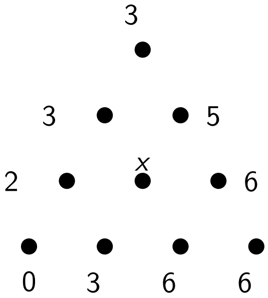




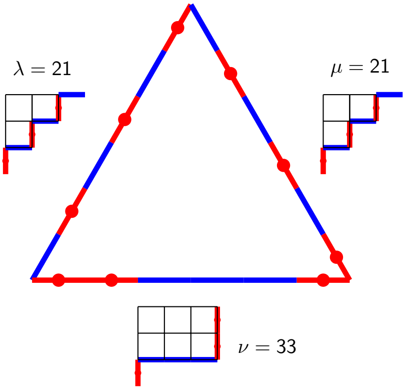
$$c^{\nu}_{\lambda\mu} = 1$$

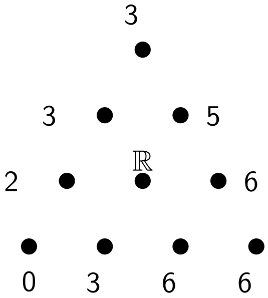
Decision



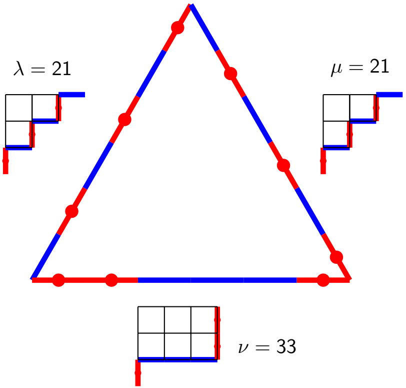


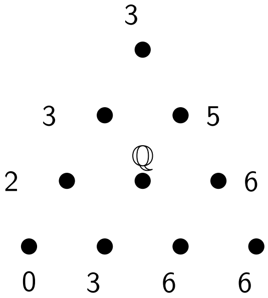
Decision



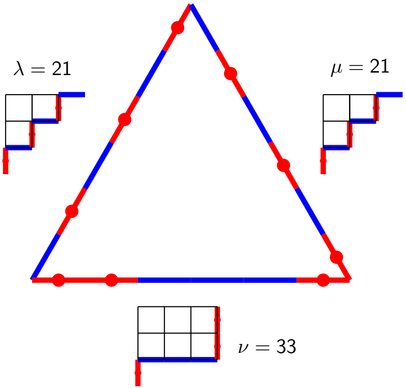


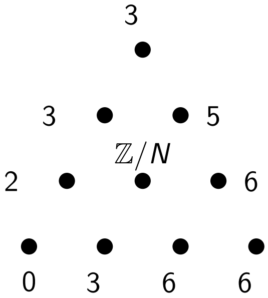
Decision





Decision





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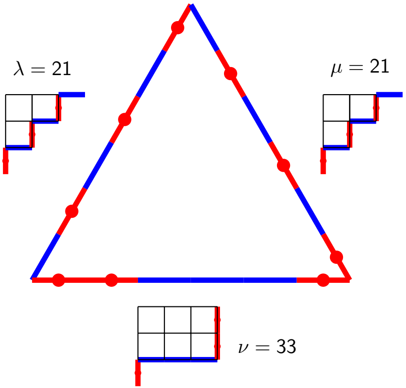

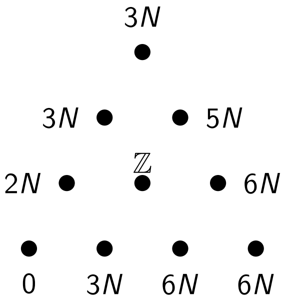

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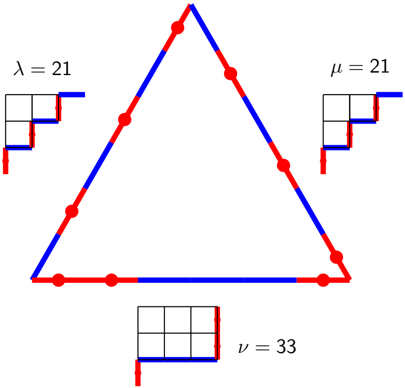
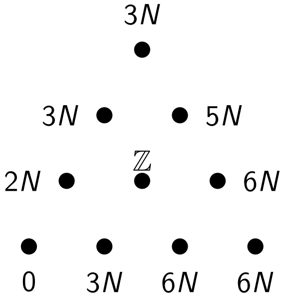
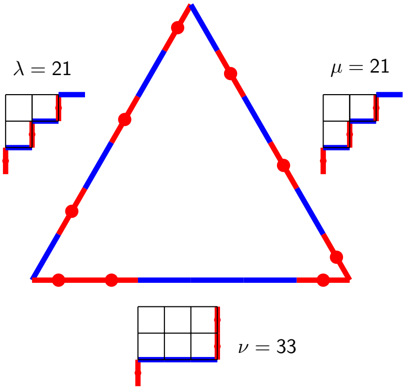
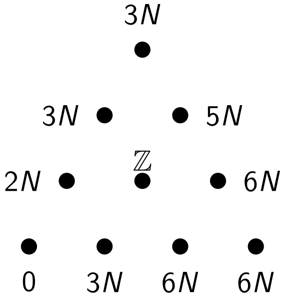




$$c^{N\nu}_{N\lambda,N\mu} > 0$$

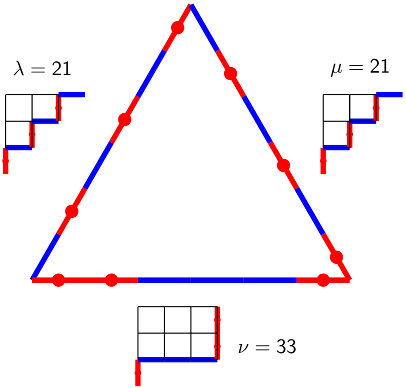
Decision

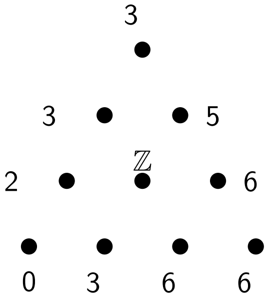




$$c^{N\nu}_{N\lambda, N\mu} > 0 \implies c^{\nu}_{\lambda\mu} > 0$$

Decision





$$c^{N\nu}_{N\lambda,N\mu} > 0 \implies c^{\nu}_{\lambda\mu} > 0$$