On q-γ-positivity

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Based on joint work with John Shareshian and with Christian Krattenthaler

γ-positivity

A polynomial $f(t) = \sum_{i=0}^{n} a_i t^i \in \mathbb{R}[t]$ is

- palindromic if $a_i = a_{n-i}$ for all i
- unimodal if for some c

$$a_0 \leq a_1 \leq \cdots \leq a_c \geq \cdots \geq a_{n-1} \geq a_n$$

Example: $f(t) = 1 + 26t + 66t^2 + 26t^3 + t^4$.

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If $\gamma_k \geq 0$ for all k then $f(t)$ said to be γ-positive.

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γ-positive $\Longrightarrow$ palindromic and unimodal

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$$\begin{array}{rcl} 1t^0(1+t)^4 & = & 1 + 4t + \ 6t^2 + \ 4t^3 + t^4 \\ 22\,t^1(1+t)^2 & = & 22t + 44t^2 + 22t^3 \\ 16\,t^2(1+t)^0 & = & 16t^2 \end{array}$$

γ-positivity

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γ-positive $\implies$ palindromic and unimodal

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So

$$f(t) = 1t^0(1+t)^4 + 22t^1(1+t)^2 + 16t^2(1+t)^0.$$

Thus $f(t)$ is γ-positive.

Eulerian polynomials

$$A_n(t) := \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{exc}(\sigma)}$$

				1				
			1		1			
		1		4		1		
	1		11		11		1	
1		26		66		26		1

Foata & Schutzenberger (1970):

$$A_n(t) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k}\, t^k (1+t)^{n-1-2k},$$

where $\gamma_{n,k} = |\{\sigma \in \mathfrak{S}_n : \sigma$ has no double descents, no final descent & $\mathrm{des}(\sigma) = k\}|$.

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3.2.14 has a double descent

124.3 has a final descent

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$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 4 \quad 1 \\ 1 \quad 11 \quad 11 \quad 1 \\ 1 \quad 26 \quad 66 \quad 26 \quad 1 \end{array}$$

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$\mathfrak{S}_3$	des
123	0
~~132~~	
213	1
~~231~~	
312	1
~~321~~	

$$\begin{aligned} A_3(t) &= 1t^0(1+t)^2 + 2t^1(1+t)^0 \\ &= 1 + 4t + t^2. \end{aligned}$$

Narayana polynomials

$$N_n(t) := \sum_{\sigma \in \mathfrak{S}_n(312)} t^{\mathrm{des}(\sigma)}$$

1			
1	1		
1	3	1	
1	6	6	1

$\mathfrak{S}_n(\tau)$ denotes the set of permutations in $\mathfrak{S}_n$ that avoid the pattern τ.

41253 has the pattern 312

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1			
1	1		
1	3	1	
1	6	6	1

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1		6		6		1

Simion & Ullman (1991): poset theoretic proof of γ-positivity.
Postnikov, Reiner, Williams (2008):

$$N_n(t) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k}\, t^k (1+t)^{n-1-2k},$$

where $\gamma_{n,k} = |\{\sigma \in \mathfrak{S}_n(312) : \sigma$ has no double descents, no final descent & $\mathrm{des}(\sigma) = k\}|$.

The proof involves a generalization of a construction of Shapiro, Woan, and Getu (1983) involving peaks and valleys of permutations.

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$$\begin{aligned} N_3(t) &= 1t^0(1+t)^2 + 1t^1(1+t)^0 \\ &= 1 + 3t + t^2. \end{aligned}$$

Gal's conjecture

The *h-polynomial* of a d-dimensional convex polytope $\mathcal{P}$ is defined by

$$h_{\mathcal{P}}(t) := \sum_{j=0}^{d} f_{d-1-j}(t-1)^j$$

where f_i is the number of faces of $\mathcal{P}$ of dimension i.

Dehn-Sommerville equations: The h-polynomial of every simplicial convex polytope is palindromic.

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Dehn-Sommerville equations: The h-polynomial of every simplicial convex polytope is palindromic.

Stanley (1980): The h-polynomial of every simplicial convex polytope is unimodal.
This is part of the celebrated g-theorem of Billera, Lee and Stanley.

Gal's Conjecture

Gal's Conjecture (2005)

The h-polynomial of a flag simplicial convex polytope $\mathcal{P}$ is γ-positive.

True for

- Barycentric subdivision (Karu 2006)
- Finite Coxeter complex (Brenti 1994, Chow 2008)
- Dual to chordal nestohedron (Postnikov-Reiner-Williams 2008)

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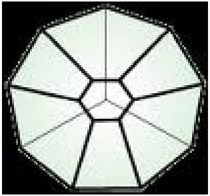
Examples:

- $\mathcal{P}_n$ is permutohedron: $h_{\mathcal{P}_n^*}(t) = A_n(t)$
- $\mathcal{A}_n$ is associahedron: $h_{\mathcal{A}_n^*}(t) = N_n(t)$
- $\mathcal{S}_n$ is stellohedron: $h_{\mathcal{S}_n^*}(t) = 1 + t\sum_{m=1}^{n} \binom{n}{m} A_m(t)$

The stellohedron (Postnikov-Reiner-Williams (2008))

Let Δ_n be the n-simplex in $\mathbb{R}^n$ with vertices $0, e_1, \ldots, e_n$.

The stellohedron $\mathcal{S}_n$ is obtained from Δ_n by truncating the faces that don't contain 0 in increasing order of dimension.



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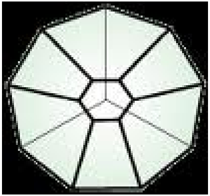

The dual $\mathcal{S}_n^*$ is a simplicial polytope. The dual of truncation is stellar subdivision. So $\mathcal{S}_n^*$ is obtained from Δ_n by stellar subdivision of the faces of Δ_n that contain 0 in decreasing order of dimension.

The formula

$$h_{\mathcal{S}_n^*}(t) = 1 + t \sum_{m=1}^{n} \binom{n}{m} A_m(t)$$

and a formula for the γ-coefficients are given in the Postnikov-Reiner-Williams paper.

q-Eulerian polynomials

$$A_n(q,t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)}$$

$n \backslash j$	0	1	2	3	4
1	1				
2	1	1			
3	1	$2+q+q^2$	1		
4	1	$3+2q+3q^2+2q^3+q^4$	$3+2q+3q^2+2q^3+q^4$	1	
5	1	$4+3q+5q^2+\dots$	$6+6q+11q^2+\dots$	$4+3q+5q^2+\dots$	1

Shareshian and MW (2010): $A_n(q,t) = \sum_{j=0}^{n-1} a_{n,j}(q)t^j$ is palindromic and q-unimodal in the sense that

- $a_{n,j}(q) = a_{n,n-1-j}(q)$ for $0 \le j \le n-1$
- $a_{n,j+1}(q) - a_{n,j}(q) \in \mathbb{N}[q]$ for $0 \le j < \frac{n}{2}$

q-γ-positivity

Let NDD be the set of words over $\mathbb{P}$ with no double descents.
Shareshian and MW (2010):

$$A_n(q,t) = \sum_{\substack{\sigma \in NDD \cap \mathfrak{S}_n \\ \sigma(n-1) < \sigma(n)}} q^{\mathrm{inv}(\sigma)} t^{\mathrm{des}(\sigma)} (1+t)^{n-1-2\mathrm{des}(\sigma)}$$

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Shareshian and MW (2013): (q-analog of the h-polynomial of the dual stellohedron)

$$1 + t\sum_{m=1}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q A_m(q,t) = \sum_{\sigma \in NDD \cap \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} t^{\mathrm{des}(\sigma)} (1+t)^{n-2\mathrm{des}(\sigma)}$$

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Thus both polynomials are of the form

$$\sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_{n,k}(q) t^k (1+t)^{d-2k} \quad \text{where } \gamma_{n,k}(q) \in \mathbb{N}[q]$$

That is, both are q-γ-positive. Consequently both are palindromic and q-unimodal.

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Postnikov, Reiner & Williams (2008):

$$1 + t\sum_{m=1}^{n} \binom{n}{m} A_m(t) = \sum_{\substack{\sigma \in NDD \cap \mathfrak{S}_{n+1} \\ \sigma(n) < \sigma(n+1) \\ \sigma(1) < \sigma(2) < \cdots < \sigma(k) = n+1}} t^{\mathrm{des}(\sigma)} (1+t)^{n-2\mathrm{des}(\sigma)}$$

Outline of proof of $A_n(q,t)$ formula

$$A_n(q,t) = \sum_{\substack{\sigma \in NDD \cap \mathfrak{S}_n \\ \sigma(n-1) < \sigma(n)}} q^{\mathrm{inv}(\sigma)} t^{\mathrm{des}(\sigma)} (1+t)^{n-1-2\mathrm{des}(\sigma)}$$

We prove a symmetric function analog of the formula.

Step 1. We define a symmetric function $Q_{n,j}(\mathbf{x})$ satisfying

$$\sum_{j=0}^{n-1} \mathrm{ps}_q(Q_{n,j}(\mathbf{x}))t^j = A_n(q,t)$$

where ps_q denotes stable principal specialization,

$$\mathrm{ps}_q(f(x_1, x_2, \dots)) = f(1, q, q^2, \dots) \prod_{i=1}^{n} (1 - q^i)$$

Step 2. Let $H(z) = \sum_{n \geq 0} h_n(\mathbf{x}) z^n$. We prove

$$\sum_{n \geq 0} \sum_{j=0}^{n-1} Q_{n,j}(\mathbf{x}) t^j z^n = \frac{(1-t)H(z)}{H(tz) - tH(z)},$$

which specializes to

$$\sum_{n \geq 0} A_n(q,t) z^n = \frac{(1-t)\exp_q(z)}{\exp_q(tz) - t\exp_q(z)}.$$

Outline of proof of $A_n(q, t)$ formula

Step 3. We use the formula of Gessel,

$$\frac{(1-t)H(z)}{H(tz)-tH(z)} = \sum_{n\geq 0} \sum_{\substack{w \in NDD_n \\ w(n-1) < w(n)}} x_w \, t^{\mathrm{des}(w)}(1+t)^{n-1-2\mathrm{des}(w)} \, z^n,$$

where $x_w = x_{w(1)}x_{w(2)}\dots x_{w(n)}$. Thus

$$\sum_{j=0}^{n-1} Q_{n,j}t^j = \sum_{\substack{w \in NDD_n \\ w(n-1) < w(n)}} x_w \, t^{\mathrm{des}(w)}(1+t)^{n-1-2\mathrm{des}(w)}$$

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where $\mathcal{D}_{n,k}$ is the set of skew hooks of size n for which k columns have size 2 and remaining $n-2k$ columns, including last column, have size 1.

$779.1558.26 \in NDD_9$

$$\begin{array}{cccccc} & & & & 2 & 6 \\ & & 1 & 5 & 5\ 8 & \\ 7 & 7 & 9 & & & \end{array}$$

$$\begin{array}{|c|c|c|c|c|c|}\hline & & & & 2 & 6 \\ \hline & & 1 & 5 & 5 & \\ \hline 7 & 7 & 9 & & & \\ \hline \end{array}$$

$\in SSYT_D, \quad D \in \mathcal{D}_{9,2}$

Outline of proof of $A_n(q,t)$ formula

Step 4. Specialize the formula

$$\sum_{j=0}^{n-1} Q_{n,j} t^j = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{D \in \mathcal{D}_{n,k}} s_D(\mathbf{x}) \; t^k (1+t)^{n-1-2k}.$$

.

$$\mathrm{ps}_q(s_D(\mathbf{x})) = \sum_{T \in SYT_D} q^{\mathrm{maj}(T)} = \sum_{\sigma \in \mathrm{DES}_D} q^{\mathrm{maj}(\sigma^{-1})} = \sum_{\sigma \in \mathrm{DES}_D} q^{\mathrm{inv}(\sigma)}$$

479.1358.26 $\in$ $NDD \cap \mathfrak{S}_9$

					2	6
		1	3	5	8	
4	7	9				

$\in SYT_D, \quad D \in \mathcal{D}_{9,2}$

So

$$A_n(q,t) = \sum_{\substack{\sigma \in NDD \cap \mathfrak{S}_n \\ \sigma(n-1) < \sigma(n)}} q^{\mathrm{inv}(\sigma)} t^{\mathrm{des}(\sigma)} (1+t)^{n-1-2\mathrm{des}(\sigma)}$$

Outline of proof of $1 + t\sum_{m=1}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q A_m(q,t)$ formula

$$1 + t \sum_{m=1}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q A_m(q,t) = \sum_{\sigma \in NDD \cap \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} t^{\mathrm{des}(\sigma)} (1+t)^{n-2\mathrm{des}(\sigma)}$$

Let

$$R_n(\mathbf{x},t) := 1 + t \sum_{m=1}^{n} \left(\sum_{j=0}^{m-1} Q_{m,j}(\mathbf{x}) t^j \right) h_{n-m}(\mathbf{x}).$$

Then

$$R_n(\mathbf{x},t) = 1 + t \sum_{m=1}^{n} \sum_{\substack{w \in NDD_m \\ w(m-1) < w(m)}} x_w \, t^{\mathrm{des}(w)} (1+t)^{m-1-2\mathrm{des}(w)} h_{n-m}(\mathbf{x}).$$

By manipulating this, we obtain

$$R_n(\mathbf{x},t) = \sum_{w \in NDD_n} x_w t^{\mathrm{des}(w)} (1+t)^{n-2\mathrm{des}(w)}$$

By specializing this, as before, we obtain the result.

Geometric interpretation (Shareshian and MW)

Let $\mathcal{P}$ be a d-dimensional simplicial convex polytope in $\mathbb{R}^d$. Let $\mathcal{V}_P$ be the toric variety associated with $\mathcal{P}$.

Danilov (1978):
$$h_{\mathcal{P}}(t) = \sum_{j=0}^{d} \dim H^{2j}(\mathcal{V}_{\mathcal{P}}) t^j$$

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$$h_{\mathcal{P}}(t) = \sum_{j=0}^{d} \dim H^{2j}(\mathcal{V}_{\mathcal{P}}) t^j$$

The symmetric group acts simplicially on the dual permutohedron $\mathcal{P}_n^*$ and the dual stellohedron $\mathcal{S}_n^*$. This induces a representation of $\mathfrak{S}_n$ on each $H^{2j}(\mathcal{V}_{\mathcal{P}_n^*})$ and on each $H^{2j}(\mathcal{V}_{\mathcal{S}_n^*})$.

Procesi (1990), Stanley (1989):

$$\sum_{n\geq 0} \sum_{j=0}^{n-1} \mathrm{ch}\, H^{2j}(\mathcal{V}_{\mathcal{P}_n^*}) t^j z^n = \frac{(1-t)H(z)}{H(tz) - tH(z)},$$

where ch denotes Frobenius characteristic.

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where ch denotes Frobenius characteristic.

We use Procesi's technique to prove

$$\mathrm{ch}\, H^{2j}(\mathcal{V}_{\mathcal{S}_n^*}) = \sum_{m=1}^{n} h_{n-m}(\mathbf{x})\ \mathrm{ch}\, H^{2(j-1)}(\mathcal{V}_{\mathcal{P}_m^*}).$$

Geometric interpretation (Shareshian and MW)

It follows that

$$\mathrm{ch}\, H^{2j}(\mathcal{V}_{\mathcal{P}_n^*}) = Q_{n,j}(\mathbf{x})$$

and

$$\mathrm{ch}\, H^{2j}(\mathcal{V}_{\mathcal{S}_n^*}) = \sum_{m=1}^{n} h_{n-m}(\mathbf{x}) Q_{m,j-1}(\mathbf{x})$$

By specializing we obtain

$$\sum_{j=0}^{n-1} \mathrm{ps}_q(\mathrm{ch}\, H^{2j}(\mathcal{V}_{\mathcal{P}_n^*}))\ t^j = A_n(q,t)$$

and

$$\sum_{j=0}^{n} \mathrm{ps}_q(\mathrm{ch}\, H^{2j}(\mathcal{V}_{\mathcal{S}_n^*}))\ t^j = 1 + t \sum_{m=1}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q A_m(q,t).$$

Instances of equivariant Gal's conjecture

Let S^D be the Specht module of skew shape D. The Schur-γ-positivity formulas for $\sum Q_{n,j}(\mathbf{x})t^j$ and $R_n(\mathbf{x},t)$ yield:

$$\sum_{j=0}^{n-1} H^{2j}(\mathcal{V}_{\mathcal{P}_n^*})t^j = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{D\in\mathcal{D}_{n,k}} S^D(\mathbf{x}) \; t^k(1+t)^{n-1-2k}$$

where $\mathcal{D}_{n,k}$ is the set of skew hooks of size n for which k columns have size 2 and remaining $n-2k$ columns, including last column, have size 1.

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Equivariant Gal's conjecture: Let G be a finite group acting simplicially on a flag simplicial convex polytope $\mathcal{P}$. This induces a representation of G on $H^{2j}(\mathcal{V}_{\mathcal{P}})$. There exist G-modules $\Gamma_{\mathcal{P},k}$ such that

$$\sum_{j=0}^{d} H^{2j}(\mathcal{V}_{\mathcal{P}})t^j = \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} \Gamma_{\mathcal{P},k}\ t^k(1+t)^{d-2k}$$

q-Narayana polynomials

The Narayana numbers have a closed form formula

$$N_n(t) = \sum_{j=0}^{n-1} \frac{1}{n} \binom{n}{j} \binom{n}{j+1} t^j.$$

Recall that the Narayana numbers refine the Catalan numbers

$$N_n(1) = C_n.$$

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The Fürlinger-Hofbauer q-Narayana polynomials are defined by

$$N_n(q,t) := \sum_{j=0}^{n-1} q^{j(j+1)} \frac{1}{[n]_q} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} n \\ j+1 \end{bmatrix}_q t^j.$$

Fürlinger and Hofbauer (1985) (or MacMahon) showed that

$$N_n(q,1) = C_n(q) := \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q.$$

Palindromicity and unimodality?

$$N_n(q,t) := \sum_{j=0}^{n-1} q^{j(j+1)} \frac{1}{[n]_q} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} n \\ j+1 \end{bmatrix}_q t^j.$$

Coefficients of t^j in $N_n(q,t)$

$n \backslash j$	0	1	2	3
1	1			
2	1	q^2		
3	1	$q^2(1+q+q^2)$	q^6	
4	1	$q^2(1+q+2q^2+q^3+q^4)$	$q^6(1+q+2q^2+q^3+q^4)$	q^{12}
5	1	$q^2(1+q+2q^2+2q^3+\dots)$	$q^6(1+q+3q^2+3q^3+\dots)$	$q^{12}(1+q+2q^2+2$

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q-γ-positivity of q-Narayama polynomials

Simion & Ullman (1991)

$$N_n(t) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{2k} C_k \; t^k (1+t)^{n-1-2k}.$$

They construct a decomposition of the noncrossing partition lattice into symmetric boolean intervals.

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$$N_n(q,t) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} q^{k(k+1)} \gamma_{n,k}(q) \, t^k \prod_{i=k+1}^{n-k-1} (1 + q^{2i} t),$$

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Touchard's identity for Catalan numbers C_n

$$C_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{2k} C_k \; 2^{n-1-2k}$$

Andrews (2010):

$$C_n(q) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} q^{2k(k+1)} \begin{bmatrix} n-1 \\ 2k \end{bmatrix}_q C_k(q) \; \frac{\prod_{i=k+2}^{n}(1+q^i)}{\prod_{i=1}^{k}(1+q^i)}$$

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$t = 1$ of Krattenthaler-MW and Dilks:

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Andrews' proof relies on a q-analog of Chu-Vandermonde summation. Our proof relies on this and other q-hypergeometric techniques. It is much more involved.

q-γ-positivity of q-Narayama polynomials

Theorem (Krattenthaler & MW (2014), Dilks (2014))

$$N_n(q,t) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} q^{k(k+1)} \gamma_{n,k}(q)\, t^k \prod_{i=k+1}^{n-k-1} (1 + q^{2i} t),$$

where

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- Is there a combinatorial or geometric interpretation?
- What does this say about q-unimodality?

Definition of q-γ-positive

A polynomial $P(q,t) \in \mathbb{N}[q][t]$ of degree d in t is *q-γ-positive* of index $r \in \mathbb{N}$ if $\exists\ \gamma_j(q) \in \mathbb{N}[q]$, such that

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Examples:

$$N_n(q,t) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} q^{k(k+1)} \gamma_{n,k}(q)\, t^k \prod_{i=k+1}^{n-1-k} (1 + q^{2i} t),$$

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$$A_n(q,t) = \sum_{\substack{\sigma \in NDD \cap \mathfrak{S}_n \\ \sigma(n-1) < \sigma(n)}} q^{\mathrm{inv}(\sigma)} t^{\mathrm{des}(\sigma)} (1+t)^{n-1-2\mathrm{des}(\sigma)}$$

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A consequence: q-unimodality

Theorem (Krattenthaler and MW)

Suppose $P(q,t) \in \mathbb{N}[q][t]$ is q-γ-positive of index r. If $\tilde{P}(q,t)$ is obtained from $P(q,t)$ by multiplying the coefficient of t^j by $q^{-r\binom{j+1}{2}}$ for each j then $\tilde{P}(q,t)$ is a palindromic and q-unimodal polynomial in $\mathbb{N}[q][t]$.

Example 1: Since

$$N_n(q,t) := \sum_{j=0}^{n-1} q^{j(j+1)} \frac{1}{[n]_q} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} n \\ j+1 \end{bmatrix}_q t^j$$

is q-γ-positive of index 2,

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is palindromic and q-unimodal.

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Example 2: Since

$$\sum_{j=0}^{n} q^{\binom{j+1}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q t^j$$

is q-γ-positive of index 1, we get the well-known result that

$$\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q t^j$$

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Example 3: Han, Jouhet, and Zeng (2011) derived a formula which implies that Carlitz's q-Eulerian polynomial

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} t^{\mathrm{des}(\sigma)}$$

is q-γ-positive of index 1. It follows that

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma) - \binom{\mathrm{des}(\sigma)+1}{2}} t^{\mathrm{des}(\sigma)}$$

is palindromic and q-unimodal.

Super q-Catalan numbers (Warnaar and Zudilin)

For $n, s \geq 0$, define the super q-Catalan numbers

$$C_n^{(s)}(q) := \begin{bmatrix} 2s \\ s \end{bmatrix}_q \begin{bmatrix} 2n \\ n \end{bmatrix}_q \begin{bmatrix} n+s \\ s \end{bmatrix}_q^{-1}.$$

- $C_n^{(1)}(q) = (1+q)C_n(q)$.
- $C_n^{(0)}(1)$ is the type B Catalan number.
- Catalan (1874) showed $C_n^{(s)}(1)$ is an integer.
- Gessel (1992) obtained a generalization of Touchard's identity for $C_n^{(s)}(1)$.
- Warnaar and Zudilin (2011) obtained a q-analog of Gessel's identity, which showed $C_n^{(s)}(q) \in \mathbb{N}[q]$.

Super q-Narayana polynomials (Krattenthaler and MW)

For $n \geq s$, define the super q-Narayana polynomials

$$N_n^{(s)}(q,t) := \begin{bmatrix} 2s \\ s \end{bmatrix}_q \sum_{j=0}^{n-s} q^{j(j+1)} \begin{bmatrix} n \\ s \end{bmatrix}_q^{-1} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} n \\ j+s \end{bmatrix}_q t^j.$$

- Note $N_n^{(1)}(q,t) = (1+q)N_n(q,t)$.
- $N_n^{(0)}(1,t)$ is the type B Narayana polynomial.
- Gessel proved $N_n^{(s)}(1,t) \in \mathbb{N}[t]$ by deriving a γ-positivity formula.

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Krattenthaler and MW (2014):

- $N_n^{(s)}(q,q^{s-1}) = C_n^{(s)}(q)$
- A q-analog of Gessel's γ-positivity formula yielding $N_n^{(s)}(q,t) \in \mathbb{Z}[q,t]$
- Positivity conjecture*: $N_n^{(s)}(q,t) \in \mathbb{N}[q,t]$.

*Proved shortly after the talk

Super q-Narayana polynomials

Theorem (Krattenthaler and MW (2014))

$$N_n^{(s)}(q,t) = \sum_{k=0}^{\lfloor \frac{n-s}{2} \rfloor} q^{k(k+1)} \gamma_{n,k}^{(s)}(q)\ t^k \prod_{i=k+1}^{n-k-s} (1+q^{2i}t),$$

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We show that $\gamma_{n,k}^{(s)}(q) \in \mathbb{Z}[q]$. Hence $N_n^{(s)}(q,t) \in \mathbb{Z}[q,t]$.

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However positivity of $\gamma_{n,k}^{(s)}(q)$ fails for $n=6, k=s=2$.

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But positivity of $N_n^{(s)}(q,t)$ seems to hold anyway.*

*Proved this and q-unimodality of $\tilde{N}_n^{(s)}$ shortly after the talk.