Counting Permutations with Even Valleys and Odd Peaks

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Introduction

Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation of $[n] = \{1, 2, \ldots, n\}$. Then i is a peak if

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j is a valley of π if

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Example (peaks in blue, valleys in red):

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 4 & 5 & 2 & 6 & 7 & 3 \end{pmatrix}$$

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n	0	1	2	3	4	5	6	7	8	9
f(n)	1	1	2	4	13	50	229	1238	7614	52706

For example, with $n = 3$ the 4 permutations are

$$123, 213, 312, 321$$

(all except 132 and 231)

Sometimes exponential generating functions for certain classes of permutations defined by restrictions on descents have nice reciprocals. So let's look at the reciprocal of $\sum_{n=0}^{\infty} f(n)x^n/n!$. We find that

$$\left(\sum_{n=0}^{\infty} f(n)x^n/n!\right)^{-1}$$
$$= 1 - x + 2\frac{x^3}{3!} - 5\frac{x^4}{4!} + 61\frac{x^6}{6!} - 272\frac{x^7}{7!} + 7936\frac{x^9}{9!} - \cdots$$

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The coefficients are Euler numbers:

$$\sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \sec x + \tan x$$
$$= 1 + x + \frac{x^2}{2!} + 2\frac{x^3}{3!} + 5\frac{x^4}{4!} + 16\frac{x^5}{5!} + 61\frac{x^6}{6!}$$
$$+ 272\frac{x^7}{7!} + 1385\frac{x^8}{8!} + \cdots$$

So it seems that

$$\sum_{n=0}^{\infty} f(n)\frac{x^n}{n!} =$$
$$\left(1 - E_1 x + E_3\frac{x^3}{3!} - E_4\frac{x^4}{4!} + E_6\frac{x^6}{6!} - E_7\frac{x^7}{7!} + \cdots\right)^{-1}.$$

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This is reminiscent of the generating function

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Is this just a coincidence?

Another form of the generating function is

$$\left(1 - E_1 x + E_3 \frac{x^3}{3!} - E_4 \frac{x^4}{4!} + E_6 \frac{x^6}{6!} - E_7 \frac{x^7}{7!} + \cdots\right)^{-1} = \frac{3\sin\left(\frac{1}{2}x\right) + 3\cosh\left(\frac{1}{2}\sqrt{3}x\right)}{3\cos\left(\frac{1}{2}x\right) - \sqrt{3}\sinh\left(\frac{1}{2}\sqrt{3}x\right)}.$$

Descents of permutations

Let $\pi = \pi_1 \cdots \pi_n$ be a permutation of $[n]$. A descent of π is an i such that $\pi_i > \pi_{i+1}$. The descents of a permutation split it into increasing runs, which are maximal consecutive increasing subsequences.

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For example the permutation 14528736 has descent set $\{3, 5, 6\}$:

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We'll write $D(L)$ for the descent set corresponding to L.

Let $L = (L_1, \ldots, L_k)$ be a composition of n. Then we write $\binom{n}{L}$ for the multinomial coefficient

$$\frac{n!}{L_1! \cdots L_k!}$$

Lemma. The number of permutations of n with descent set contained in $D(L)$ is $\binom{n}{L}$.

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Lemma. The number of permutations of n with descent set contained in $D(L)$ is $\binom{n}{L}$.

Proof by example. Take $L = (3, 1, 2)$, so $n = 6$ and $D(L) = \{3, 4\}$. We take 1,2,. . . , 6 and put them into 3 boxes with three in the first, one in the second, and two in the third:

$$|\,2\,4\,5\,|\,1\,|\,3\,6\,|$$

Let $L = (L_1, \ldots, L_k)$ be a composition of n. Then we write $\binom{n}{L}$ for the multinomial coefficient

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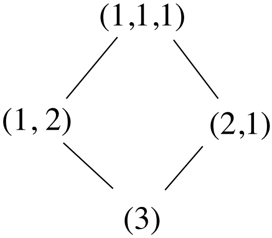
$$|\,2\,4\,5\,|\,1\,|\,3\,6\,|$$

We arrange the numbers in each box in increasing order and then remove the bars to get

$$2\,4\,5\,1\,3\,6$$

which has descent set contained in $\{3, 4\}$.

We can find the number of permutations of n with a given descent set by inclusion-exclusion. Let us partial order compositions of n by reverse refinement, corresponding to ordering descent sets by inclusion. So the compositions of 3 are ordered as



Let $\beta(L)$ be the number of permutations of $[n]$ with descent composition L, where L is a composition of n.

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Then $\binom{n}{L} = \sum_{K \leq L} \beta(K)$, so by inclusion-exclusion

$$\beta(L) = \sum_{K \leq L} (-1)^{l(L)-l(K)} \binom{n}{K},$$

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where $l(L)$ is the number of parts of L.

We can count all sorts of sets of permutations defined by descent sets by added up $\beta(L)$ for appropriate L. We get exponential generating functions since $\binom{n}{L}$ is the coefficient of $x^n/n!$ in

$$\frac{x^L}{L!} := \frac{x^{L_1}}{L_1!} \cdots \frac{x^{L_k}}{L_k!}.$$

Alternating descents

Following Denis Chebikin (2008), we define an alternating descent of a permutation π to be an odd descent or an even ascent. For example the alternating descents of 3 7 4 2 1 5 6 8 are 3 and 6:

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Nicolaescu’s question is closely connected to alternating runs of permutations, because a permutation has all valleys even and all peaks odd if and only if all of its alternating runs have length less than 3.

This is because an odd valley or an even peak corresponds to an alternating run of length at least 3:

$$\begin{pmatrix} 1\,2\,3 & \cdots & 6\,7\,8 & \cdots \\ 1\,6\,2 & & 5\,3\,4 & \end{pmatrix}$$

As Chebikin observed, the number of permutations of $[n]$ with alternating descent set contained in $D(L)$ is

$$\binom{n}{L}_E := \frac{n!}{L_1! \cdots L_k!} E_{L_1} \cdots E_{L_k}$$

We prove this by putting the numbers $1, \ldots, n$ into boxes with L_i in the ith box, and then arranging the numbers in each box into either an up-down or down-up permutation, depending on the parity of the starting position.

Now let $\hat{\beta}(L)$ be the number of permutations of $[n]$ with alternating descent composition L, where L is a composition of n. Then just as before, $\binom{n}{L}_E = \sum_{K \le L} \hat{\beta}(K)$, so by inclusion-exclusion

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$$\hat{\beta}(L) = \sum_{K \le L} (-1)^{l(L)-l(K)} \binom{n}{K}_E.$$

Note that $\binom{n}{L}_E$ is the coefficient of $x^n/n!$ in

$$E_{L_1} \frac{x^{L_1}}{L_1!} \cdots E_{L_k} \frac{x^{L_k}}{L_k!}.$$

Therefore, if we can find an exponential generating function for some class of permutations determined by descent sets, obtained by adding up $\beta(L)x^L/L!$ for appropriate L, then by changing $x^n/n!$ to $E_n x^n/n!$ we should get the corresponding result for alternating descent sets. This will explain the connection between

$$\left(1 - x + \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^7}{7!} + \cdots\right)^{-1}$$

and

$$\left(1 - E_1 x + E_3\frac{x^3}{3!} - E_4\frac{x^4}{4!} + E_6\frac{x^6}{6!} - E_7\frac{x^7}{7!} + \cdots\right)^{-1}.$$

Unfortunately, just changing to $x^n/n!$ with $E_n x^n/n!$ doesn't really work:

Unfortunately, just changing to $x^n/n!$ with $E_n x^n/n!$ doesn't really work: we would like a linear map that takes

$$\frac{x^{L_1}}{L_1!} \cdots \frac{x^{L_k}}{L_k!}.$$

to

$$E_{L_1} \frac{x^{L_1}}{L_1!} \cdots E_{L_k} \frac{x^{L_k}}{L_k!}.$$

but we can't do this since these products aren't linearly independent.

For example,

$$\frac{x^2}{2!} \cdot \frac{x^3}{3!} = 10\frac{x^5}{5!},$$

but

$$E_2\frac{x^2}{2!} \cdot E_3\frac{x^3}{3!} \neq 10E_5\frac{x^5}{5!}$$

To make this work, and to compute the generating functions more efficiently, we use noncommutatative symmetric functions, which were studied extensively by Gelfand, Krob, Lascoux, Leclerc, Retakh, and Thibon in 1995 and further studied by Thibon and others since then. But we will need only some very simple properties of the noncommutative symmetric functions most of which were known earlier.

We work with formal power series in noncommuting variables $X_1, X_2, \ldots$. We define the complete noncommutative symmetric functions

$$\mathbf{h}_n = \sum_{i_1 \le \cdots \le i_n} X_{i_1} X_{i_2} \cdots X_{i_n}.$$

For any composition $L = (L_1, \ldots, L_k)$ of n we define

$$\mathbf{h}_L = \mathbf{h}_{L_1} \cdots \mathbf{h}_{L_k}$$

which can be written as

$$\mathbf{h}_L = \sum_L X_{i_1} X_{i_2} \cdots X_{i_n}$$

where the sum is over all $(i_1, \ldots, i_n)$ satisfying

$$\underbrace{i_1 \le \cdots \le i_{L_1}}_{L_1}, \underbrace{i_{L_1+1} \le \cdots \le i_{L_1+L_2}}_{L_2}, \cdots, \underbrace{i_{L_1+\cdots+L_{k-1}+1} \le \cdots \le i_n}_{L_k}.$$

We define the ribbon noncommutative symmetric functions

$$\mathbf{r}_L = \sum_L X_{i_1} X_{i_2} \cdots X_{i_n}$$

where the sum is over all $(i_1, \ldots, i_n)$ satisfying

$$\underbrace{i_1 \le \cdots \le i_{L_1}}_{L_1} > \underbrace{i_{L_1+1} \le \cdots \le i_{L_1+L_2}}_{L_2} > \cdots > \underbrace{i_{L_1+\cdots+L_{k-1}+1} \le \cdots \le i_n}_{L_k}.$$

Then

$$\mathbf{h}_L = \sum_{K \le L} \mathbf{r}_K,$$

so by inclusion-exclusion,

$$\mathbf{r}_L = \sum_{K \le L} (-1)^{l(L)-l(K)} \mathbf{h}_K$$

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When we make the X_i commute, we get ordinary symmetric functions: $\mathbf{h}_L$ becomes the ordinary complete symmetric function h_L, and $\mathbf{r}_L$ becomes the ribbon (skew-hook, zigzag) skew Schur function.

We can define a homomorphism from noncommutative symmetric functions to exponential generating functions by

$$\Phi(\mathbf{h}_n) = \frac{x^n}{n!}$$

so

$$\Phi(\mathbf{h}_L) = \frac{x^{L_1}}{L_1!} \cdots \frac{x^{L_k}}{L_k!} = \binom{n}{L} \frac{x^n}{n!}.$$

Note that for a noncommutative symmetric function f, $\Phi(f)$ is the coefficient of $x_1 x_2 \cdots x_n$ in the result of replacing $X_1, X_2, \ldots$ with commuting variables $x_1, x_2, \ldots$.
Then by inclusion-exclusion,

$$\Phi(\mathbf{r}_L) = \beta(L) \frac{x^n}{n!}.$$

Similarly, we can define another homomorphism from noncommutative symmetric functions to exponential generating functions by

$$\hat{\Phi}(\mathbf{h}_n) = E_n \frac{x^n}{n!}$$

so by the inclusion-exclusion formula for $\hat{\beta}$ we have

$$\hat{\Phi}(\mathbf{r}_L) = \hat{\beta}(L) \frac{x^n}{n!}.$$

This means that if we can find the noncommutative symmetric function generating function for some class of words determined by their descent sets (i.e., we can express it in terms of the $\mathbf{r}_L$), then by applying Φ and $\hat{\Phi}$ we can get the corresponding exponential generating functions for permutations with the corresponding descent sets or alternating descent sets.

This means that if we can find the noncommutative symmetric function generating function for some class of words determined by their descent sets (i.e., we can express it in terms of the $\mathbf{r}_L$), then by applying Φ and $\hat{\Phi}$ we can get the corresponding exponential generating functions for permutations with the corresponding descent sets or alternating descent sets.

There are many formulas to which we can apply Φ and $\hat{\Phi}$, but one that gives us what we need for Nicolaescu's problem is the following:

Theorem 1. (G, Jackson and Aleliunas; 1977) Let $w_1, w_2, \ldots$ be arbitrary commuting weights. Then

$$\sum_L w_L \mathbf{r}_L = \left(\sum_{n=0}^{\infty} a_n \mathbf{h}_n\right)^{-1}$$

where the sum on the left is over all compositions $L = (L_1, \ldots, L_k)$, and $w_L = w_{L_1} \cdots w_{L_k}$.
Here $a_0, a_1, \ldots$ are defined by

$$\sum_{n=0}^{\infty} a_n z^n = \left(\sum_{n=0}^{\infty} w_n z^n\right)^{-1},$$

where $w_0 = 1$.

If we take $w_i = 1$ for $i < m$ and $w_i = 0$ for $i \geq m$ in Theorem 1 then we get

$$\sum_L \mathbf{r}_L = \left(\sum_{n=0}^{\infty} (\mathbf{h}_{mn} - \mathbf{h}_{mn+1}) \right)^{-1},$$

where the sum on the left is over compositions L with all parts less than m.

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where the sum on the left is over compositions L with all parts less than m.

Applying Φ gives David and Barton's result that the exponential generating function for permutations in which every increasing run has length less than m is

$$\left(1 - x + \frac{x^m}{m!} - \frac{x^{m+1}}{(m+1)!} + \frac{x^{2m}}{(2m)!} - \frac{x^{2m+1}}{(2m+1)!} + \cdots\right)^{-1},$$

Applying $\hat{\Phi}$ gives

$$\left[\sum_{n=0}^{\infty}\left(E_{mn}\frac{x^{mn}}{(mn)!} - E_{mn+1}\frac{x^{mn+1}}{(mn+1)!}\right)\right]^{-1}$$

as the exponential generating function for permutations in which every alternating run has length less than m; the case $m = 3$ is the solution to Nicolaescu's problem.

Two more special cases of Theorem 1

If we set $w_m = 1$ and $w_n = 0$ for $n \neq m$ then we find that

$$\sum_{n=0}^{\infty} \mathbf{r}_{(m^n)} = \left(\sum_{n=0}^{\infty} (-1)^n \, \mathbf{h}_{mn} \right)^{-1},$$

Applying Φ gives Carlitz's 1973 result that

$$\left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{mn}}{(mn)!} \right)^{-1}$$

is the exponential generating function for permutations in which every increasing run has length m.

Applying $\hat{\Phi}$ shows that

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counts permutations in which every alternating run has length m.

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For $m = 4$, these are permutations of $[4n]$ with descent set $\{2, 6, 10, \ldots, 4n-2\}$.

If we set $w_n = t$ for all $n \geq 1$ in Theorem 1, then we are counting words by the number of increasing runs (which is one more than the number of descents), and we get

$$\sum_{L} t^{l(L)} \mathbf{r}_L = (1-t) \left[1 - t \sum_{n=0}^{\infty} (1-t)^n \, \mathbf{h}_n \right]^{-1},$$

where the first sum is over all compositions L.

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where the first sum is over all compositions L.

Applying Φ gives the well-known generating function for the Eulerian polynomials,

$$1 + \sum_{n=1}^{\infty} A_n(t)\frac{x^n}{n!} = \frac{1-t}{1-te^{(1-t)x}},$$

where

$$A_n(t) = \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{des}(\pi)+1},$$

and $\mathrm{des}(\pi)$ is the number of descents of π.

Applying $\hat{\Phi}$ gives

$$1+\sum_{n=1}^{\infty}\hat{A}_n(t)\frac{x^n}{n!}=\frac{1-t}{1-t\big(\sec(1-t)x+\tan(1-t)x\big)},$$

where

$$\hat{A}_n(t)=\sum_{\pi\in\mathfrak{S}_n}t^{\mathrm{altdes}(\pi)+1}$$

and altdes(π) is the number of alternating descents of π, a result due to Chebikin.

The major index and alternating major index

We define the major index maj(π) of a permutation π to be the sum of the descents of π, and we define the alternating major index altmaj(π) to be the sum of the alternating descents of π.

Then there is an analogue of Theorem 1 that allows us to count permutations by major index and alternating major index.

I'll just state the results:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \frac{\sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{des}(\pi)} q^{\mathrm{maj}(\pi)}}{(1-t)(1-tq)\cdots(1-tq^n)} = \sum_{k=0}^{\infty} t^k \prod_{j=0}^{k} e^{xq^j}$$

(MacMahon/Riordan 1954/Carlitz 1975)

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \frac{\sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{altdes}(\pi)} q^{\mathrm{altmaj}(\pi)}}{(1-t)(1-tq)\cdots(1-tq^n)}$$

$$= \sum_{k=0}^{\infty} t^k \prod_{j=0}^{k} \left(\sec(xq^j) + \tan(xq^j)\right) \quad \text{(Remmel)}$$

The remaining material was not presented during my talk.

More homomorphisms

There are other homomorphisms from the ring of noncommutative symmetric functions that enable us to count permutations by other parameters.

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There are other homomorphisms from the ring of noncommutative symmetric functions that enable us to count permutations by other parameters.

The simplest one allows us to count permutations by inversions. An inversion of a permutation π is a pair (i, j) with $i < j$ such that $\pi_i > \pi_j$. We define a homomorphism Φ_1 on noncommutative symmetric function by

$$\Phi_1(\mathbf{h}_n) = \frac{x^n}{(q)_n},$$

where $(q)_n = (1-q)(1-q^2)\cdots(1-q^n)$. Then if L is a composition of n, we have

$$\Phi_1(\mathbf{r}_L) = \sum_{C(\pi)=L} q^{\mathrm{inv}(\pi)} \frac{x^n}{(q)_n}.$$

We define the inverse major index $\mathrm{imaj}(\pi)$ to be $\mathrm{maj}(\pi^{-1})$. Then it's also true that

$$\Phi_1(\mathbf{r}_L) = \sum_{C(\pi)=L} q^{\mathrm{imaj}(\pi)} \frac{x^n}{(q)_n}.$$

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Thus, for example,

$$\left(1 - x + \frac{x^3}{(q)_3} - \frac{x^4}{(q)_4} + \frac{x^6}{(q)_6} - \frac{x^7}{(q)_7} + \cdots\right)^{-1}$$

is the Eulerian generating function for permutations with no increasing runs of length 3 by inversions or by inverse major index.

More generally, if we make the variables X_i commute, we get the quasi-symmetric generating function for counting permutations by their inverse descent set, from which we can count by the number of inverse descents, or the number of inverse peaks, etc.

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These homomorphisms may be defined more naturally on a larger algebra than the noncommutative symmetric functions, the Malvenuto-Reutenauer algebra (also called the algebra of free quasi-symmetric functions).

To define it, we first define the standardization $\mathrm{st}(w)$ of the word w to be the permutation obtained by replacing the smallest entry by 1, the next smallest by 2, and so on, where repeated letters are ordered from left to right, so for example

$$\mathrm{st}(1\,1\,5\,2\,1\,2) = 1\,2\,6\,4\,3\,5.$$

Then for each $\pi \in \mathfrak{S}_n$ we define a series $[\pi]$ in the noncommuting variables $X_1, X_2, \ldots$ by

$$[\pi] = \sum_{\mathrm{st}(i_1 \cdots i_n) = \pi} X_{i_1} \cdots X_{i_n}$$

For example,

$$[132] = \sum_{a \leq b < c} X_a X_c X_b$$

and

$$[12 \cdots n] = \mathbf{h}_n.$$

The Malvenuto-Reutenauer algebra is spanned by the $[\pi]$ with $\pi \in S_n$ for some n.

The Malvenuto-Reutenauer algebra is spanned by the $[\pi]$ with $\pi \in S_n$ for some n.

It's easy to describe how these basis elements multiply: if $\pi \in \mathfrak{S}_m$ and $\sigma \in \mathfrak{S}_n$ then

$$[\pi][\sigma] = \sum [\tau]$$

where the sum is over all $\tau = \tau_1 \cdots \tau_{m+n} \in \mathfrak{S}_{m+n}$ for which $\mathrm{st}(\tau_1 \cdots \tau_m) = \pi$ and $\mathrm{st}(\tau_{m+1} \cdots \tau_{m+n}) = \sigma$.

Then the algebra of noncommutative symmetric functions is the subalgebra of the Malvenuto-Reutenauer algebra generating by the $\mathbf{h}_n = [1 \cdots n]$.

Then the homomorphism Φ_1 extends in two different ways to MR, we can define either

$$\Phi_{1\mathrm{a}}([\pi]) = q^{\mathrm{inv}(\pi)} \frac{x^n}{(q)_n}$$

or

$$\Phi_{1\mathrm{b}}([\pi]) = q^{\mathrm{imaj}(\pi)} \frac{x^n}{(q)_n},$$

where $\pi \in \mathfrak{S}_n$.

We have the following relations between the algebras that we've discussed:

$$\begin{array}{ccc} \text{NCSF} & \longrightarrow & \text{MR} \\ \big\downarrow & & \big\downarrow \\ \text{SF} & \longrightarrow & \text{QSF} \end{array}$$