Ehrhart series, unimodality, and integrally closed reflexive polytopes

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Lattice polytopes and Ehrhart series

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Theorem (E. Ehrhart (rationality), R. Stanley (non-negativity))

If P is d-dimensional, then

$$E_P(t) = \frac{\sum_{j=0}^d h_j^* t^j}{(1-t)^{d+1}} =: \frac{h_P^*(t)}{(1-t)^{d+1}},$$

where the h_j^*'s are non-negative integers.

Let e_i denote the i-th standard basis vector in $\mathbb{R}^d$.

Example

Denote by Lef_d the convex hull of the vectors

$$e_1, \ldots, e_d, -de_1 - \sum_{k=2}^{d}(d+1)e_k ,$$

Then

$$h^*_{Lef_d}(t) = 1 + (d+2)t + (d+2)t^2 + \cdots + (d+2)t^{d-1} + t^d$$

(reasonably straightforward computation).

Example

Denote by C_d the convex hull of the vectors

$$-\sum_{i=1}^{d} e_i, \quad e_1 - \sum_{i=2}^{d} e_i, \quad e_2 - \sum_{i=3}^{d} e_i, \quad \vdots \quad e_{d-1} - e_d, \quad e_d .$$

Then

$$h^*_{C_d}(t) = (1+t)^d$$

(immediate consequence of work by M. Beck, T. McAllister, and P. Jayawant on affine free sums).

Example

Denote by $LecHall_d$ the convex hull of the vectors

$$\begin{aligned} e_d, \quad & (d-1)e_{d-1} + de_d, \\ & (d-2)e_{d-2} + (d-1)e_{d-1} + de_d, \\ & \vdots \\ & 2e_2 + \cdots + de_d \\ & e_1 + 2e_2 + \cdots + de_d \, . \end{aligned}$$

Then

$$h^*_{LecHall_d}(t) = \sum_{\pi \in \mathfrak{S}_{d-1}} t^{\mathrm{des}(\pi)}$$

(implicit in work of S. Corteel, S. Lee, and C. Savage, not straightforward). The integer points in $LecHall_d$ are precisely the lecture hall partitions satisfying $\lambda_d - \lambda_{d-1} = 1$.

A few observations

- The polytopes C_d, Lef_d, and $LecHall_d$ are all simplices, i.e. the convex hull of $d+1$ points in $\mathbb{R}^d$.

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- The polytopes C_d, Lef_d, and $LecHall_d$ are all simplices, i.e. the convex hull of $d+1$ points in $\mathbb{R}^d$.
- These h^*-polynomials have unimodal coefficients.
- C_d and Lef_d are *reflexive* and *integrally closed*, and $LecHall_d$ is a lattice pyramid over an integrally closed reflexive simplex.

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The latter two questions are (very) open. In general, it isn't clear what is the "right" question to ask.

Integral closure

Definition

A lattice polytope P in $\mathbb{R}^d$ is *integrally closed* if for every $m \in \mathbb{N}$ and every

$$z \in mP \cap \mathbb{Z}^d \,,$$

there exist

$$x_1, \ldots, x_m \in P \cap \mathbb{Z}^d \text{ such that } x_1 + \cdots + x_m = z \,.$$

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- There is a closely-related notion of P being *normal* — while integral closure implies normality, the converse does not always hold.
- In the literature, the distinction between these two notions is often blurred.
- Existence of a unimodular triangulation of P implies integral closure.

Reflexive polytopes

Definition

A lattice polytope P containing 0 in its interior is *reflexive* if any of the following (equivalent) conditions hold:

- $h_i^* = h_{d-i}^*$ for all $i = 0, \ldots, d$.
- The dual of P is also a lattice polytope.
- The facets of P are each lattice distance 1 from the origin.

Equivalences due to T. Hibi.

Interesting properties of reflexives

Theorem (C. Haase & I. Melnikov)

Every lattice polytope is lattice equivalent to a face of some reflexive polytope.

Theorem (J. Lagarias & G. Ziegler)

In a fixed dimension d, there are finitely many reflexive polytopes (up to unimodular equivalence).

By computer calculations due to M. Kreuzer and H. Skarke, the number of reflexives in each dimension is:

$d = 2 : 16$

$d = 3 : 4,319$

$d = 4 : 473,800,776$

$d \geq 5$: unknown

Regular unimodular triangulations

Theorem (W. Bruns & T. Römer)

If P is

- *reflexive and*
- *admits a regular unimodular triangulation (hence is integrally closed),*

then $h_P^*(t)$ is the h-polynomial of the boundary complex of a simplicial polytope. Thus, $h_P^*(t)$ is unimodal.

Note: This actually holds for any Gorenstein polytope P.

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- There is a simple technique to compute h^*-vectors for simplices, and interesting h^*-polynomials occur here.
- Lattice simplices are a rich source of (surprising) examples in Ehrhart theory, e.g.
  - Reeve's tetrahedron showing no analogue of Pick's theorem is possible in $\mathbb{R}^3$
  - A. Higashitani's recent counterexamples to conjectures on Ehrhart polynomial roots
  - S. Payne's examples of non-h^*-unimodal reflexives

Free sums

Definition

Suppose $P, Q \subseteq \mathbb{R}^n$ are lattice polytopes. Call $P \oplus Q := \mathrm{conv}\{P \cup Q\}$ a *free sum* if, up to unimodular equivalence, $P \cap Q = \{0\}$ and the affine spans of P and Q are orthogonal coordinate subspaces of $\mathbb{R}^n$.

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Theorem (M. Beck, T. McAllister, & P. Jayawant)

If $P, Q \subseteq \mathbb{R}^n$ are reflexive polytopes such that $0 \in P^\circ$ and $P \oplus Q = \mathrm{conv}\{P \cup Q\}$ is a free sum, then

$$h^*_{P\oplus Q}(t) = h^*_P(t)h^*_Q(t).$$

Free sums and unimodality

Lemma

Suppose $P \subseteq \mathbb{R}^d$ and $Q \subseteq \mathbb{R}^n$ are full-dimensional reflexive simplices with $0 \in P^\circ$ and $\{v_0, \ldots, v_n\}$ denoting the vertices of Q. Then for each $i = 0, 1, \ldots, n$ the polytope formed by

$$P *_i Q := conv(P \times 0^n) \cup (0^d \times Q - v_i) \subseteq \mathbb{R}^{d+n}$$

is a free sum and is a reflexive simplex.

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Theorem (BB & R. Davis)

If P and Q are any integrally closed simplices with $0 \in P^\circ$ and P reflexive, then $P *_i Q$ is integrally closed.

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Corollary

If P and Q are integrally closed, reflexive, h^*-unimodal simplices with $0 \in P^\circ$, then so is $P *_i Q$ for each i.

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Experimentally, very few.
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To make such computations, we need a tool to check for free sum decompositions.

For each reflexive simplex P, it is known that one can assign a type vector $(q_0, q_1, \ldots, q_d, \lambda)$ (details omitted).

Type vectors decompose for free sums

Theorem (BB & R. Davis)

If

- $P = conv\{v_0, \ldots, v_n\} \subseteq \mathbb{R}^n$ *of type* $((p_0, \ldots, p_n), \lambda)$ *and*
- $Q = conv\{w_0, \ldots, w_m\} \subseteq \mathbb{R}^m$ *of type* $((q_0, \ldots, q_m), \mu)$

are full-dimensional reflexive simplices, then $P *_i Q$ *is a reflexive simplex of type*

$$\left(\frac{1}{d}(q_i p_0, q_i p_1, \ldots, q_i p_n, sq_0, sq_1, \ldots, \widehat{sq_i}, \ldots, sq_m), d\right),$$

where $s = \sum_{j=0}^{n} p_j$ *and* $d = \gcd(q_i, \sum_{j=0}^{n} p_j)$.

Lefschetz elements

For any integrally closed lattice simplex $P \subseteq \mathbb{R}^n$ with vertices $\{v_0, \ldots, v_n\}$, the Hilbert series of the 0-(Krull)-dimensional algebra

$$R_P := \mathbb{C}[x^a z | a \in P \cap \mathbb{Z}^n]/(x^{v_0} z, \ldots, x^{v_n} z) ,$$

graded by the exponent on z, is equal to $h^*_P(t)$. Here $x^a := x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$.

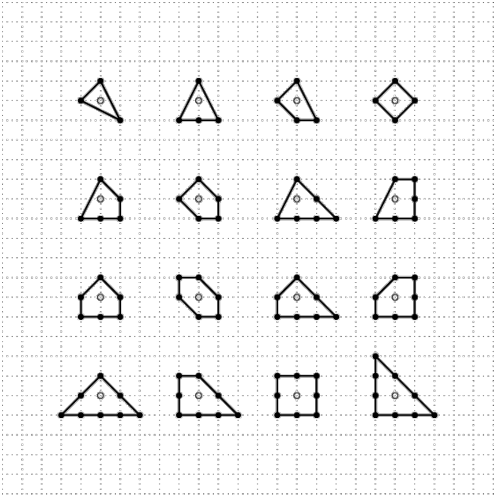
Existence of a *weak Lefschetz element* in R_P implies unimodality of $h^*_P(t)$.

Theorem (BB & R. Davis)

For every $d \geq 3$, R_{Lef_d} does not admit a weak Lefschetz element.

It is possible that quotienting the semigroup algebra generated by Lef_d by a different system of parameters would admit a weak Lefschetz element.

Thanks for listening!



The 16 reflexives in dimension 2. (Figure by F. Rodriguez Villegas.)