Weak and strong moments of l_r-norms of log-concave vectors

Rafał Latała (based on the joint work with Marta Strzelecka)

University of Warsaw

Minneapolis, April 14 2015

Log-concave measures/vectors

A measure μ on a locally convex linear space F is called *logarithmically concave* (*log-concave* in short) if for any compact nonempty sets $K, L \subset F$ and $\lambda \in [0, 1]$,

$$\mu(\lambda K + (1 - \lambda)L) \geq \mu(K)^{\lambda} \mu(L)^{1-\lambda}.$$

A random vector with values in F is called log-concave if its distribution is logarithmically concave.

By the result of Borell an n-dimensional vector with a full dimensional support is log-concave iff it has a log-concave density, i.e. a density of the form e^{-h}, where h is a convex function with values in $(-\infty, \infty]$.

Log-concave measures/vectors

A measure μ on a locally convex linear space F is called *logarithmically concave* (*log-concave* in short) if for any compact nonempty sets $K, L \subset F$ and $\lambda \in [0,1]$,

$$\mu(\lambda K + (1-\lambda)L) \geq \mu(K)^{\lambda}\mu(L)^{1-\lambda}.$$

A random vector with values in F is called log-concave if its distribution is logarithmically concave.

By the result of Borell an n-dimensional vector with a full dimensional support is log-concave iff it has a log-concave density, i.e. a density of the form e^{-h}, where h is a convex function with values in $(-\infty, \infty]$.

Examples of log-concave vectors

- Gaussian vectors
- Vectors with independent log-concave coordinates (in particular vectors with product exponential distribution)
- Vectors uniformly distributed on convex bodies
- Affine images of log-concave vectors
- Sums of independent log-concave vectors
- Weak limits of log-concave vectors

It may be shown that the class of log-concave distributions on $\mathbb{R}^n$ is the smallest class that contains uniform distributions on convex bodies and is closed under affine transformations and weak limits.

Examples of log-concave vectors

- Gaussian vectors
- Vectors with independent log-concave coordinates (in particular vectors with product exponential distribution)
- Vectors uniformly distributed on convex bodies
- Affine images of log-concave vectors
- Sums of independent log-concave vectors
- Weak limits of log-concave vectors

It may be shown that the class of log-concave distributions on $\mathbb{R}^n$ is the smallest class that contains uniform distributions on convex bodies and is closed under affine transformations and weak limits.

Isotropic vectors

Let $X = (X_1, \ldots, X_n)$ be a random vector in $\mathbb{R}^n$ such that $\mathbb{E}|X|^2 < \infty$. We say that the distribution of X is *isotropic*, if

$$\mathbb{E}X_i = 0 \text{ and } \mathbb{E}X_iX_j = \delta_{i,j} \text{ for all } 1 \le i,j \le n.$$

If $\mathbb{E}|X|^2 < \infty$ and X has a full dimensional support then there exists an affine transformation T such that TX is isotropic.

Here and in the sequel $|x| = \|x\|_2$, where

$$\|x\|_r = \left(\sum_{i \ge n} |x_i|^r\right)^{1/r} \quad \text{for } x \in \mathbb{R}^d, r \ge 1.$$

Remark. By the result of Borell, for any log-concave vector X and any seminorm $\|\cdot\|$, $\mathbb{E}\|X\|^p < \infty$. Moreover

$$(\mathbb{E}\|X\|^p)^{1/p} \le C\frac{p}{q}(\mathbb{E}\|X\|^q)^{1/q} \quad \text{for } p \ge q \ge 1.$$

By C we denote universal constants (that may differ from line to line).

Isotropic vectors

Let $X = (X_1, \ldots, X_n)$ be a random vector in $\mathbb{R}^n$ such that $\mathbb{E}|X|^2 < \infty$. We say that the distribution of X is *isotropic*, if

$$\mathbb{E}X_i = 0 \text{ and } \mathbb{E}X_iX_j = \delta_{i,j} \text{ for all } 1 \leq i,j \leq n.$$

If $\mathbb{E}|X|^2 < \infty$ and X has a full dimensional support then there exists an affine transformation T such that TX is isotropic. Here and in the sequel $|x| = \|x\|_2$, where

$$\|x\|_r = \left(\sum_{i \geq n} |x_i|^r\right)^{1/r} \quad \text{for } x \in \mathbb{R}^d, r \geq 1.$$

Remark. By the result of Borell, for any log-concave vector X and any seminorm $\|\cdot\|$, $\mathbb{E}\|X\|^p < \infty$. Moreover

$$(\mathbb{E}\|X\|^p)^{1/p} \leq C\frac{p}{q}(\mathbb{E}\|X\|^q)^{1/q} \quad \text{for } p \geq q \geq 1.$$

By C we denote universal constants (that may differ from line to line).

Isotropic vectors

Let $X = (X_1, \ldots, X_n)$ be a random vector in $\mathbb{R}^n$ such that $\mathbb{E}|X|^2 < \infty$. We say that the distribution of X is *isotropic*, if

$$\mathbb{E}X_i = 0 \text{ and } \mathbb{E}X_iX_j = \delta_{i,j} \text{ for all } 1 \leq i,j \leq n.$$

If $\mathbb{E}|X|^2 < \infty$ and X has a full dimensional support then there exists an affine transformation T such that TX is isotropic.
Here and in the sequel $|x| = \|x\|_2$, where

$$\|x\|_r = \left(\sum_{i \geq n} |x_i|^r\right)^{1/r} \quad \text{for } x \in \mathbb{R}^d, r \geq 1.$$

Remark. By the result of Borell, for any log-concave vector X and any seminorm $\|\cdot\|$, $\mathbb{E}\|X\|^p < \infty$. Moreover

$$(\mathbb{E}\|X\|^p)^{1/p} \leq C\frac{p}{q}(\mathbb{E}\|X\|^q)^{1/q} \quad \text{for } p \geq q \geq 1.$$

By C we denote universal constants (that may differ from line to line).

Paouris inequality

One of the fundamental properties of log-concave vectors is the Paouris inequality.

Theorem (Paouris'06)

Fo any log-concave vector X in $\mathbb{R}^n$,

$$(\mathbb{E}|X|^p)^{1/p} \le C\left((\mathbb{E}|X|^2)^{1/2} + \sigma_X(p)\right) \qquad \text{for } p \ge 1,$$

where

$$\sigma_X(p) := \sup_{\|t\|_2 \le 1} \left(\mathbb{E}\left|\sum_{i=1}^n t_i X_i\right|^p\right)^{1/p}.$$

Equivalently, in terms of tails we have

$$\mathbb{P}(|X| \ge Ct\mathbb{E}|X|) \le \exp\left(-\sigma_X^{-1}(t\mathbb{E}|X|)\right) \qquad \text{for } t \ge 1,$$

Paouris inequality

One of the fundamental properties of log-concave vectors is the Paouris inequality.

Theorem (Paouris'06)

Fo any log-concave vector X in $\mathbb{R}^n$,

$$(\mathbb{E}|X|^p)^{1/p} \leq C\left((\mathbb{E}|X|^2)^{1/2} + \sigma_X(p)\right) \qquad \text{for } p \geq 1,$$

where

$$\sigma_X(p) := \sup_{\|t\|_2 \leq 1} \left(\mathbb{E}\left|\sum_{i=1}^n t_i X_i\right|^p\right)^{1/p}.$$

Equivalently, in terms of tails we have

$$\mathbb{P}(|X| \geq Ct\mathbb{E}|X|) \leq \exp\left(-\sigma_X^{-1}(t\mathbb{E}|X|)\right) \qquad \text{for } t \geq 1,$$

Paouris inequality in the isotropic case

For an isotropic vector$X$,

$$\mathbb{E}|X| \le (\mathbb{E}|X|^2)^{1/2} = \sqrt{n}, \quad \text{and} \quad \sigma_X(2) = 1.$$

So if X is isotropic log-concave then

$$\sigma_X(p) \le p \quad \text{for} \quad p \ge 1.$$

Hence we have the following weaker form of the Paouris inequality.

Corollary

Fo any isotropic log-concave vector X in $\mathbb{R}^n$,

$$(\mathbb{E}|X|^p)^{1/p} \le C\left(\sqrt{n} + p\right) \qquad \text{for } p \ge 1.$$

and

$$\mathbb{P}(|X| \ge Ct\sqrt{n}) \le \exp\left(-t\sqrt{n}\right) \qquad \text{for } t \ge 1.$$

Paouris inequality in the isotropic case

For an isotropic vector$X$,

$$\mathbb{E}|X| \leq (\mathbb{E}|X|^2)^{1/2} = \sqrt{n}, \quad \text{and} \quad \sigma_X(2) = 1.$$

So if X is isotropic log-concave then

$$\sigma_X(p) \leq p \quad \text{for} \quad p \geq 1.$$

Hence we have the following weaker form of the Paouris inequality.

Corollary

Fo any isotropic log-concave vector X in $\mathbb{R}^n$,

$$(\mathbb{E}|X|^p)^{1/p} \leq C\left(\sqrt{n} + p\right) \qquad \text{for } p \geq 1.$$

and

$$\mathbb{P}(|X| \geq Ct\sqrt{n}) \leq \exp\left(-t\sqrt{n}\right) \qquad \text{for } t \geq 1.$$

Example

Let $Y = \sqrt{n} g U$, where U has a uniform distribution on S^{n-1} and g is the standard normal $\mathcal{N}(0,1)$ r.v., independent of U. Then it is easy to see that Y is isotropic, rotationally invariant and for any seminorm on $\mathbb{R}^n$

$$(\mathbb{E}\|Y\|^p)^{1/p} = \sqrt{n}(\mathbb{E}|g|^p)^{1/p}(\mathbb{E}\|U\|^p)^{1/p} \sim \sqrt{pn}(\mathbb{E}\|U\|^p)^{1/p} \text{ for } p \geq 1.$$

In particular this implies that for any $t \in \mathbb{R}^n$,

$$\left(\mathbb{E}\left|\sum_{i=1}^n t_i Y_i\right|^p\right)^{1/p} \leq C\frac{p}{q}\left(\mathbb{E}\left|\sum_{i=1}^n t_i Y_i\right|^q\right)^{1/q} \text{ for } p \geq q \geq 1.$$

Therefore

$$(\mathbb{E}|Y|^p)^{1/p} \sim \sqrt{pn}, \quad (\mathbb{E}|Y|^2)^{1/2} = \sqrt{n}, \quad \sigma_Y(p) \leq Cp$$

and for $1 \ll p \ll n$, $(\mathbb{E}|Y|^p)^{1/p} \gg (\mathbb{E}|Y|^2)^{1/2} + \sigma_Y(p)$.

It would be very valuable to have a nice characterization of random vectors which satisfy the Paouris inequality.

Example

Let $Y = \sqrt{n} g U$, where U has a uniform distribution on S^{n-1} and g is the standard normal $\mathcal{N}(0,1)$ r.v., independent of U. Then it is easy to see that Y is isotropic, rotationally invariant and for any seminorm on $\mathbb{R}^n$

$$(\mathbb{E}\|Y\|^p)^{1/p} = \sqrt{n}(\mathbb{E}|g|^p)^{1/p}(\mathbb{E}\|U\|^p)^{1/p} \sim \sqrt{pn}(\mathbb{E}\|U\|^p)^{1/p} \text{ for } p \geq 1.$$

In particular this implies that for any $t \in \mathbb{R}^n$,

$$\left(\mathbb{E}\left|\sum_{i=1}^n t_i Y_i\right|^p\right)^{1/p} \leq C\frac{p}{q}\left(\mathbb{E}\left|\sum_{i=1}^n t_i Y_i\right|^q\right)^{1/q} \text{ for } p \geq q \geq 1.$$

Therefore

$$(\mathbb{E}|Y|^p)^{1/p} \sim \sqrt{pn}, \quad (\mathbb{E}|Y|^2)^{1/2} = \sqrt{n}, \quad \sigma_Y(p) \leq Cp$$

and for $1 \ll p \ll n$, $(\mathbb{E}|Y|^p)^{1/p} \gg (\mathbb{E}|Y|^2)^{1/2} + \sigma_Y(p)$.

It would be very valuable to have a nice characterization of random vectors which satisfy the Paouris inequality.

Example

Let $Y = \sqrt{n} g U$, where U has a uniform distribution on S^{n-1} and g is the standard normal $\mathcal{N}(0,1)$ r.v., independent of U. Then it is easy to see that Y is isotropic, rotationally invariant and for any seminorm on $\mathbb{R}^n$

$$(\mathbb{E}\|Y\|^p)^{1/p} = \sqrt{n}(\mathbb{E}|g|^p)^{1/p}(\mathbb{E}\|U\|^p)^{1/p} \sim \sqrt{pn}(\mathbb{E}\|U\|^p)^{1/p} \text{ for } p \geq 1.$$

In particular this implies that for any $t \in \mathbb{R}^n$,

$$\left(\mathbb{E}\left|\sum_{i=1}^n t_i Y_i\right|^p\right)^{1/p} \leq C\frac{p}{q}\left(\mathbb{E}\left|\sum_{i=1}^n t_i Y_i\right|^q\right)^{1/q} \text{ for } p \geq q \geq 1.$$

Therefore

$$(\mathbb{E}|Y|^p)^{1/p} \sim \sqrt{pn}, \quad (\mathbb{E}|Y|^2)^{1/2} = \sqrt{n}, \quad \sigma_Y(p) \leq Cp$$

and for $1 \ll p \ll n$, $(\mathbb{E}|Y|^p)^{1/p} \gg (\mathbb{E}|Y|^2)^{1/2} + \sigma_Y(p)$.

It would be very valuable to have a nice characterization of random vectors which satisfy the Paouris inequality.

Conjecture about weak and strong moments

It is natural to ask whether the Paouris inequality may be generalized to non-Euclidean norms. One may risk the following conjecture.

Conjecture

There exists a universal constant C such that for any log-concave vector X with values in a normed space $(F, \|\ \|)$,

$$(\mathbb{E}\|X\|^p)^{1/p} \leq C\left(\mathbb{E}\|X\| + \sup_{\varphi\in F^*,\|\varphi\|_*\leq 1}(\mathbb{E}|\varphi(X)|^p)^{1/p}\right) \quad \text{for } p \geq 1.$$

Remark. Obviously for $p \geq 1$, $(\mathbb{E}\|X\|^p)^{1/p} \geq \mathbb{E}\|X\|$ and strong moments dominate weak moments, i.e.

$$(\mathbb{E}\|X\|^p)^{1/p} \geq \sup_{\varphi\in F^*,\|\varphi\|_*\leq 1}(\mathbb{E}|\varphi(X)|^p)^{1/p}.$$

Conjecture about weak and strong moments

It is natural to ask whether the Paouris inequality may be generalized to non-Euclidean norms. One may risk the following conjecture.

Conjecture

There exists a universal constant C such that for any log-concave vector X with values in a normed space $(F, \| \ \|)$,

$$(\mathbb{E}\|X\|^p)^{1/p} \leq C\left(\mathbb{E}\|X\| + \sup_{\varphi\in F^*, \|\varphi\|_*\leq 1} (\mathbb{E}|\varphi(X)|^p)^{1/p}\right) \quad \text{for } p \geq 1.$$

Remark. Obviously for $p \geq 1$, $(\mathbb{E}\|X\|^p)^{1/p} \geq \mathbb{E}\|X\|$ and strong moments dominate weak moments, i.e.

$$(\mathbb{E}\|X\|^p)^{1/p} \geq \sup_{\varphi\in F^*, \|\varphi\|_*\leq 1} (\mathbb{E}|\varphi(X)|^p)^{1/p}.$$

Main result

Theorem

Let X be a log-concave vector with values in a normed space $(F, \| \ \|)$ which may be isometrically embedded in l_r for some $r \in [2, \infty)$. Then for $p \geq 1$,

$$(\mathbb{E}\|X\|^p)^{1/p} \leq Cr \left(\mathbb{E}\|X\| + \sup_{\varphi \in F^*, \|\varphi\|_* \leq 1} (\mathbb{E}|\varphi(X)|^p)^{1/p} \right).$$

Remark. Let X and F be as above. Then by Chebyshev's inequality we obtain large deviation estimate for $\|X\|$:

$$\mathbb{P}(\|X\| \geq Crt\mathbb{E}\|X\|) \leq \exp\left(-\sigma_{X,F}^{-1}(t\mathbb{E}\|X\|)\right) \qquad \text{for } t \geq 1,$$

where

$$\sigma_{X,F}(p) := \sup_{\varphi \in F^*, \|\varphi\|_* \leq 1} (\mathbb{E}\varphi(X)^p)^{1/p} \qquad \text{for } p \geq 1$$

denotes the weak p-th moment of X.

Main result

Theorem

Let X be a log-concave vector with values in a normed space $(F, \| \ \|)$ which may be isometrically embedded in l_r for some $r \in [2, \infty)$. Then for $p \geq 1$,

$$(\mathbb{E}\|X\|^p)^{1/p} \leq Cr\left(\mathbb{E}\|X\| + \sup_{\varphi \in F^*, \|\varphi\|_* \leq 1} (\mathbb{E}|\varphi(X)|^p)^{1/p}\right).$$

Remark. Let X and F be as above. Then by Chebyshev's inequality we obtain large deviation estimate for $\|X\|$:

$$\mathbb{P}(\|X\| \geq Crt\mathbb{E}\|X\|) \leq \exp\left(-\sigma_{X,F}^{-1}(t\mathbb{E}\|X\|)\right) \quad \text{for } t \geq 1,$$

where

$$\sigma_{X,F}(p) := \sup_{\varphi \in F^*, \|\varphi\|_* \leq 1} (\mathbb{E}\varphi(X)^p)^{1/p} \quad \text{for } p \geq 1$$

denotes the weak p-th moment of X.

Isomorhic embeddings

Remark. If $i\colon F \to l_r$ is an isomorphic embedding and $\lambda = \|i\|_{F\to l_r}\|i^{-1}\|_{i(F)\to F}$, then we may define another norm on F by $\|x\|' := \|i(x)\|/\|i\|_{F\to l_r}$. Obviously $(F, \|\ \|')$ isometrically embeds in l_r, moreover $\|x\|' \le \|x\| \le \lambda\|x\|'$ for $x \in F$. Hence the previous theorem gives

$$
\begin{aligned}
(\mathbb{E}\|X\|^p)^{1/p} &\le \lambda(\mathbb{E}(\|X\|')^p)^{1/p} \\
&\le Cr\lambda\left(\mathbb{E}\|X\|' + \sup_{\varphi\in F^*,\|\varphi\|'_*\le 1}(\mathbb{E}|\varphi(X)|^p)^{1/p}\right) \\
&\le Cr\lambda\left(\mathbb{E}\|X\| + \sup_{\varphi\in F^*,\|\varphi\|_*\le 1}(\mathbb{E}|\varphi(X)|^p)^{1/p}\right).
\end{aligned}
$$

Reduction to finite dimension

Since log-concavity is preserved under linear transformations and, by Hahn-Banach theorem, any linear functional on a subspace of l_r is a restriction of a functional on the whole l_r with the same norm, it is enough to prove Theorem 2 for $F = l_r$. An easy approximation argument shows that we may consider finite dimensional spaces l_r^n. To simplify the notation for an n-dimensional vector X and $p \geq 1$ we write

$$\sigma_{r,X}(p) := \sup_{\|t\|_{r'} \leq 1} \left(\mathbb{E} \left| \sum_{i=1}^{n} t_i X_i \right|^p \right)^{1/p},$$

where r' denotes the Hölder's dual of r, i.e. $r' = \frac{r}{r-1}$.

Theorem

Let X be a log-concave vector in $\mathbb{R}^n$ and $r \in [2, \infty)$. Then

$$(\mathbb{E}\|X\|_r^p)^{1/p} \leq Cr \left(\mathbb{E}\|X\|_r + \sigma_{r,X}(p) \right) \qquad \text{for } p \geq 1.$$

Reduction to finite dimension

Since log-concavity is preserved under linear transformations and, by Hahn-Banach theorem, any linear functional on a subspace of l_r is a restriction of a functional on the whole l_r with the same norm, it is enough to prove Theorem 2 for $F = l_r$. An easy approximation argument shows that we may consider finite dimensional spaces l_r^n. To simplify the notation for an n-dimensional vector X and $p \geq 1$ we write

$$\sigma_{r,X}(p) := \sup_{\|t\|_{r'} \leq 1} \left(\mathbb{E} \left| \sum_{i=1}^{n} t_i X_i \right|^p \right)^{1/p},$$

where r' denotes the Hölder's dual of r, i.e. $r' = \frac{r}{r-1}$.

Theorem

Let X be a log-concave vector in $\mathbb{R}^n$ and $r \in [2, \infty)$. Then

$$(\mathbb{E}\|X\|_r^p)^{1/p} \leq Cr \left(\mathbb{E}\|X\|_r + \sigma_{r,X}(p) \right) \qquad \textit{for } p \geq 1.$$

Modified bound for l_r-norms

Proof of the main result is based on the following estimate.

Theorem

Suppose that $r \in [2, \infty)$ and X is a log-concave n-dimensional random vector. Let

$$d_i := (\mathbb{E}X_i^2)^{1/2}, \quad d := \left(\sum_{i=1}^n d_i^r\right)^{1/r}. \tag{1}$$

Then for $p \geq r$ and $t \geq Cr \log\left(\frac{d}{\sigma_{r,X}(p)}\right)$,

$$\mathbb{E}\left(\sum_{i=1}^n |X_i|^r \mathbb{1}_{\{|X_i| \geq td_i\}}\right)^{p/r} \leq (Cr\sigma_{r,X}(p))^p.$$

Modified bound implies comparison of weak and strong moments in l_r^n.

Since $(\mathbb{E}\|X\|_r^p)^{1/p} \leq Cp\mathbb{E}\|X\|_r$, we may assume that $p \geq r$. Let d_i and d be definde by (1). Then

$$d^2 = \|(\mathbb{E}X_i^2)\|_{r/2} \leq \mathbb{E}\|(X_i^2)\|_{r/2} = \mathbb{E}\|X\|_r^2 \leq (C\mathbb{E}\|X\|_r)^2.$$

Set

$$\tilde{p} := \inf\{q \geq p\colon\ \sigma_{r,X}(q) \geq d\}.$$

Modified bound applied with $\tilde{p}$ instead of p and $t = 0$ yields

$$\begin{aligned}(\mathbb{E}\|X\|_r^p)^{1/p} &\leq (\mathbb{E}\|X\|_r^{\tilde{p}})^{1/\tilde{p}} \leq Cr\sigma_{r,X}(\tilde{p}) = Cr\max\{d, \sigma_{r,X}(p)\} \\ &\leq Cr(\mathbb{E}\|X\|_r + \sigma_{r,X}(p)).\end{aligned}$$

Modified bound implies comparison of weak and strong moments in l_r^n.

Since $(\mathbb{E}\|X\|_r^p)^{1/p} \le Cp\mathbb{E}\|X\|_r$, we may assume that $p \ge r$. Let d_i and d be definde by (1). Then

$$d^2 = \|(\mathbb{E}X_i^2)\|_{r/2} \le \mathbb{E}\|(X_i^2)\|_{r/2} = \mathbb{E}\|X\|_r^2 \le (C\mathbb{E}\|X\|_r)^2.$$

Set

$$\tilde{p} := \inf\{q \ge p\colon\ \sigma_{r,X}(q) \ge d\}.$$

Modified bound applied with $\tilde{p}$ instead of p and $t = 0$ yields

$$\begin{aligned}(\mathbb{E}\|X\|_r^p)^{1/p} \le (\mathbb{E}\|X\|_r^{\tilde{p}})^{1/\tilde{p}} \le Cr\sigma_{r,X}(\tilde{p}) = Cr\max\{d, \sigma_{r,X}(p)\} \\ \le Cr(\mathbb{E}\|X\|_r + \sigma_{r,X}(p)).\end{aligned}$$

Modified bound implies comparison of weak and strong moments in l_r^n.

Since $(\mathbb{E}\|X\|_r^p)^{1/p} \leq Cp\mathbb{E}\|X\|_r$, we may assume that $p \geq r$. Let d_i and d be definde by (1). Then

$$d^2 = \|(\mathbb{E}X_i^2)\|_{r/2} \leq \mathbb{E}\|(X_i^2)\|_{r/2} = \mathbb{E}\|X\|_r^2 \leq (C\mathbb{E}\|X\|_r)^2.$$

Set

$$\tilde{p} := \inf\{q \geq p\colon \ \sigma_{r,X}(q) \geq d\}.$$

Modified bound applied with $\tilde{p}$ instead of p and $t = 0$ yields

$$\begin{aligned}(\mathbb{E}\|X\|_r^p)^{1/p} \leq (\mathbb{E}\|X\|_r^{\tilde{p}})^{1/\tilde{p}} \leq Cr\sigma_{r,X}(\tilde{p}) = Cr\max\{d, \sigma_{r,X}(p)\} \\ \leq Cr(\mathbb{E}\|X\|_r + \sigma_{r,X}(p)).\end{aligned}$$

Idea of the proof of the modified bound

Random vector $-X$ is also log-concave, has the same values of d_i and $\sigma_{r,-X} = \sigma_{r,X}$. Hence it is enough to show that

$$\mathbb{E}\left(\sum_{i=1}^{n} X_i^r \mathbb{1}_{\{X_i \geq td_i\}}\right)^{p/r} \leq (Cr\sigma_{r,X}(p))^p$$

for $t \geq Cr\log(d/\sigma_{r,X}(p))$.

It is easy to reduce to the case when $t \geq Cr$ and $l = p/r$ is a positive integer. For $l = 1, 2, \ldots$ we have

$$\mathbb{E}\left(\sum_{i=1}^{n} X_i^r \mathbb{1}_{\{X_i \geq td_i\}}\right)^{l} \leq \mathbb{E}\left(\sum_{i=1}^{n}\sum_{k=0}^{\infty} 2^{(k+1)r}(td_i)^r \mathbb{1}_{\{X_i \geq 2^k td_i\}}\right)^{l}$$

$$= (2t)^{rl} \sum_{i_1,\ldots,i_l=1}^{n} \sum_{k_1,\ldots,k_l=0}^{\infty} 2^{(k_1+\ldots+k_l)r} d_{i_1}^r \ldots d_{i_l}^r \mathbb{P}(B_{i_1,k_1\ldots,i_l,k_l}),$$

where

$$B_{i_1,k_1\ldots,i_l,k_l} := \{X_{i_1} \geq 2^{k_1} td_{i_1}, \ldots, X_{i_l} \geq 2^{k_l} td_{i_l}\}.$$

Idea of the proof of the modified bound

Random vector $-X$ is also log-concave, has the same values of d_i and $\sigma_{r,-X} = \sigma_{r,X}$. Hence it is enough to show that

$$\mathbb{E}\left(\sum_{i=1}^{n} X_i^r \mathbb{1}_{\{X_i \geq td_i\}}\right)^{p/r} \leq (Cr\sigma_{r,X}(p))^p$$

for $t \geq Cr\log(d/\sigma_{r,X}(p))$.
It is easy to reduce to the case when $t \geq Cr$ and $l = p/r$ is a positive integer. For $l = 1, 2, \ldots$ we have

$$\mathbb{E}\left(\sum_{i=1}^{n} X_i^r \mathbb{1}_{\{X_i \geq td_i\}}\right)^l \leq \mathbb{E}\left(\sum_{i=1}^{n}\sum_{k=0}^{\infty} 2^{(k+1)r}(td_i)^r \mathbb{1}_{\{X_i \geq 2^k td_i\}}\right)^l$$

$$= (2t)^{rl} \sum_{i_1,\ldots,i_l=1}^{n} \sum_{k_1,\ldots,k_l=0}^{\infty} 2^{(k_1+\ldots+k_l)r} d_{i_1}^r \ldots d_{i_l}^r \mathbb{P}(B_{i_1,k_1\ldots,i_l,k_l}),$$

where

$$B_{i_1,k_1\ldots,i_l,k_l} := \{X_{i_1} \geq 2^{k_1} td_{i_1}, \ldots, X_{i_l} \geq 2^{k_l} td_{i_l}\}.$$

Idea of the proof of the modified bound

Random vector $-X$ is also log-concave, has the same values of d_i and $\sigma_{r,-X} = \sigma_{r,X}$. Hence it is enough to show that

$$\mathbb{E}\left(\sum_{i=1}^{n} X_i^r \mathbb{1}_{\{X_i \geq td_i\}}\right)^{p/r} \leq (Cr\sigma_{r,X}(p))^p$$

for $t \geq Cr\log(d/\sigma_{r,X}(p))$.

It is easy to reduce to the case when $t \geq Cr$ and $l = p/r$ is a positive integer. For $l = 1, 2, \ldots$ we have

$$\mathbb{E}\left(\sum_{i=1}^{n} X_i^r \mathbb{1}_{\{X_i \geq td_i\}}\right)^{l} \leq \mathbb{E}\left(\sum_{i=1}^{n}\sum_{k=0}^{\infty} 2^{(k+1)r}(td_i)^r \mathbb{1}_{\{X_i \geq 2^k td_i\}}\right)^{l}$$

$$= (2t)^{rl} \sum_{i_1,\ldots,i_l=1}^{n} \sum_{k_1,\ldots,k_l=0}^{\infty} 2^{(k_1+\ldots+k_l)r} d_{i_1}^r \ldots d_{i_l}^r \mathbb{P}(B_{i_1,k_1\ldots,i_l,k_l}),$$

where

$$B_{i_1,k_1\ldots,i_l,k_l} := \{X_{i_1} \geq 2^{k_1} td_{i_1}, \ldots, X_{i_l} \geq 2^{k_l} td_{i_l}\}.$$

Idea of the proof of modified bound II

So we are to show that

$$m(l) := \sum_{k_1,\ldots,k_l=0}^{\infty} \sum_{i_1,\ldots,i_l=1}^{n} 2^{(k_1+\ldots+k_l)r} d_{i_1}^r \ldots d_{i_l}^r \mathbb{P}(B_{i_1,k_1,\ldots,i_l,k_l}).$$

$$\leq \left(\frac{Cr\sigma_{r,X}(rl)}{t} \right)^{rl}$$

for $t \geq Cr \max \left\{ 1, \log \left(\frac{d}{\sigma_{r,X}(rl)} \right) \right\}$.

This is done by dividing terms in $m(l)$ into a number of groups and estimating each of them.

Idea of the proof of modified bound II

So we are to show that

$$m(l) := \sum_{k_1,\ldots,k_l=0}^{\infty} \sum_{i_1,\ldots,i_l=1}^{n} 2^{(k_1+\ldots+k_l)r} d_{i_1}^r \ldots d_{i_l}^r \mathbb{P}(B_{i_1,k_1,\ldots,i_l,k_l}).$$

$$\leq \left(\frac{Cr\sigma_{r,X}(rl)}{t} \right)^{rl}$$

for $t \geq Cr \max\left\{1, \log\left(\frac{d}{\sigma_{r,X}(rl)}\right)\right\}$.
This is done by dividing terms in $m(l)$ into a number of groups and estimating each of them.

Crucial technical estimate

Proposition

Let X, r, d_i and d be as before and $A := \{X \in K\}$, where K is a convex set in $\mathbb{R}^n$ satisfying $0 < \mathbb{P}(A) \leq 1/e$. Then for every $t \geq r$,

$$\sum_{i=1}^{n} \mathbb{E}|X_i|^r \mathbb{1}_{A \cap \{X_i \geq td_i\}} \leq C^r \mathbb{P}(A) \left(r^r \sigma_{r,X}^r (-\log(\mathbb{P}(A))) + (dt)^r e^{-t/C} \right).$$

The proof is based on the fact that vector Y distributed as X conditioned on the set $A = \{X \in K\}$, i.e.

$$\mathbb{P}(Y \in B) = \frac{\mathbb{P}(X \in B)}{\mathbb{P}(X \in B \cap K)},$$

is again log-concave if K is convex. To see that $\mathbb{E}Y_i^2$ cannot be large for too many i's we use the Paouris inequality for X.

Crucial technical estimate

Proposition

Let X, r, d_i and d be as before and $A := \{X \in K\}$, where K is a convex set in $\mathbb{R}^n$ satisfying $0 < \mathbb{P}(A) \leq 1/e$. Then for every $t \geq r$,

$$\sum_{i=1}^{n} \mathbb{E}|X_i|^r \mathbb{1}_{A \cap \{X_i \geq td_i\}} \leq C^r \mathbb{P}(A) \left(r^r \sigma_{r,X}^r (-\log(\mathbb{P}(A))) + (dt)^r e^{-t/C} \right).$$

The proof is based on the fact that vector Y distributed as X conditioned on the set $A = \{X \in K\}$, i.e.

$$\mathbb{P}(Y \in B) = \frac{\mathbb{P}(X \in B)}{\mathbb{P}(X \in B \cap K)},$$

is again log-concave if K is convex. To see that $\mathbb{E}Y_i^2$ cannot be large for too many i's we use the Paouris inequality for X.

Case $r = \infty$

Recall the general conjecture about comparison of weak and strong moments

$$(\mathbb{E}\|X\|^p)^{1/p} \leq C\left(\mathbb{E}\|X\| + \sup_{\varphi\in F^*,\|\varphi\|_*\leq 1}(\mathbb{E}|\varphi(X)|^p)^{1/p}\right) \quad \text{for } p \geq 1. \tag{2}$$

Since every separable Banach space embeds in l_∞ it is enough to prove (2) in l_∞^n. It is known, but under the additional assumption that X is isotropic.

Theorem

Let X be an isotropic log-concave vector in $\mathbb{R}^n$. Then for any $a_1, \ldots, a_n$ and $p \geq 1$,

$$(\mathbb{E}\max_i |a_i X_i|^p)^{1/p} \leq C\left(\mathbb{E}\max_i |a_i X_i| + \max_i (\mathbb{E}|X_i|^p)^{1/p}\right)$$

The proof is completely different than in the case of l_r-norms. It uses exponential concentration of log-concave vectors.

Case $r = \infty$

Recall the general conjecture about comparison of weak and strong moments

$$(\mathbb{E}\|X\|^p)^{1/p} \leq C\left(\mathbb{E}\|X\| + \sup_{\varphi\in F^*,\|\varphi\|_*\leq 1}(\mathbb{E}|\varphi(X)|^p)^{1/p}\right) \quad \text{for } p \geq 1. \tag{2}$$

Since every separable Banach space embeds in l_∞ it is enough to prove (2) in l_∞^n. It is known, but under the additional assumption that X is isotropic.

Theorem

Let X be an isotropic log-concave vector in $\mathbb{R}^n$. Then for any $a_1, \ldots, a_n$ and $p \geq 1$,

$$(\mathbb{E}\max_i |a_i X_i|^p)^{1/p} \leq C\left(\mathbb{E}\max_i |a_i X_i| + \max_i (\mathbb{E}|X_i|^p)^{1/p}\right)$$

The proof is completely different than in the case of l_r-norms. It uses exponential concentration of log-concave vectors.

Case $r = \infty$

Recall the general conjecture about comparison of weak and strong moments

$$(\mathbb{E}\|X\|^p)^{1/p} \le C\left(\mathbb{E}\|X\| + \sup_{\varphi\in F^*,\|\varphi\|_*\le 1}(\mathbb{E}|\varphi(X)|^p)^{1/p}\right) \quad \text{for } p \ge 1. \tag{2}$$

Since every separable Banach space embeds in l_∞ it is enough to prove (2) in l_∞^n. It is known, but under the additional assumption that X is isotropic.

Theorem

Let X be an isotropic log-concave vector in $\mathbb{R}^n$. Then for any $a_1,\ldots,a_n$ and $p \ge 1$,

$$(\mathbb{E}\max_i |a_iX_i|^p)^{1/p} \le C\left(\mathbb{E}\max_i |a_iX_i| + \max_i(\mathbb{E}|X_i|^p)^{1/p}\right)$$

The proof is completely different than in the case of l_r-norms. It uses exponential concentration of log-concave vectors.

Exponential concentration

Let μ be a measure on $\mathbb{R}^n$. We say that μ *satisfies the exponential concentration with constant* α if for any Borel set A,

$$\mu(A) \geq \frac{1}{2} \;\Rightarrow\; \mu(A + \alpha t B_2^n) \geq 1 - e^{-t} \text{ for } t \geq 0.$$

The fundamental open problem (Kannan-Lovász-Simonovits conjecture) states that every isotropic log-concave measure satisfies exponential concentration with universal α.
Klartag proved it with $\alpha \leq Cn^{1/2-\varepsilon}$ with $\varepsilon \geq 1/30$. This was improved by Eldan (with the use of the result of Guédon-E.Milman) to $\alpha \leq Cn^{1/3} \log^{1/2}(n+1)$.

Exponential concentration

Let μ be a measure on $\mathbb{R}^n$. We say that μ *satisfies the exponential concentration with constant* α if for any Borel set A,

$$\mu(A) \geq \frac{1}{2} \; \Rightarrow \; \mu(A + \alpha t B_2^n) \geq 1 - e^{-t} \text{ for } t \geq 0.$$

The fundamental open problem (Kannan-Lovász-Simonovits conjecture) states that every isotropic log-concave measure satisfies exponential concentration with universal α.
Klartag proved it with $\alpha \leq Cn^{1/2-\varepsilon}$ with $\varepsilon \geq 1/30$. This was improved by Eldan (with the use of the result of Guédon-E.Milman) to $\alpha \leq Cn^{1/3}\log^{1/2}(n+1)$.

Exponential concentration

Let μ be a measure on $\mathbb{R}^n$. We say that μ *satisfies the exponential concentration with constant* α if for any Borel set A,

$$\mu(A) \geq \frac{1}{2} \;\Rightarrow\; \mu(A + \alpha t B_2^n) \geq 1 - e^{-t} \text{ for } t \geq 0.$$

The fundamental open problem (Kannan-Lovász-Simonovits conjecture) states that every isotropic log-concave measure satisfies exponential concentration with universal α.
Klartag proved it with $\alpha \leq Cn^{1/2-\varepsilon}$ with $\varepsilon \geq 1/30$. This was improved by Eldan (with the use of the result of Guédon-E.Milman) to $\alpha \leq Cn^{1/3} \log^{1/2}(n+1)$.

Optimal concentration inequalities

For a probability measure μ on $\mathbb{R}^n$ define

$$\Lambda_\mu(y) = \log \int e^{\langle y,z \rangle} d\mu(z), \quad \Lambda_\mu^*(x) = \sup_y (\langle y, x \rangle - \Lambda_\mu(y))$$

and

$$B_\mu(t) = \{x \in \mathbb{R}^n \colon \ \Lambda_\mu(x) \leq t\}.$$

We say that μ *satifies the optimal concentration inequality* with constant α if for any Borel set A,

$$\mu(A) \geq \frac{1}{2} \ \Rightarrow \ \mu(A + \alpha B_\mu(t)) \geq 1 - e^{-t} \text{ for } t \geq 1.$$

Proposition

If the law of an n-dimensional random vectors X satisfies the optimal concentration inequality with constant α then

$$(\mathbb{E}\|X\|^p)^{1/p} \leq \mathbb{E}\|X\| + C\alpha \sup_{\varphi \in F^*, \|\varphi\|_* \leq 1} (\mathbb{E}|\varphi(X)|^p)^{1/p} \quad \textit{for } p \geq 1.$$

Optimal concentration inequalities

For a probability measure μ on $\mathbb{R}^n$ define

$$\Lambda_\mu(y) = \log \int e^{\langle y,z\rangle} d\mu(z), \quad \Lambda^*_\mu(x) = \sup_y(\langle y,x\rangle - \Lambda_\mu(y))$$

and

$$B_\mu(t) = \{x \in \mathbb{R}^n\colon\ \Lambda_\mu(x) \le t\}.$$

We say that μ *satisfies the optimal concentration inequality* with constant α if for any Borel set A,

$$\mu(A) \ge \frac{1}{2} \ \Rightarrow\ \mu(A + \alpha B_\mu(t)) \ge 1 - e^{-t} \text{ for } t \ge 1.$$

Proposition

If the law of an n-dimensional random vectors X satisfies the optimal concentration inequality with constant α then

$$(\mathbb{E}\|X\|^p)^{1/p} \le \mathbb{E}\|X\| + C\alpha \sup_{\varphi\in F^*,\|\varphi\|_*\le 1} (\mathbb{E}|\varphi(X)|^p)^{1/p} \quad \textit{for } p \ge 1.$$

Optimal concentration inequalities

For a probability measure μ on $\mathbb{R}^n$ define

$$\Lambda_\mu(y) = \log \int e^{\langle y,z\rangle} d\mu(z), \quad \Lambda^*_\mu(x) = \sup_y(\langle y,x\rangle - \Lambda_\mu(y))$$

and

$$B_\mu(t) = \{x \in \mathbb{R}^n\colon\ \Lambda_\mu(x) \le t\}.$$

We say that μ *satifies the optimal concentration inequality* with constant α if for any Borel set A,

$$\mu(A) \ge \frac{1}{2} \ \Rightarrow\ \mu(A + \alpha B_\mu(t)) \ge 1 - e^{-t} \text{ for } t \ge 1.$$

Proposition

If the law of an n-dimensional random vectors X satisfies the optimal concentration inequality with constant α then

$$(\mathbb{E}\|X\|^p)^{1/p} \le \mathbb{E}\|X\| + C\alpha \sup_{\varphi\in F^*,\|\varphi\|_*\le 1} (\mathbb{E}|\varphi(X)|^p)^{1/p} \quad \textit{for } p \ge 1.$$

Example - Talagrand's two level concentration

In the case when $\mu = \nu^n$ is the product exponential measure (i.e. the measure with the density $2^{-n}\exp(-\sum_{i\le n}|x_i|)$ it is easy to check that for $t \ge 1$,

$$B_{\nu^n}(t) \sim tB_1^n + \sqrt{t}B_2^n.$$

The optimal concentration inequality in this case is the Talagrand two-level concentration

$$\nu^n(A) \ge \frac{1}{2} \;\Rightarrow\; \nu^n(A + tB_1^n + \sqrt{t}B_2^n) \ge 1 - e^{-t/C} \quad \text{for } t \ge 1.$$

Optimal concentration inequalities are strictly related to infimum convolution inequalities.

Example - Talagrand's two level concentration

In the case when $\mu = \nu^n$ is the product exponential measure (i.e. the measure with the density $2^{-n}\exp(-\sum_{i\leq n}|x_i|)$ it is easy to check that for $t \geq 1$,

$$B_{\nu^n}(t) \sim tB_1^n + \sqrt{t}B_2^n.$$

The optimal concentration inequality in this case is the Talagrand two-level concentration

$$\nu^n(A) \geq \frac{1}{2} \;\Rightarrow\; \nu^n(A + tB_1^n + \sqrt{t}B_2^n) \geq 1 - e^{-t/C} \quad \text{for } t \geq 1.$$

Optimal concentration inequalities are strictly related to infimum convolution inequalities.

Optimal concentration inequalities - examples

Examples of vectors that satisfy the optimal concentration inequality with universal constant

- Gaussian vectors
- vectors with independent log-concave coordinates
- rotationally invariant log-concave vectors
- uniform distributions on B_r^n-balls
- log-concave vectors with densities of the form $\exp(-g(\|x\|_r))$, $1 \le r < \infty$, $g\colon [0,\infty) \to (-\infty,\infty]$ convex increasing.

Corollary

For all random vectors listed above

$$(\mathbb{E}\|X\|^p)^{1/p} \le \mathbb{E}\|X\| + C \sup_{\varphi\in F^*,\|\varphi\|_*\le 1} (\mathbb{E}|\varphi(X)|^p)^{1/p} \quad \text{for } p \ge 1.$$

Optimal concentration inequalities - examples

Examples of vectors that satisfy the optimal concentration inequality with universal constant

- Gaussian vectors
- vectors with independent log-concave coordinates
- rotationally invariant log-concave vectors
- uniform distributions on B_r^n-balls
- log-concave vectors with densities of the form $\exp(-g(\|x\|_r))$, $1 \leq r < \infty$, $g\colon [0,\infty) \to (-\infty,\infty]$ convex increasing.

Corollary

For all random vectors listed above

$$(\mathbb{E}\|X\|^p)^{1/p} \leq \mathbb{E}\|X\| + C \sup_{\varphi\in F^*, \|\varphi\|_* \leq 1} (\mathbb{E}|\varphi(X)|^p)^{1/p} \quad \text{for } p \geq 1.$$

Unconditional vectors

We say that a random vector $X = (X_1, \ldots, X_n)$ has *unconditional distribution* if the distribution of $(\eta_1 X_1, \ldots, \eta_n X_n)$ is the same as X for any choice of signs $\eta_1, \ldots, \eta_n$.

Theorem

Let X be an n-dimensional isotropic, unconditional, log-concave vector and $Y = (Y_1, \ldots, Y_n)$, where Y_i are independent symmetric exponential r.v's with variance 1 (i.e. with the density $2^{-1/2}\exp(-\sqrt{2}|x|)$). Then for any norm $\| \ \|$ on $\mathbb{R}^n$ and $p \geq 1$,

$$(\mathbb{E}\|X\|^p)^{1/p} \leq C\Big(\mathbb{E}\|Y\| + \sup_{\|\varphi\|_* \leq 1} (\mathbb{E}|\varphi(X)|^p)^{1/p}\Big).$$

Proof is based on the Talagrand two-sided estimate of $\mathbb{E}\|Y\|$ and the Bobkov-Nazarov bound for the joint d.f. of X, which implies $(\mathbb{E}|\varphi(X)|^p)^{1/p} \leq C(\mathbb{E}|\varphi(Y)|^p)^{1/p}$ for $p \geq 1$.

Unconditional vectors

We say that a random vector $X = (X_1, \ldots, X_n)$ has *unconditional distribution* if the distribution of $(\eta_1 X_1, \ldots, \eta_n X_n)$ is the same as X for any choice of signs $\eta_1, \ldots, \eta_n$.

Theorem

Let X be an n-dimensional isotropic, unconditional, log-concave vector and $Y = (Y_1, \ldots, Y_n)$, where Y_i are independent symmetric exponential r.v's with variance 1 (i.e. with the density $2^{-1/2}\exp(-\sqrt{2}|x|)$). Then for any norm $\| \ \|$ on $\mathbb{R}^n$ and $p \geq 1$,

$$(\mathbb{E}\|X\|^p)^{1/p} \leq C\Big(\mathbb{E}\|Y\| + \sup_{\|\varphi\|_* \leq 1} (\mathbb{E}|\varphi(X)|^p)^{1/p}\Big).$$

Proof is based on the Talagrand two-sided estimate of $\mathbb{E}\|Y\|$ and the Bobkov-Nazarov bound for the joint d.f. of X, which implies $(\mathbb{E}|\varphi(X)|^p)^{1/p} \leq C(\mathbb{E}|\varphi(Y)|^p)^{1/p}$ for $p \geq 1$.

Unconditional vectors

We say that a random vector $X = (X_1, \ldots, X_n)$ has *unconditional distribution* if the distribution of $(\eta_1 X_1, \ldots, \eta_n X_n)$ is the same as X for any choice of signs $\eta_1, \ldots, \eta_n$.

Theorem

Let X be an n-dimensional isotropic, unconditional, log-concave vector and $Y = (Y_1, \ldots, Y_n)$, where Y_i are independent symmetric exponential r.v's with variance 1 (i.e. with the density $2^{-1/2}\exp(-\sqrt{2}|x|)$). Then for any norm $\| \ \|$ on $\mathbb{R}^n$ and $p \geq 1$,

$$(\mathbb{E}\|X\|^p)^{1/p} \leq C\Big(\mathbb{E}\|Y\| + \sup_{\|\varphi\|_* \leq 1} (\mathbb{E}|\varphi(X)|^p)^{1/p}\Big).$$

Proof is based on the Talagrand two-sided estimate of $\mathbb{E}\|Y\|$ and the Bobkov-Nazarov bound for the joint d.f. of X, which implies $(\mathbb{E}|\varphi(X)|^p)^{1/p} \leq C(\mathbb{E}|\varphi(Y)|^p)^{1/p}$ for $p \geq 1$.

Unconditional vectors ctd

Using the easy estimate $\mathbb{E}\|Y\| \leq C \log n \, \mathbb{E}\|X\|$ we get.

Corollary

For any n-dimensional unconditional, log-concave vector X, any norm $\| \ \|$ on $\mathbb{R}^n$ and $p \geq 1$ one has

$$(\mathbb{E}\|X\|^p)^{1/p} \leq C\Big(\log n \, \mathbb{E}\|X\| + \sup_{\|\varphi\|_* \leq 1} (\mathbb{E}|\varphi(X)|^p)^{1/p}\Big).$$

The Maurey-Pisier result implies $\mathbb{E}\|Y\| \leq C\mathbb{E}\|X\|$ in spaces with nontrivial cotype.

Corollary

Let X be as above, $2 \leq q < \infty$ and $F = (\mathbb{R}^n, \| \ \|)$ has a q-cotype constant bounded by $\beta < \infty$.

$$(\mathbb{E}\|X\|^p)^{1/p} \leq C(q,\beta)\Big(\mathbb{E}\|X\| + \sup_{\|\varphi\|_* \leq 1} (\mathbb{E}|\varphi(X)|^p)^{1/p}\Big).$$

Unconditional vectors ctd

Using the easy estimate $\mathbb{E}\|Y\| \le C \log n \, \mathbb{E}\|X\|$ we get.

Corollary

For any n-dimensional unconditional, log-concave vector X, any norm $\| \ \|$ on $\mathbb{R}^n$ and $p \ge 1$ one has

$$(\mathbb{E}\|X\|^p)^{1/p} \le C\Big(\log n \, \mathbb{E}\|X\| + \sup_{\|\varphi\|_* \le 1} (\mathbb{E}|\varphi(X)|^p)^{1/p}\Big).$$

The Maurey-Pisier result implies $\mathbb{E}\|Y\| \le C\mathbb{E}\|X\|$ in spaces with nontrivial cotype.

Corollary

Let X be as above, $2 \le q < \infty$ and $F = (\mathbb{R}^n, \| \ \|)$ has a q-cotype constant bounded by $\beta < \infty$.

$$(\mathbb{E}\|X\|^p)^{1/p} \le C(q,\beta)\Big(\mathbb{E}\|X\| + \sup_{\|\varphi\|_* \le 1} (\mathbb{E}|\varphi(X)|^p)^{1/p}\Big).$$

Questions

The conjecture

$$(\mathbb{E}\|X\|^p)^{1/p} \leq C\left(\mathbb{E}\|X\| + \sup_{\varphi\in F^*,\|\varphi\|_*\leq 1}(\mathbb{E}|\varphi(X)|^p)^{1/p}\right) \quad \text{for } p \geq 1.$$

seems to be rather hard in full generality. One may try to considerfirst some simpler open cases:

- l_r-norms with $1 \leq r \leq 2$
- Orlicz norms or more general unconditional norms
- Unconditional log-concave vectors

It is also not clear if one may improve the Paoris inequality to

$$(\mathbb{E}|X|^p)^{1/p} \leq \mathbb{E}|X| + C\sigma_X(p) \quad \text{for } p \geq 1.$$

Questions

The conjecture

$$(\mathbb{E}\|X\|^p)^{1/p} \leq C\left(\mathbb{E}\|X\| + \sup_{\varphi\in F^*,\|\varphi\|_*\leq 1}(\mathbb{E}|\varphi(X)|^p)^{1/p}\right) \quad \text{for } p \geq 1.$$

seems to be rather hard in full generality. One may try to considerfirst some simpler open cases:

- l_r-norms with $1 \leq r \leq 2$
- Orlicz norms or more general unconditional norms
- Unconditional log-concave vectors

It is also not clear if one may improve the Paoris inequality to

$$(\mathbb{E}|X|^p)^{1/p} \leq \mathbb{E}|X| + C\sigma_X(p) \quad \text{for } p \geq 1.$$

References

- R. Eldan, *Thin shell implies spectral gap up to polylog via a stochastic localization scheme*, Geom. Funct. Anal. 23 (2013), 532–569.
- B. Klartag, *Power-law estimates for the central limit theorem for convex sets*, J. Funct. Anal. 245 (2007), 284–310.
- R. Latała, *Weak and strong moments of random vectors*, in: Marcinkiewicz centenary volume, 115–121, Banach Center Publ. 95, Polish Acad. Sci. Inst. Math., Warsaw, 2011.
- R. Latała, M. Strzelecka, *Weak and strong moments of l_r-norms of log-concave vectors*, arXiv:1501.01649.
- R. Latała, J. O. Wojtaszczyk, *On the infimum convolution inequality*, Studia Math. 189 (2008), 147-187.
- G. Paouris, *Concentration of mass on convex bodies*, Geom. Funct. Anal. **16** (2006), 1021–1049.
- M. Talagarand, *A new isoperimetric inequality and the concentration of measure phenomenon*, in Israel Seminar (GAFA), Lecture Notes in Math. 1469, 94–124, Springer, Berlin 1991

Thank you for your attention!