Curvature-Dimension Condition for Non-Conventional Dimensions

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Setup - Weighted Riemannian Manifold

- (M^n, g) Riemannian manifold: smooth, complete, connected, oriented; with C^2 boundary ∂M; d - induced geodesic distance.
- M is geodesically convex ($\Rightarrow (M, d)$ is geodesic space).
- $\mu = \Psi \cdot \mathrm{Vol}_g$: $\Psi \in C^2$ and $\Psi = \exp(-V) > 0$ on M.

Def (Bakry–Émery generalized Ricci tensor): given $N \in (-\infty, \infty]$

$$\mathrm{Ric}_{g,\mu,N} := \mathrm{Ric}_g + \nabla_g^2 V - \frac{1}{N-n} \nabla_g V \otimes \nabla_g V = \mathrm{Ric}_g - (N-n) \frac{\nabla_g^2 \Psi^{\frac{1}{N-n}}}{\Psi^{\frac{1}{N-n}}}.$$

- $N = +\infty \Rightarrow \mathrm{Ric}_{g,\mu,\infty} = \mathrm{Ric}_{g,\mu} = \mathrm{Ric}_g + \nabla_g^2 V$ (Lichnerowicz).
- $N = n \Rightarrow \Psi \equiv c$, $\mathrm{Ric}_{g,\mu,n} = \mathrm{Ric}_g$.

Def: (M^n, g, μ) satisfies Curvature-Dimension condition CD(ρ, N) if:

$$\exists \rho \in \mathbb{R} \qquad \mathrm{Ric}_{g,\mu,N} \geq \rho g \ \text{ on } M.$$

Remark: extended to (geodesic) mm-setting by Lott–Sturm–Villani.

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Examples of CD(ρ, N) Spaces

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$$\mathrm{Ric}_g + \nabla_g^2 V - \frac{1}{N-n} \nabla_g V \otimes \nabla_g V = \mathrm{Ric}_g - (N-n) \frac{\nabla_g^2 \Psi^{\frac{1}{N-n}}}{\Psi^{\frac{1}{N-n}}} \geq \rho g .$$

- ρ = generalized Ricci Curvature lower-bound.
- N = gener. Dimension upper-bound; traditionally $N \in [n, \infty]$.

Examples of Convex Spaces ($\rho = 0$):

- $(K, |\cdot|, Leb) \in CD(0, n)$ - uniform measure on convex set $K \subset \mathbb{R}^n$.
- $(\mathbb{R}^n, |\cdot|, \mu) \in CD(0, \infty)$ - μ is log-concave measure.
- Borell's class of $\frac{1}{N}$-concave measures, i.e. $(N-n)\Psi^{\frac{1}{N-n}}$ concave on convex support M^n, is exactly $CD(0, N)$ in Euclidean space $(\mathbb{R}^n, |\cdot|)$.
- $N < 0$ above yields $1/\Psi^{\frac{1}{n-N}}$ convex, i.e. heavy-tailed measures: (e.g. Cauchy $c_n(1+|x|^2)^{-\frac{n+\alpha}{2}} dx$, $\alpha > 0$ is $CD(0, -\alpha)$).

Examples of Strictly Convex Spaces ($\rho > 0$):

- $(S^n, g, \mathrm{Vol}_{S^n}) \in CD(n-1, n)$ - canonical n-sphere.
- $(\mathbb{R}^n, |\cdot|, c_n \exp(-|x|^2/2) dx) \in CD(1, \infty)$ - Gaussian measure.

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Properties of CD(ρ, N) Spaces

Study various properties of CD(ρ, N) spaces:

- Bonnet–Myers: $\rho > 0 \;\Rightarrow\;$ diam$(M) \leq \pi/\sqrt{K}$, where:

$$K := \frac{\rho}{N-1} \quad \text{“generalized sectional-curvature"}\;.$$

- Bishop–Gromov volume comparison inequalities.
- Harnack inequalities / Heat-Kernel estimates.
- Topology: $\pi_1(M)$, Betti numbers, ...
- Brunn–Minkowski type inequalities.
- Poincaré inequality, Spectral estimates.
- Sobolev / Log-Sobolev / Transport-Entropy inequalities.
- Isoperimetric / Concentration inequalities.
- etc...

Extension of CD Theory to Non-Conventional Dimn's

- Borell, Brascamp–Lieb '75 - $\frac{1}{N}$-concave measures on $\mathbb{R}^n$, $\frac{1}{N} \in [-\infty, \frac{1}{n}]$, i.e. $CD(0, N)$. Studied on $\mathbb{R}^n$ by Bobkov, Ledoux.
- Ohta–Takatsu '11-'13 - N-dim entropies, $N \in (-\infty, 0] \cup [n, \infty]$.
- Kolesnikov-M. '13 - Poincaré-type inqs (Lichnerowicz, Brascamp–Lieb, Bobkov–Ledoux, Colesanti), new Brunn–Minkowski inq, $N \in (-\infty, 0] \cup [n, \infty]$.
- Ohta '13 - Lichnerowicz ($\partial M = \emptyset$), extend Brunn–Minkowski inq of Cordero–McCann–Shmuckenschläger / Lott–Sturm–Villani, $N \in (-\infty, 0) \cup [n, \infty]$.
- M. '14 - Isoperimetric Inequalities for $N \in (-\infty, 1) \cup [n, \infty]$.
- Klartag '14 - Extends localization method from Euclidean to manifold setting, $N \in (-\infty, 1) \cup [n, \infty]$.
- M. '15 - How can we go beyond this range? (in progress).

Isoperimetric Inequalities - Definitions

Metric-Measure Space: (Ω, d, μ), (Ω, d) Polish, μ Borel.
In our weighted manifold setting $(\Omega, d, \mu) = (M, g, \mu)$.

Isoperimetric Inqs compare between $\mu(A)$ and $\mu^+(A)$ (Minkowski's exterior boundary measure):

$\mu^+(A) := \liminf_{\varepsilon \to 0} \frac{\mu(A^d_\varepsilon) - \mu(A)}{\varepsilon}$,

$A^d_\varepsilon := \{x \in \Omega; d(x, A) < \varepsilon\}$.

Isoperimetric profile: $\mathcal{I} = \mathcal{I}(\Omega, d, \mu) : [0, \mu(\Omega)] \to \mathbb{R}_+$ defined:

$$\mathcal{I}(v) := \inf \left\{ \mu^+(A) \; ; \; \mu(A) = v \right\} \; , \; \mu^+(A) \geq \mathcal{I}(\mu(A)) \; .$$

On $(\mathbb{R}, |\cdot|, \mu)$, also define flat profile $\mathcal{I}^\flat = \mathcal{I}^\flat(\mathbb{R}, |\cdot|, \mu)$:

$$\mathcal{I}^\flat(v) := \inf \left\{ \mu^+(A) \; ; \; \mu(A) = v \; , \; A = (-\infty, \xi] \text{ or } A = [\xi, \infty) \right\} \; .$$

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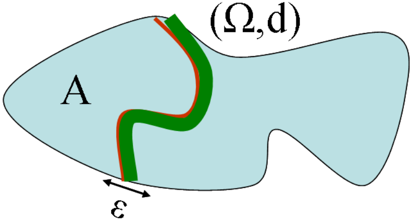

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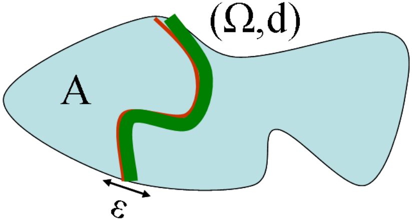

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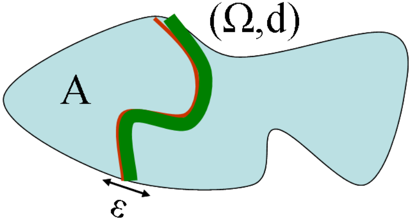

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Isoperimetric Inequalities - Ctd

Classical isoperimetric inequality in $(\mathbb{R}^n, |\cdot|, Leb)$:
Euclidean balls minimize boundary measure: $\mathcal{I}(v) = c_n v^{\frac{n-1}{n}}$.

Isoperimetric inequality = lower bound on $\mathcal{I} : [0, \mu(\Omega)] \to \mathbb{R}_+$.

Further motivation for studying isoperimetric inequalities:
Isoperimetric Inqs $\Rightarrow$ Sobolev Inqs $\Rightarrow$ Concentration Inqs.

From here on: μ is a probability measure, $\mu(\Omega) = 1$.
Recall: Isoperimetric Profiles $\mathcal{I}, \mathcal{I}^\flat$ defined on $[0, \mu(\Omega)] = [0, 1]$.

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Key Tool - Jacobian Comparison

Theorem (Generalized Heintze–Karcher $N \in (-\infty, 1) \cup [n, \infty]$; HK $N = n$, Bayle $N \in (n, \infty)$, Morgan $N = \infty$, M. $N \in (-\infty, 1)$)

Let S denote C^2 hypersurface in $(M^n, g, \mu = \Psi \cdot Vol_g)$ with unit normal ν_x, and $Ric_{g,\mu,N} \geq \rho g$ on $S_r^+ := \{\exp_x(t\nu_x); x \in S, t \in [0, r]\}$. Then:

$$\mu(S_r^+) \leq \int_S \int_0^r J_{H_{S,\mu}(x),\rho,N}(t)dt \;\; \Psi(x) dVol_S(x) \; ,$$

$$H_{S,\mu}(x) := tr \; II_S^\nu(x) + \langle \nu_x, \nabla_g \log \Psi(x) \rangle \;\; \textit{(generalized mean-curvature)} \, .$$

Idea: write Jacobian of $\gamma = \exp_x(t\nu_x) : S \times \mathbb{R} \to M$ as $J_x(t) = J_G J_W$:

$$\nu := \nabla_t \gamma \qquad -(\log J_G)'' - \frac{1}{n-1}((\log J_G)')^2 \geq Ric_g(\nu, \nu) \; ,$$

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Explicit Description of $J_{H,\rho,N}$

$J_{H,\rho,N}$ is a **model** $CD(\rho, N)$ density: solves the following CD ODE, on maximal interval containing the origin where such a solution exists:

$$-(\log J)'' - \frac{((\log J)')^2}{N-1} = -(N-1)\frac{(J^{\frac{1}{N-1}})''}{J^{\frac{1}{N-1}}} = \rho\ ,\ J(0) = 1\ ,\ J'(0) = H\ .$$

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$$J_{H,\rho,N}(t) := \begin{cases} \left(\left(c_K(t) + \frac{H}{N-1} s_K(t)\right)_+\right)^{N-1} & N \notin \{1, \infty\} \\ \exp(Ht - \frac{\rho}{2} t^2) & N = \infty \end{cases} .$$

$$s_K(t) := \begin{cases} \frac{\sin(\sqrt{K}t)}{\sqrt{K}} & K > 0 \\ t & K = 0 \\ \frac{\sinh(\sqrt{-K}t)}{\sqrt{-K}} & K < 0 \end{cases} ,\quad c_K(t) := \begin{cases} \cos(\sqrt{K}t) & K > 0 \\ 1 & K = 0 \\ \cosh(\sqrt{-K}t) & K < 0 \end{cases} ,$$

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Gromov–Lévy Program on CD(ρ, N) weighted-mnfld

Consider A of given $\mu(A) = v$ on which $\mu^+(A)$ is minimal $(= \mathcal{I}(v))$.

Existence & regularity of **isop minimizers** (Geometric Measure Th.: Almgren, Bombieri, De Giorgi, Federer, Fleming, Giusti, Gonzalez–Massari–Tamanini, Morgan, Simons):

- $\overline{\partial A \cap \mathring{M}} = \partial_s A \cup \partial_r A$; $\partial_s A$ - singular (Hausdorff dim $\leq n-8$), $\partial_r A$ regular hypersurface (as smooth as Ψ), CMC: $H_{\partial_r A,\mu} \equiv H_\mu(A)$.
- Normal rays from $\partial_r A$ sweep out entire $\mathring{M} \setminus \partial_s A$ (when $\partial M = \emptyset$, due to Gromov '80).
- By Gen. Heintze–Karcher: $\exists a + b \leq \text{diam}(M) =: D \in (0, \infty]$:

$$v = \mu(A) \leq \mu((\partial_r A)^-_a) \leq \mu^+(A) \int_0^a J_{-H_\mu(A),\rho,N}(t)dt ,$$

$$1 - v = \mu(M \setminus A) \leq \mu((\partial_r A)^+_b) \leq \mu^+(A) \int_0^b J_{H_\mu(A),\rho,N}(t)dt .$$

$$\Rightarrow \quad \mu^+(A) \geq \inf_{H \in \mathbb{R}, a+b=D} \max \left(\frac{v}{\int_{-a}^0 J_{H,\rho,N}(t)}, \frac{1-v}{\int_0^b J_{H,\rho,N}(t)} \right) =: \mathcal{GL}^\flat_{\rho,N,D}(v).$$

Isoperimetric Inequality for CDD(ρ, N, D)

Assume CDD(ρ, N, D) = CD(ρ, N) & diam(M) $\leq D \in (0, \infty]$, $\rho \in \mathbb{R}$.

Thm (Isoperimetric Inequality for CDD, M. '12,'14)

1. If $N \in (-\infty, 1) \cup [n, \infty]$ then for all $v \in (0, 1)$:

$$\mathcal{I}(M^n, g, \mu)(v) \geq \mathcal{GL}^{\flat}_{\rho,N,D}(v) \ .$$

2. If $N \in (-\infty, 0] \cup [n, \infty]$ or $D = \infty$ then $\mathcal{GL}^{\flat}_{\rho,N,D} = \mathcal{I}^{\flat}_{\rho,N,D}$,

$$\mathcal{I}^{\flat}_{\rho,N,D}(v) := \inf_{\substack{a,b>0 \\ a+b=D}} \inf_{H \in \mathbb{R}} \left\{ \mathcal{I}^{\flat}(J_{H,\rho,N}, [-a,b])(v) ; \int_{-a}^{b} J_{H,\rho,N}(t)dt < \infty \right\} ,$$

where the inner infimum over an empty set of H's is 0.

Notation: interval $L \subset \mathbb{R}$, $\mathcal{I}^{\flat}(f, L) = \mathcal{I}^{\flat}(\mathbb{R}, |\cdot|, \frac{1}{\int_L f(x)dx} f(x) 1_L(x) dx)$.

Thm (Sharpness, M. '12)

Part (2) automatically yields sharpness for $n = 1$, $\forall v \in [0, 1]$. In fact, sharpness verified for all $n \geq 2$, $N \in [n, \infty]$, $v \in [0, 1]$, ρ and D.

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Recall the "generalized sectional-curvature" $K := \rho/(N-1)$.

Case S1 - $N \in [n, \infty)$, $\rho > 0$, $D \geq \pi/\sqrt{K}$ ($N = n$ recovers Lévy–Gromov, $N > n$ Bayle):

$$\mathcal{I}(M, g, \mu) \geq \mathcal{I}^\flat\left(\sin(\sqrt{K}t)^{N-1}, [0, \pi/\sqrt{K}]\right) \quad N\text{-dim sphere with Ric} = \rho\,.$$

Case S2 - $N = \infty$, $\rho > 0$, $D = \infty$ (recovers Sudakov–Tsirelson, Borell, Bakry–Ledoux):

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Case S3 - $N \in (-\infty, 1)$, $\rho > 0$, $D = \infty$ (new '14):

$$\mathcal{I}(M, g, \mu) \geq \mathcal{I}^\flat\left(\cosh(\sqrt{K}t)^{N-1}, \mathbb{R}\right) \quad (-\infty, 1) \ni N\text{-dim sphere with Ric} = \rho\,.$$

In all other cases, no single model space, but one-parameter family of 1-D spaces:
Case F1A - $N \in [n, \infty)$, $\rho > 0$, $D < \pi/\sqrt{K}$ ('12, improves Bérard–Besson–Gallot):

$$\mathcal{I}(M, g, \mu)(v) \geq \min_{\xi \in [0, \frac{\pi}{\sqrt{K}} - D]} \mathcal{I}^\flat\left(\sin(\sqrt{K}t)^{N-1}, [\xi, \xi + D]\right)(v)\,.$$

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Case F1B - $N \in (-\infty, 0]$, $\rho < 0$, $D < \pi/\sqrt{K}$ (new '14):

$$\mathcal{I}(M, g, \mu) \geq \inf_{\xi \in (0, \frac{\pi}{\sqrt{K}} - D)} \mathcal{I}^\flat \left(\sin(\sqrt{K} t)^{N-1}, [\xi, \xi + D] \right) .$$

Case F3AB - $\{N \in [n, \infty), \rho < 0\}$ or $\{N \in (-\infty, 0], \rho > 0\}$, $D < \infty$ (new '12,'14):

$$\mathcal{I}(M, g, \mu) \geq \min \left\{ \begin{array}{c} \inf_{\xi > 0} \mathcal{I}^\flat(\sinh(\sqrt{-K} t)^{N-1}, [\xi, \xi + D]) , \\ \mathcal{I}^\flat(\exp(\sqrt{-K} t)^{N-1}, [0, D]) , \\ \inf_{\xi \in \mathbb{R}} \mathcal{I}^\flat(\cosh(\sqrt{-K} t)^{N-1}, [\xi, \xi + D]) \end{array} \right\} .$$

Case F4 - $N \in (-\infty, 0] \cup [n, \infty)$, $\rho = 0$, $D < \infty$ (new '12,'14)

$$\mathcal{I}(M^n, g, \mu) \geq \min \left\{ \begin{array}{c} \inf_{\xi > 0} \mathcal{I}^\flat(t^{N-1}, [\xi, \xi + D]), \\ \mathcal{I}^\flat(1, [0, D])) \end{array} \right\} .$$

Case F5 - $N = \infty$, $\rho = 0$, $D < \infty$ ('12)

$$\mathcal{I}(M^n, g, \mu) \geq \min_{H \geq 0} \mathcal{I}^\flat(\exp(Ht), [0, D]) .$$

Positive Curvature, Negative Dimension

Let (M^n, g, μ) satisfy $\mathrm{CD}(\rho, N)$, $\rho > 0$, $N \in (-\infty, 1)$.
Example: $([0,\infty), |\cdot|, \exp(-t)dt)$ satisfies $\mathrm{CD}(1, 0)$.

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Two-Level Concentration Property:

$$\mu(A) \geq 1/2 \;\; \Rightarrow \;\; 1 - \mu(A_r) \leq \begin{cases} C_N^{(1)} \frac{\exp(-c\rho r^2)}{\sqrt{\rho} r} & r \in \left[0, \frac{1}{\sqrt{K}}\right] \\ C_N^{(2)} \exp(-c\sqrt{(1-N)\rho}\, r) & \text{otherwise} \end{cases} .$$

Contrary to $N \in [n, \infty]$, no log-Sobolev inequality, but spectral-gap:
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$$\lambda_1 = \lambda_1(M^n, g, \mu) \geq \rho \frac{N}{N-1} \quad \forall N \in (-\infty, 0) \cup [n, \infty].$$

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Transition at $N = -1$: first eigenfunction $\sinh(\sqrt{K}t) \notin L^2(\mu)$.

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$$\lambda_1 = \lambda_1(M^n, g, \mu) \geq \rho \frac{N}{N-1} \quad \forall N \in (-\infty, 0) \cup [n, \infty].$$

Kolesnikov–M. '13: this is sharp for $N \in (-\infty, -1] \cup [n, \infty]$.
Transition at $N = -1$: first eigenfunction $\sinh(\sqrt{K}t) \notin L^2(\mu)$.

M. '14: However, spectral-gap for all $N \in (-\infty, 1)$:

$$\text{Cheeger's inequality} \;\;\Rightarrow\;\; \lambda_1 \geq \rho \frac{1}{4(1-N)(\int_0^\infty \cosh^{N-1}(t)dt)^2} .$$

Zero Curvature, Negative Dimension

Given $N \in (-\infty, 0) \cup [1, \infty]$ define the N-dim Cheeger constant:

$$D_{Che,N}(\Omega, d, \mu) := \inf_{v \in (0,1)} \frac{\mathcal{I}_{(\Omega,d,\mu)}(v)}{\min(v, 1-v)^{\frac{N-1}{N}}} .$$

If $D_{Che,N} > 0$, this implies bdd diameter ($N \in [1, \infty)$), exponential concentration ($N = \infty$), or polynomial concentration ($N \in (-\infty, 0)$):

$$\mu(A) \geq 1/2 \;\Rightarrow\; 1 - \mu(A_r) \leq \left(\left(\frac{1}{2}\right)^{\frac{1}{N}} - \frac{D_{Che,N}}{N} r \right)_+^N .$$

Thm (M. '10 '14): Let (M^n, g, μ) satisfy CD$(0, N)$, $N \in (-\infty, 0) \cup [n, \infty]$

- Arbitrarily weak concentration implies N-dim isoperimetry:

$$\forall r > 0 \quad D_{Che,N} \geq 2^{\frac{N-1}{N}} \inf_{\mu(A) \geq 1/2} \frac{\mu(A_r) - 1/2}{r} .$$

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- diam(M) $\leq D \;\Rightarrow\; D_{Che,N} \geq \frac{1}{D}$ (In $\mathbb{R}^n$, due to Bobkov '07).

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Zero Curvature, Negative Dimension - Ctd

Thm (M. '14): Let (M^n, g, μ) satisfy CD$(0, N)$, $N \in (-\infty, 0)$

- Weak Sobolev inequalities:

$$\|f\|_{L^{p,p\frac{N-1}{N}}(\mu)} \leq D_{Che,N}^{-1} 2^{-\frac{1}{N}} p \left(\frac{q}{p}\right)^{\frac{1}{q}} \||\nabla f|\|_{L^{q,p\frac{N-1}{N}}(\mu)} ,$$

for all $\frac{N}{N-1} \leq p \leq -N$, $\frac{1}{q} = \frac{1}{p} + \frac{1}{N}$ and $\mathrm{med}_\mu(f) = 0$.

- Nash-type inequalities:

$$\|f\|_{L^p(\mu)} \leq D_{Che,N}^{-\frac{N}{N-p}} 2^{-\frac{1}{N-p}} p^{\frac{N}{N-p}} \||\nabla f|\|_{L^p(\mu)}^{\frac{N}{N-p}} \|f\|_{L^\infty(\mu)}^{-\frac{p}{N-p}} ,$$

for all $p \geq 1$ and $\mathrm{med}_\mu(f) = 0$.

- In fact, they are all equivalent, to within numeric $C_{N,p,q} > 0$.

The "forbidden range" $N \in [1, n)$

Observation: when $N \in [1, n)$ all Isop. / Conc. properties break down.

Example: $(\mathbb{R}, |\cdot|, \cosh(t/\sqrt{\varepsilon})^{-\varepsilon} dt)$ satisfies $CD(1, 1-\varepsilon)$.
Has arbitrarily bad properties as $\varepsilon \to 0$.

Emulate it as an n-dimensional weighted-manifold of revolution:
Consider $M^n := \mathbb{R} \times S^{n-1}$ with:

$$g_\delta := dt^2 + \delta^2 \cosh(t/\sqrt{\varepsilon})^2 g_{S^{n-1}} \;;$$

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When $\delta > 0$ small enough, $(M^n, g_\delta, \mu_\delta)$ satisfies $CD(1, 1-\varepsilon)$.

Now note that $CD(\rho, N)$ is monotone in $\frac{1}{N-n}$:

$$\mathrm{Ric}_g + \nabla_g^2 V - \frac{1}{N-n} \nabla_g V \otimes \nabla_g V \geq \rho g.$$

Hence, $(M^n, g_\delta, \mu_\delta)$ satisfies $CD(1, N)$ for all $N \in [1, n)$.

In contrast to $N \in [n, \infty]$, where $CD(1, N)$ has Gaussian properties.

Reason: Cauchy–Schwarz not applicable.
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Avoiding Cauchy–Schwarz

Idea: assume $J = J_0 J_1$ on $\mathbb{R}$, so that each J_i satisfies:

$$J_i(0) = 1 \ , \ J_i'(0) = H_i \ , \ -(\log J_i)'' - \frac{1}{\mathcal{N}_i}((\log J_i)')^2 \geq \rho_i \quad , \quad i = 0, 1.$$

If $\mathcal{N}_0, \mathcal{N}_1 > 0$ or $\{\mathcal{N}_0 \mathcal{N}_1 < 0 \text{ and } \mathcal{N}_0 + \mathcal{N}_1 < 0\}$, apply C–S:

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and by the maximum principle: $J \leq J_{H_0+H_1, \rho_0+\rho_1, \mathcal{N}_0+\mathcal{N}_1}$.

However, if C–S inapplicable: use $J = J_0 J_1 \leq J_{H_0,\rho_0,\mathcal{N}_0} J_{H_1,\rho_1,\mathcal{N}_1}$.
Moreover, may yield improved estimates, even if C–S is applicable.

Caveats: 1. Cannot use Gromov–Lévy program, since now H_0, H_1 will no longer be constant on $\partial_r A$. But can use Localization (Payne–Weinberger, Gromov–V. Milman, Kannan–Lovász–Simonovits, Klartag '14).
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Sample Result

Let $(\mathbb{R}^n, |\cdot|, \mu)$ satisfy $\mathrm{CD}(N-1, N)$ and $N \in (n, \infty)$.

By Gromov–Lévy–Bayle: $\mathcal{I}(v) \geq \mathcal{I}^\flat(\sin^{N-1}(t), [0,\pi])(v)$.

M. '12 - this is **sharp** for $\mathrm{CD}(N-1, N)$ n-dim **weighted manifolds**.

M. '14 - this is **never** sharp in Euclidean space $\mathbb{R}^n$, $n \geq 2$.
In fact, can be **strictly** improved to:

$$\mathcal{I}(v) \geq \inf_{H_0, H_1 \in \mathbb{R}} \mathcal{I}^\flat \left(J_{H_0,0,n-1} J_{H_1,N-1,N-n}, \mathbb{R}\right)(v)$$

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$$C_{H_0,H_1,\delta} := \left\{(t,y) \in \mathbb{R} \times \mathbb{R}^{n-1} ; |y| \leq \delta \left(1 + \frac{H_0}{n-1} t\right)_+ \right\},$$

endowed with measure $\mu = J_{H_1,N-1,N-n}(t) \Pi_{i=1}^{n-1} J_{0,N-1,N-n}(y_i) dt dy$.

Graded Curvature-Dimension Condition - Definition

More generally than decoupling geometry and measure, we propose:

Definition - GCD($\rho_0, N_0, \lambda_0; \rho_1, N_1, \lambda_1$), $\lambda_0, \lambda_1 \geq 0, \lambda_0 + \lambda_1 = 1$

$(M^n, g, \mu = \Psi \cdot \mathrm{Vol}_g)$ satisfies Graded Curvature-Dimension Condition GCD($\rho_0, N_0, \lambda_0; \rho_1, N_1, \lambda_1$) if we can write $\Psi = \Psi_0 \Psi_1$ with $\Psi_0, \Psi_1 > 0$ and C^2 smooth on M, so that each $(M^n, g, \Psi_i \mathrm{Vol}_g)$ satisfies:

$$\lambda_i \mathrm{Ric}_g - \nabla_g^2 \log \Psi_i - \frac{1}{N - \lambda_i n} \nabla_g \log \Psi_i \otimes \nabla_g \log \Psi_i \geq \rho_i g.$$

Note that $CD(\rho, N) = GCD(\rho, N, 1; 0, 0, 0)$.

Cauchy–Schwarz: GCD is stronger than CD in appropriate range.

Namely, GCD($\rho_0, N_0, \lambda_0; \rho_1, N_1, \lambda_1$) $\Rightarrow$ CD($\rho_0 + \rho_1, N_0 + N_1$) if:

$N_0 = \lambda_0 n$; or $N_1 = \lambda_1 n$; or

$N_0 \in (\lambda_0 n, \infty]$ and $N_1 \in (\lambda_1 n, \infty]$; or

$(N_0 - \lambda_0 n)(N_1 - \lambda_1 n) < 0$ and $N_0 + N_1 < n$.

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Graded Curvature-Dimension Condition - Sample Thm

Thm (M. '14, in progress)

Let (M^n, g, μ) satisfy GCD$(\rho_0, N_0, \lambda_0; \rho_1, N_1, \lambda_1)$, with $N_i \in (-\infty, \lambda_i) \cup [\lambda_i n, \infty]$ and $\mathrm{diam}(M) \leq D \in (0, \infty]$. Then denoting:

$$\mathcal{N}_i = N_i - \lambda_i \quad , \quad J^{(i)}_{H_i} := J_{H_i, \rho_i, \mathcal{N}_i},$$

we have:

$$\mathcal{I}(v) \geq \inf_{\substack{a, b > 0 \\ a + b = D \\ H_0, H_1 \in \mathbb{R}}} \max \left(\frac{v}{\int_{-a}^{0} J^{(0)}_{H_0}(t) J^{(1)}_{H_1}(t) dt}, \frac{1 - v}{\int_{0}^{b} J^{(0)}_{H_0}(t) J^{(1)}_{H_1}(t) dt} \right),$$

assuming the function on RHS is concave.