Regularization of L_1 functions under the convolution operator in Gaussian space

Ronen Eldan, Weizmann Institute of Science
Joint work with James Lee

IMA, April 2015

The setting

- Our setting is $\mathbb{R}^n$ equipped with the standard Gaussian measure γ whose density is

$$\frac{d\gamma}{dx} = \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2}$$

- We consider the convolution operator

$$P_t[f](x) := \mathbb{E}\left[f\left(\sqrt{1-t}x + \sqrt{t}\Gamma \right) \right], \quad 0 \le t \le 1$$

where Γ is a standard Gaussian random vector.

- If B_t is a standard Brownian motion then

$$P_t[f](x) = \mathbb{E}\left[f(B_1) \middle| B_{1-t} = x\sqrt{1-t} \right].$$

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Hypercontractivity

- It is well-known that the operator P_t has the following smoothing property,

Theorem - hypercontractivity of the operator P_t (Gross, Nelson,...)

For any $p > 1$ and $t > 0$, there exists a constant $q > p$ satisfying

$$\|P_t f\|_{L_q(\gamma)} \leq \|f\|_{L_p(\gamma)}.$$

for all $f \in L_p(\gamma)$

- This fact has applications to several fields such as analysis of PDEs and quantum information theory.
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What about L_1 functions?

Question (Talagrand)

Is it true that for every non-negative function f such that $\mathbb{E}f(\Gamma) = 1$ one has

$$\mathbb{P}(P_t[f](\Gamma) > \alpha) \leq \frac{C(t)}{\alpha} g(\alpha)$$

as for some function $g(\alpha)$ satisfying $g(\alpha) \to 0$ as $\alpha \to \infty$?

- Talagrand originally (in a 1989 paper) asked this question in the discrete setting, hence, the space in hand is $\{-1, 1\}^n$ and the convolution operator corresponds to re-sampling every bit, independently, with probability t.
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- It seems that the L_2 machinery or the log-Sobolev inequality, used to prove hypercontractivity, are useless in this case since we have no a-priory bound on any L_p norm.
- There's a dual formulation of the conjecture with an isopertimetric flavor:

Dual (equivalent) conjecture

There exists a function h, $\lim_{x \to 0} h(x) = +\infty$, such that: For any set $A \subset \mathbb{R}^n$, there exists a non-negaive function φ supported on A such that $\forall x, \ P_t[\varphi](x) \leq 1$ and such that $\frac{1}{\gamma(A)} \int \varphi d\gamma \geq h(\gamma(A))$.

Theorem (Ball, Barthe, Bednorz, Oleszkiewicz and Wolff 2010)

For every non-negative function f such that $\mathbb{E} f(\Gamma) = 1$ one has

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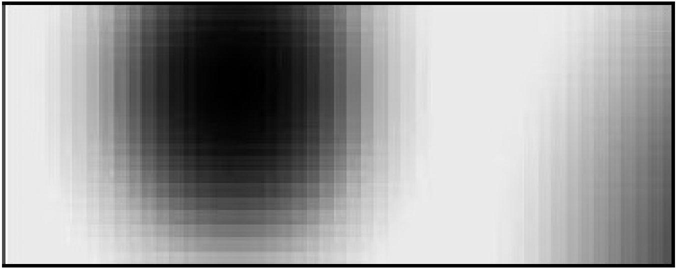
Anti-concentration of the temperature

Let g be a non-negative function such that $\mathbb{E}[g(\Gamma)] = 1$.
Suppose also that we know that

$$\mathbb{E}\left[g(\Gamma)\mathbf{1}_{\{g(\Gamma)\in[s,2s]\}}\right] \le \psi(s), \quad \forall s > 1$$

where $\psi(s) \searrow 0$ as $s \to \infty$.
We refer to this property as *anti-concentration of temperature*.



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Alternatively, if μ is the measure whose density is $\frac{d\mu}{d\gamma} = g$ and $X \sim \mu$, $Y = \log(g(X))$, so that by definition we have

$$D(g||\gamma) = \mathbb{E}[Y],$$

then anti-concentration of temperature is just the fact that

$$P(Y \in [\alpha, \alpha + 1]) \le \psi(\alpha).$$

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Then we have,

$$\begin{aligned}\mathbb{P}(g(\Gamma) > \alpha) &= \sum_{k=0}^{\infty} \mathbb{E}\left[\mathbf{1}_{\{g(\Gamma)\in[2^k\alpha,2^{k+1}\alpha]\}}\right] \\ &\leq \frac{1}{\alpha}\sum_{k=0}^{\infty} 2^{-k}\mathbb{E}\left[g(\Gamma)\mathbf{1}_{g(\Gamma)\in[2^k\alpha,2^{k+1}\alpha]}\right] \\ &\leq \frac{1}{\alpha}\sum_{k=0}^{\infty} 2^{-k}\psi(\alpha) \leq \frac{\psi(\alpha)}{\alpha}.\end{aligned}$$

A logarithmic anti-concentration result

Theorem (E., Lee 2014)

If f is a positive function satisfying $\mathbb{E}[f(\Gamma)] = 1$ and

$$\nabla^2 \log f(x) \succeq -\beta \mathrm{Id}, \quad \forall x \in \mathbb{R}^n$$

then

$$\mathbb{E}\left[f(\Gamma)\mathbf{1}_{f(\Gamma)\in[s,2s]}\right] \leq \frac{C(\beta)(\log\log(s))^4}{\sqrt{\log s}}, \quad \forall s \geq 2$$

- Since $P_t f$ is a mixture of Gaussians, and since log-convexity is preserved under mixtures, we have

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- A positive answer to the Gaussian variant of Talagrand's question is therefore an easy corollary of the theorem.
- The expression $\sqrt{\log s}$ is optimal, as demonstrated by the indicator of a half-space.
- The $\log\log$ term was very recently removed by J. Lehec.
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A naive idea for establishing anti-concentration of the temperature

- Define a measure μ by $\frac{d\mu}{d\gamma}(x) = f(x)$.
- Fix $s > 0$ and define $E = \{x; f(x) \geq s\}, \ F = \{x; f(x) \geq 2s\}$.
- We need to show that $\mu(E) - \mu(F) = o(1)$.
- Consider the set

$$E' = \left\{ x + \log 2 \frac{\nabla \log f(x)}{|\nabla \log f(x)|^2}; \ x \in E \right\}.$$

- By the assumption $\nabla^2 \log f(x) \succeq -\beta \mathrm{Id}$ it follows that

$$y \in E' \Rightarrow \log f(y) \geq \log s + \log 2 + O(1/|\nabla \log f(x)|^2)$$

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Girsanov's theorem

Let B_t be a Brownian motion with respect to a probability space (Ω, P) and suppose that

$$W_t = B_t + \int_0^t v_s ds$$

for some adapted drift v_t. **Girsanov's theorem** states that the measures associated to the processes B_t, W_t are absolutely continuous with respect to each other and that if one defines the change of measure

$$\frac{dQ}{dP} = \exp\left(-\int_0^1 \langle v_t, dB_t \rangle - \frac{1}{2} \int_0^1 |v_t|^2 dt \right)$$

then the process W_t becomes a Brownian motion with respect to the measure Q.

Föllmer's drift

- Define
$$W_t = B_t + \int_0^1 v_s ds, \quad v_t = \nabla \log P_{1-t} f(W_t).$$
- Moreover, define $M_t = P_{1-t} f(W_t)$.
- We have, using Itô's formula,
$$dM_t = \frac{\partial}{\partial t} P_{1-t} f(W_t) + \frac{1}{2} \Delta P_{1-t} f(W_t) + M_t |v_t|^2 dt + M_t \langle v_t, dB_t \rangle.$$
$$\Rightarrow M_t = \exp\left(\int_0^t \langle v_s, dB_s \rangle + \frac{1}{2} |v_s|^2 dt \right).$$
- Thus, W_t is a Brownian motion under the measure Q which satisfies $\frac{dQ}{dP} = M_1^{-1} = \frac{1}{f(W_1)}$. In other words, W_1 has the law μ. The drift v_t is known in the literature as Föllmer's drift.

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Föllmer's drift - continued

- Here's an alternative way to view this process: First, let W_t be standard Brownian motion under the measure Q.
- Consider the change of measure $\frac{dP}{dQ} = f(W_1)$.
- It turns out that under the measure P, the process W_t is a Brownian motion with a drift $v_t := \nabla \log P_{1-t} f(W_t)$.
- We learnt about this construction from a paper of **Joseph Lehec** (2011). In this paper, Lehec proves that

$$D(f||\gamma) = \inf_{u_t} \mathbb{E} \int_0^1 \frac{1}{2} |u_t|^2 dt$$

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Föllmer's drift - continued

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Perturbing the process W_t

- Fix a parameter $\delta > 0$ and consider the process:

$$X_t^\delta = W_t + \delta \int_0^t v_s ds = B_t + (1+\delta) \int_0^1 v_s ds.$$

- According to Girsanov, the process X_t^δ is a Brownian motion under the change of measure

$$\frac{dQ_\delta}{dQ} = \exp\left(-\delta \int_0^1 \langle v_t, dB_t \rangle - \frac{(1+\delta)^2 - 1}{2} \int_0^1 |v_t|^2 dt \right)$$

$$= \exp\left(-\delta \int_0^1 \langle v_t, dB_t \rangle - (\delta + \delta^2/2) \int_0^1 |v_t|^2 dt \right)$$

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The process v_t is a martingale.

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- The technical bulk of the proof consists mainly of showing that $Var\left[\left\langle v_1, \int_0^1 v_t dt \right\rangle - \int_0^1 |v_t|^2 dt\right]$ is small.
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- We can now calculate,

$$\begin{aligned}\mathbb{P}(f(W_1) > 2r) &= \mathbb{E}_P\left[f(B_1)\mathbf{1}_{\{B_1>2r\}}\right]\\ &= \mathbb{E}_{Q^\delta}\left[f(X_1^\delta)\mathbf{1}_{\{X_1^\delta>2r\}}\right]\\ &= \mathbb{E}_P\left[f(X_1^\delta)\mathbf{1}_{\{X_1^\delta>2r\}}\frac{dQ_\delta}{dP}\right]\\ &= \mathbb{E}_P\left[\frac{f(X_1^\delta)}{f(W_1)}\mathbf{1}_{\{X_1^\delta>2r\}}\frac{dQ_\delta}{dQ}\right] \approx \mathbb{P}(f(X_1^\delta) > 2r).\end{aligned}$$

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Further questions

- Is the main theorem true on a more general setting? It is natural to ask whether it is true on spaces with a Bakry-Emery Curvature-Dimension condition (such as positively curved Riemannian manifolds).
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