List Colourings of Simple Hypergraphs

Andrew Thomason (with David Saxton and Arès Méroueh)

IMA 9th Sept 2014

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Let $L : V(G) \to \mathcal{P}\{\text{colours}\}$ assign a list of colours to each $v \in G$.

G is *L-choosable* if there's a colouring with $\quad \forall v \; f(v) \in L(v)$

G is *k-list-colourable* if G is L-choosable whenever $\quad \forall v \; |L(v)| \geq k$

The *list chromatic number* of G is

$$\chi_l(G) = \min\{k : \; G \text{ is } k\text{-list-colourable}\}$$

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Theorem (Alon '00)

If G is 2-uniform, average degree d then $\chi_l(G) \geq (\frac{1}{2} + o(1)) \log_2 d$

Bounds

Alon: for graphs G, $\chi_l(G) \geq (\frac{1}{2} + o(1)) \log_2 d$

$\chi_l(K_{d,d}) \leq \log_2 d + 2$:
Suppose $|L(v)| \geq \ell = \log_2 d + 2$.
For $c \in \{\text{colours}\}$, "forbid" c either on V_1 or on V_2 at random.
For each $v \in V_i$ pick, if poss, $f(v) \in L(v)$ not forbidden on V_i.
If every v has such a choice, then f colours $K_{d,d}$.
Expected number of v with no choice is $\leq 2d(\frac{1}{2})^\ell < 1$. □
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$K^{(r)}_{d,d,\ldots,d}$ is the complete r-partite r-uniform hypergraph

$\chi_l(K^{(r)}_{d,d,\ldots,d}) \leq \log_r d + 2$:
For each $c \in \{\text{colours}\}$, "forbid" c on one of the V_i at random.
Expected number of v with no choice is $\leq rd(\frac{1}{r})^\ell < 1$. □

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G is *simple* or *linear* if $|e \cap f| \leq 1$ for all distinct edges e, f.

A *Latin square* graph is a simple d-regular subgraph of $K^{(3)}_{d,d,d}$.

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Theorem (Haxell+Pei '09)

If G is a Latin square, d large, then $\chi_l(G) = \Omega(\log d / \log\log d)$

Theorem (Haxell+Verstraëte '10)

For simple 3-uniform G, ave deg d, $\chi_l(G) = \Omega(\sqrt{\log d / \log\log d})$

Theorem (Alon+Kostochka '11)

For simple r-uniform G, ave deg d, $\chi_l(G) = \Omega((\log d)^{1/(r-1)})$

Containers

Theorem (Saxton+T '12,'14)

Let G be simple r-uniform d-regular. Then $\chi_l(G) \geq \frac{1}{r-1} \log_r d$ (bounds too for non-regular, non-simple)

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Theorem (Saxton+T '12,'14)

Let G be simple r-uniform d-regular. Let $\epsilon > 0$. Then there is a collection $\mathcal{C}$ of subsets of $V(G)$ such that

- *for every independent set I, there's a $C \in \mathcal{C}$ with $I \subset C$*
- *for every $C \in \mathcal{C}$, $e(G[C]) < \epsilon e(G)$, so $|C| < (1 - \frac{1}{r} + \epsilon)|G|$*
- $|\mathcal{C}| \leq 2^{\beta n d^{-1/(r-1)}}$ where $\beta = c(r, \epsilon) \log n$

Related results for non-regular, non-simple and non-independent. See also Balogh+Morris+Samotij '14.

An aside

Let H be a graph. Define as usual

$$\pi(H) = \lim_{n\to\infty} \text{ex(n,H)}\binom{n}{2}^{-1}$$

$$m(H) = \max_{H'\subset H} \frac{e(H')-1}{v(H')-2}.$$

Theorem (Saxton+T '14)

Let H be a graph and let $\epsilon > 0$. There exists $c > 0$ such that the following is true. Let $n \geq c$. Let q satisfy $n^{-1/m(H)} \leq q \leq 1/c$. Then there's a collection $\mathcal{C}$ of graphs on vertex set $[n]$ such that

- *for every graph I on $[n]$ with containing fewer than $q^{e(H)}n^{|H|}$ copies of H, there exists $C \in \mathcal{C}$ with $I \subset C$,*
- *every $C \in \mathcal{C}$ contains at most $\epsilon n^{|H|}$ copies of H, and so $e(C) \leq (\pi(H) + \epsilon)\binom{n}{2}$,*
- $\log |\mathcal{C}| \leq cqn^2 \log n$.

Collarings

Say $f : V(G) \to \{\text{colours}\}$ is *ok* for $L : V(G) \to \mathcal{P}\{\text{colours}\}$
if $\forall v f(v) \in L(v)$

Aim: find L so $\forall v\ |L(v)| = \ell$ and no f is ok for L — so $\chi_l(G) > \ell$

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A *collaring* is $(I_1, I_2, I_3, \ldots, I_t)$ where I_j independent (can overlap)

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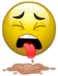

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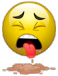

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$\Pr[(C_1, \ldots, C_t) \text{ ok for } L] \approx (1 - 2^{-\ell})^n \approx \exp(-n2^{-\ell})$

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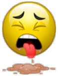

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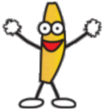

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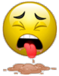

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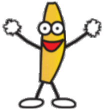

$\mathbb{E}\#$ ok Collarings $\approx \exp(tnd^{-1})\exp(-n2^{-\ell}) < 1$

(for hypergraphs $r^{-\ell}$ and $nd^{-1/(r-1)}$)

Sapozhenko's containers for d-regular graphs

Given independent set I, form a small subset T thus.
Start with $T = \emptyset$. Inspect vertices $1, 2, \ldots, n$ in order.
Add v to T if $v \in I$ and adding v to T increases $|\Gamma(T)|$ by $\geq \epsilon d$

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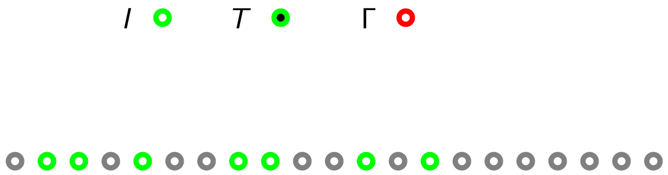

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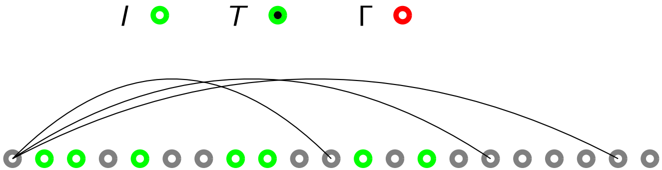

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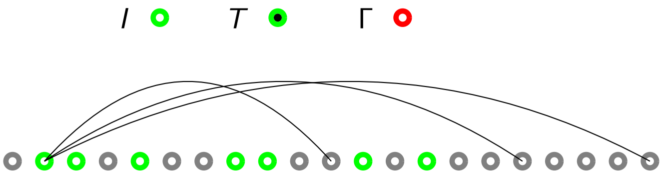

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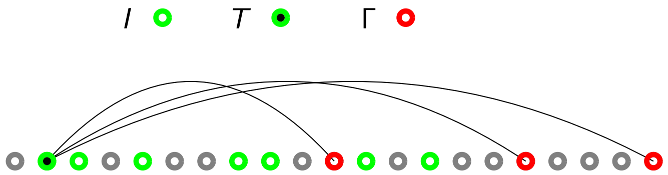

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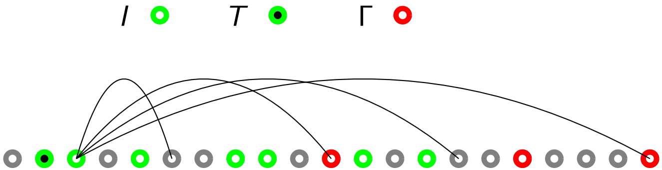

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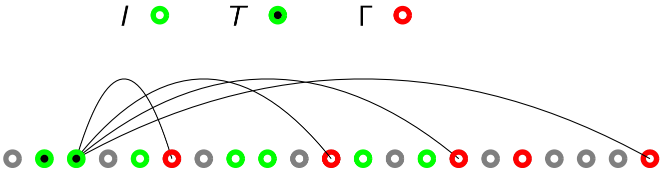

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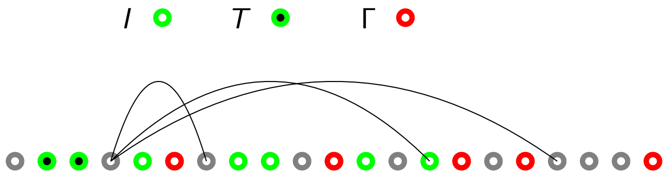

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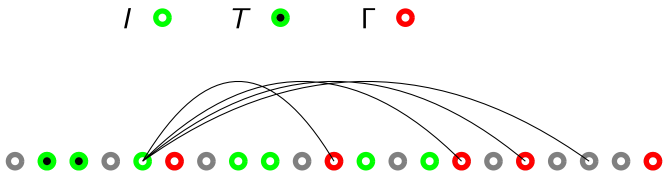

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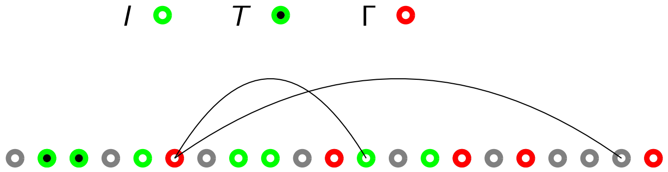

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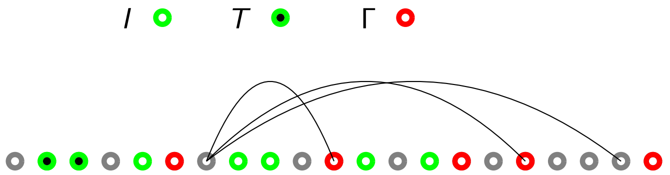

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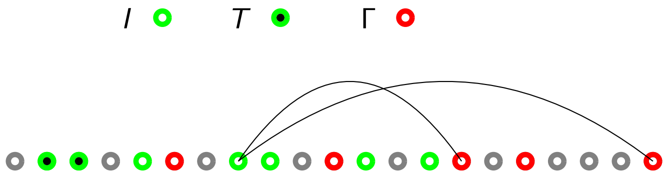

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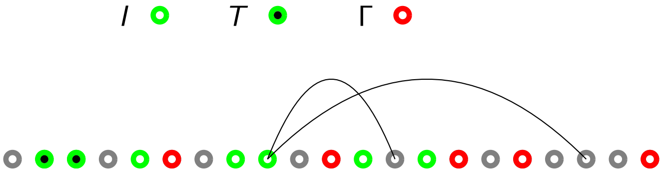

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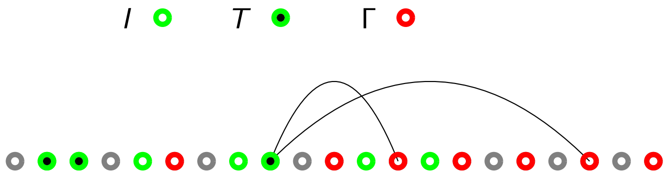

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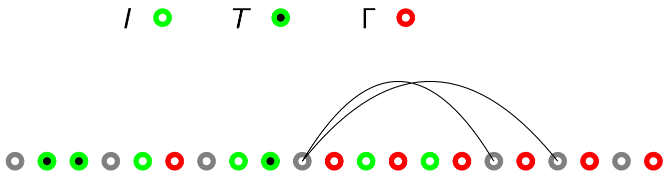

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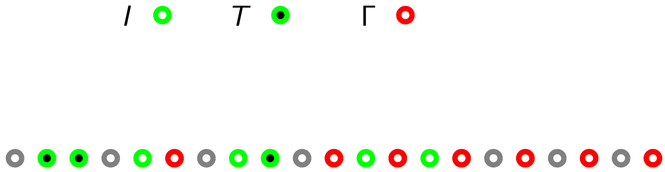

Builder builds a container C for I *just knowing T*
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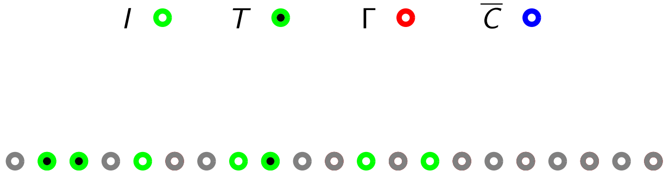

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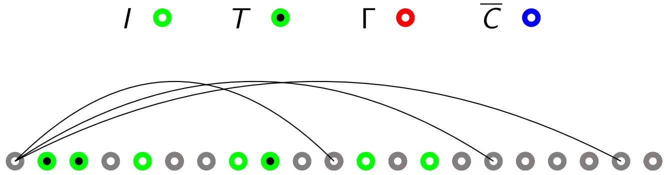

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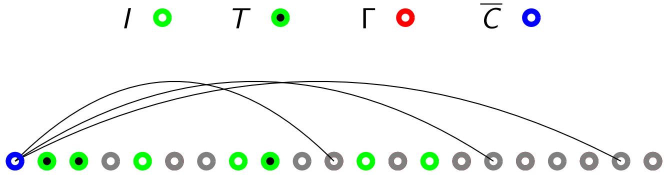

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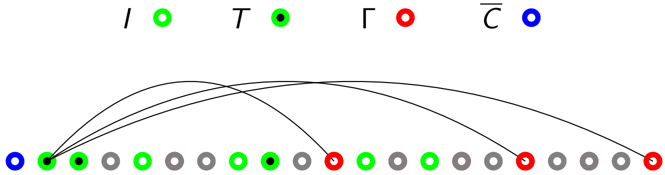

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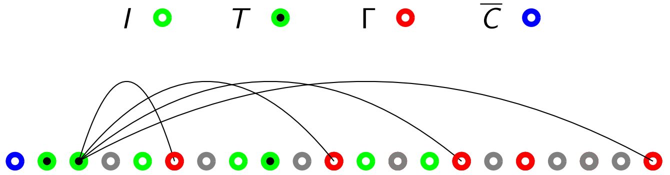

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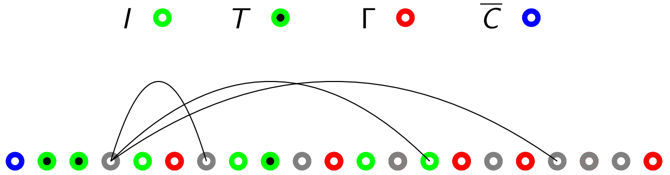

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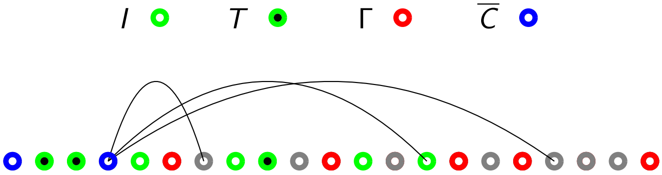

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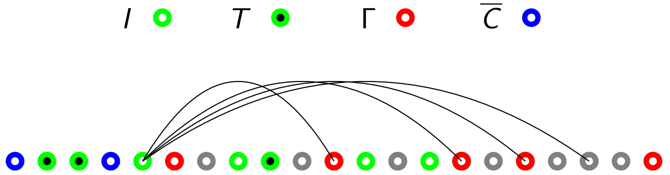

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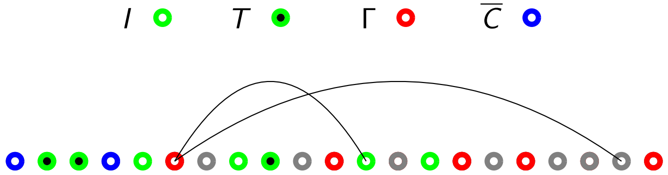

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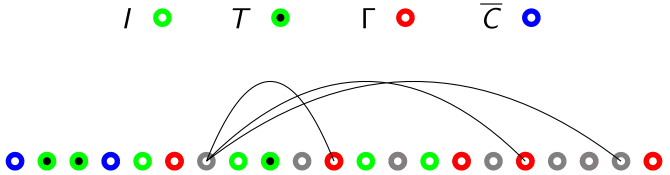

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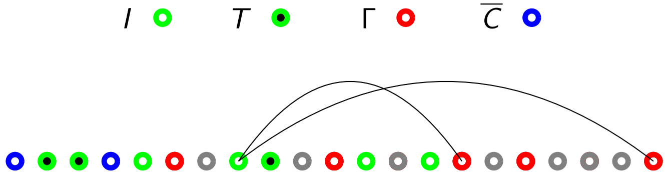

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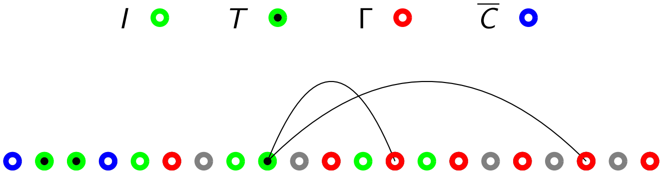

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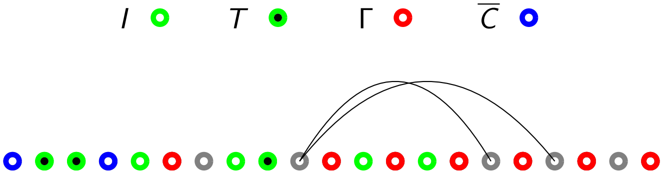

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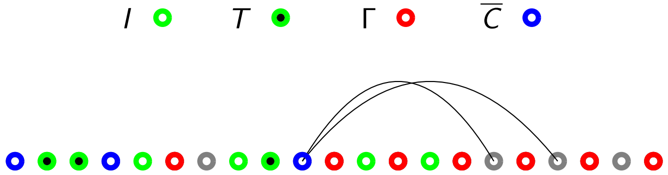

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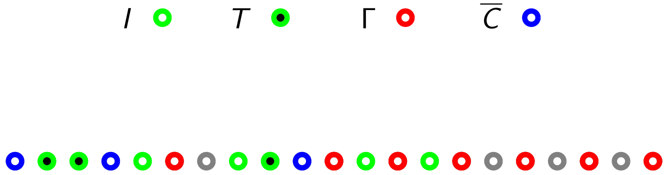

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$|T| \leq n/\epsilon d$ by construction
$e(C \setminus T) \leq \epsilon nd$ so $|C| \leq (1 - \frac{1}{2} + \epsilon)n + |T|$

Simple 3-uniform d-regular hypergraphs

For all graphs G of average degree d, $\chi_I(G) \geq (1 + o(1)) \log_2 d$.
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The upper bound came from latin square graphs G for which $\chi_I(G) \leq \chi(K^{(3)}_{d,d,d})$. Is this tight?

List colouring Latin squares

A latin square is a 3-uniform G with vertices $V_1 \sqcup V_2 \sqcup V_3$

For every two vertices u, v in different classes, there is exactly one w in the third class such that $\{u, v, w\}$ is an edge.
Thus G is simple and d-regular where $d = |V_1| = |V_2| = |V_3|$.

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$\chi_l(G) \leq 0 \cdot 92 \log_3 d$:
Suppose $|L(v)| \geq \ell = \alpha \log_3 d$, $\alpha = 0.92$
Let $q = 0.9083$
For each $c \in \{\text{colours}\}$,

- with probability q, "forbid" c on one of V_1, V_2, V_3
- with probability $1 - q$ allow c on any V_i (c is "free")

For each $v \in V_i$ pick, if poss, $f(v) \in L(v)$ forbidden on another V_j; failing that, pick, if poss, a free $f(v) \in L(v)$

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$\mathbb{E}$ number of monochromatic edges is $\leq d^2(1-q)^\ell < 1$ $\square$

Preference orderings

To make better use of the free colours, define 3 preference orders

best	$3k$	$3k$	k	$2k$
	$3k-1$	$3k-1$	$k-1$	$2k-1$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$2k+1$	$2k+1$	1	$k+1$
	$2k$	$2k$	$3k$	k
	$2k-1$	$2k-1$	$3k-1$	$k-1$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$k+1$	$k+1$	$2k+1$	1
	k	1	$k+1$	$2k+1$
	$k-1$	2	$k+2$	$2k+2$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
worst	1	k	$2k$	$3k$
	labels	V_1	V_2	V_3

Preference orderings - algorithm

Algorithm: randomly label the $\{$colours$\}$ $1, \ldots, 3k$.

$v \in V_i$ chooses $f(v) \in L(v)$ highest in the preference order for V_i.

Idea: suppose edge $\{v_1, v_2, v_3\}$ is monochromatic with colour c.
Then c is in worst third for (say) v_1 and middle third for (say) v_3.

Then $L(v_1) \cap L(v_3) = \{c\}$ else both would not choose c.

If c is relative height $1/3 - x$, $0 \leq x \leq 1/3$, in the V_1 order
then it is height $1/3 + x$ in V_3 order.
Hence $\Pr[\text{both choose } c] \leq (\frac{1}{3} - x)^\ell (\frac{1}{3} + x)^\ell \leq (\frac{1}{9})^\ell$.

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$\mathbb{E}$ number of monochromatic edges is $\leq d^2 (\frac{1}{9})^\ell < 1$ if $\ell > \log_3 d$
But bring back fixed colours and use preferences on free colours
gives $\chi_l(G) \leq 0.77 \log_3 d$

Preference orderings - bells algorithm

New algorithm: forget fixed colours.
Randomly put the $\{$colours$\}$ into $3k$ groups of size $10^6\ell$.
$v \in V_i$ selects the most preferred group having at least 10^3 in $L(v)$.
v will pick $f(v)$ from the members of $L(v)$ in its selected group.
How?

The set of vertices selecting the same group can now contain edges
But it is 10^3-degenerate. So colour with colours from the group.

Hence $\chi_l(G) \leq (\frac{1}{2} + o(1)) \log_3 d$ (modulo the structure of G).

Conclusions

Theorem (Méroueh+T)

Let G be a d-regular Latin square graph.

- $\chi_l(G) \leq 0.77 \log_3 d$
- *if* $L : V(G) \to \mathcal{P}\{colours\}$ *is random, and* $\forall v$ $|L(v)| > (\frac{1}{2} + \epsilon) \log_3 d$, *then there's an ok colouring*
- *if G is random, then for any* $L : V(G) \to \mathcal{P}\{colours\}$ *with* $\forall v$ $|L(v)| > (\frac{1}{2} + \epsilon) \log_3 d$ *there's an ok colouring*[1]

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But there's reason to believe preference orders give the best possible results and hence the "right" value.

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- $\chi_l(G) \leq 0.77 \log_3 d$
- *if $L : V(G) \to \mathcal{P}\{colours\}$ is random, and $\forall v$ $|L(v)| > (\frac{1}{2} + \epsilon) \log_3 d$, then there's an ok colouring*
- *if G is random, then for any $L : V(G) \to \mathcal{P}\{colours\}$ with $\forall v$ $|L(v)| > (\frac{1}{2} + \epsilon) \log_3 d$ there's an ok colouring*[1]

What about $r = 4$? Lower bound is $\chi_l(G) \geq \frac{1}{3} \log_4 d$.

Best known preference order method works only for $\ell \geq \frac{2}{5} \log_4 d$.

But there's reason to believe preference orders give the best possible results and hence the "right" value.

Then $\frac{1}{3} \log_4 d$ would not be optimal. Quo vadis containers?

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[1] modulo an "obvious" conjecture