Sensitivity-Based Topology and Shape Optimization for Electrical Machines subject to Nonlinear Magnetostatics

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Outline

1. Motivation and Problem Description
2. Topology Optimization
3. Shape Optimization
4. A Locally Modified Finite Element Method
5. Application to Electric Motor
6. Conclusion & Outlook
Outline

1 Motivation and Problem Description

2 Topology Optimization

3 Shape Optimization

4 A Locally Modified Finite Element Method

5 Application to Electric Motor

6 Conclusion & Outlook
Motivation

Consider electric motors as used in washing machines, computer cooling fans, assembly tools.

**Goal:** Find *optimal* design

Possible Objectives:
- Maximum torque
- Minimal torque ripple
- Smooth rotation (little noise and vibration)

**Figure:** Real world motor by Hanning Elektro-Werke GmbH & Co KG
Model Problem: Two-dimensional Model (2D Magnetostatics)

- . . . air
- . . . iron
- . . . magnets
- . . . coils
Model Problem: Two-dimensional Model (2D Magnetostatics)
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Goal: Find optimal geometry for electric motor
Motivation and Problem Description

Model Problem from 2D Magnetostatics

\[
\min_{\Omega_f} \mathcal{J}(\Omega_f) = \mathcal{J}(u(\Omega_f))
\]

subject to
\[
-\text{div} \left( \nu_{\Omega_f}(|\nabla u|) \nabla u \right) = F \quad \text{in} \ \Omega
\]
\[
u_{\Omega_f}(x, |\nabla u|) = \chi_{\Omega_f}(x) \hat{\nu}(|\nabla u|) + \chi_{\Omega_{air}}(x) \nu_0
\]

with the magnetic reluctivity

where

- \( F = J_3 - \text{div} \left( (-M_2)^T \right) \) with \( (M_1, M_2) \) magnetization
- \( \Omega_f \subset \Omega \) ... ferromagnetic subdomain
- \( \Omega_{air} = \Omega \setminus \Omega_f \)
- \( \mathbf{B} = \text{curl} \ (0, 0, u)^T \) ... magn. flux density
- \( \nu_0 = \text{const} > 0 \) reluctivity of air
- Natural Assumptions:
  - \( 0 < m \leq \hat{\nu}(s) \leq \nu_0 \)
  - \( 0 < m \leq (\hat{\nu}(s)s)' \leq M \)

\[ \hat{\nu}(|\mathbf{B}|) \]

magnetic reluctivity \( \hat{\nu} \)
Model Problem from 2D Magnetostatics

\[
\min_{\Omega_f} J(\Omega_f) = J(u(\Omega_f))
\]

s.t. \[
\begin{align*}
-\text{div} \left( \nu_{\Omega_f}(\nabla u) \right) & = F \quad \text{in } \Omega \\
\nu & = 0 \quad \text{on } \partial \Omega
\end{align*}
\] (1)

with the magnetic reluctivity

\[
\nu_{\Omega_f}(x, \nabla u) = \chi_{\Omega_f}(x) \hat{\nu}(\nabla u) + \chi_{\Omega_{air}}(x) \nu_0
\]

where

- \( F = J_3 - \text{div} \left( \left( \begin{array}{c} -M_2 \\ M_1 \end{array} \right) \right) \) with \( \left( \begin{array}{c} M_1 \\ M_2 \end{array} \right) \) magnetization
- \( \Omega_f \subset \Omega \) ... ferromagnetic subdomain
- \( \Omega_{air} = \Omega \setminus \Omega_f \)
- \( \mathbf{B} = \text{curl } ((0, 0, u)^T) \) ... magn. flux density
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Model Problem from 2D Magnetostatics

\[
\begin{align*}
\min_{\Omega_f} \quad & J(\Omega_f) = J(u(\Omega_f)) \\
\text{s.t.} \quad & -\text{div} \left( \nu_{\Omega_f} (|\nabla u|) \nabla u \right) = F \quad \text{in } \Omega \\
& u = 0 \quad \text{on } \partial \Omega
\end{align*}
\]

(1)

with the magnetic reluctivity

\[
\nu_{\Omega_f} (x, |\nabla u|) = \chi_{\Omega_f} (x) \hat{\nu}(|\nabla u|) + \chi_{\Omega_{air}} (x) \nu_0
\]

where

- \( F = J_3 - \text{div} \left( \left( \begin{array}{c} -M_2 \\ M_1 \end{array} \right) \right) \) with \( \left( \begin{array}{c} M_1 \\ M_2 \end{array} \right) \) magnetization
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Model Problem from 2D Magnetostatics

\[
\min_{\Omega_f} \mathcal{J}(\Omega_f) = \mathcal{J}(u(\Omega_f))
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s.t. \[
\begin{aligned}
-\text{div} \left( \nu_{\Omega_f}(\nabla u) \right) \nabla u &= F & \text{in } \Omega \\
u &= 0 & \text{on } \partial \Omega
\end{aligned}
\] (1)

with the magnetic reluctivity

\[
\nu_{\Omega_f}(x, |\nabla u|) = \chi_{\Omega_f}(x) \hat{\nu}(|\nabla u|) + \chi_{\Omega_{air}}(x) \nu_0
\]

where

- \( F = J_3 - \text{div} \left( \left( \begin{smallmatrix} -M_2 \\ M_1 \end{smallmatrix} \right) \right) \) with \( \left( \begin{smallmatrix} M_1 \\ M_2 \end{smallmatrix} \right) \) magnetization
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- Natural Assumptions:
  - \( 0 < m \leq \hat{\nu}(s) \leq \nu_0 \)
  - \( 0 < m \leq (\hat{\nu}(s)s)' \leq M \)
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1 Motivation and Problem Description

2 Topology Optimization
   - Overview
   - On/Off Method by Takahashi et al.
   - Topological Derivative for Nonlinear Magnetostatics
   - Comparison with On/Off Sensitivity
   - Computational Issues
   - Numerical Experiments

3 Shape Optimization

4 A Locally Modified Finite Element Method

5 Application to Electric Motor

6 Conclusion & Outlook
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6 Conclusion & Outlook
Design Optimization Approaches

- Density-Based Approaches
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- Level Set Method
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Design Optimization Approaches

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- Level Set Method
- Shape Optimization

Optimization of electrical equipment
- Sizing/Parameter Optimization

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Optimization of electrical equipment

- Sizing/Parameter Optimization
- Heuristic Methods
Design Optimization Approaches

- Density-Based Approaches
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- Shape Optimization

Optimization of electrical equipment
- Sizing/Parameter Optimization
- Heuristic Methods
- Sensitivity w.r.t. material coefficient ("On/Off Method")
Design Optimization Approaches

- Density-Based Approaches
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- Shape Optimization

Optimization of electrical equipment

- Sizing/Parameter Optimization
- Heuristic Methods
- Sensitivity w.r.t. material coefficient ("On/Off Method")

Design Optimization Approaches

- Density-Based Approaches
- Level Set Method
- Shape Optimization

Optimization of electrical equipment

- Sizing/Parameter Optimization
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Sensitivity-Based Top. Opt.: On/Off Method

\[ \min_{\Omega_f} \mathcal{J}(\Omega_f) = \mathcal{J}(u(\Omega_f)) \]

\[
\begin{cases}
-\text{div} \left( \nu_{\Omega_f}(|\nabla u|) \nabla u \right) = F & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega 
\end{cases}
\]

where \( \nu_{\Omega_f}(|\nabla u|) = \chi_{\Omega_f} \hat{\nu}(|\nabla u|) + \chi_{\Omega_{\text{air}}} \nu_0 \).

- iron/air only via \( \nu \)
- \( \hat{\nu} \ll \nu_0 \)
Sensitivity-Based Top. Opt.: On/Off Method

\[
\begin{align*}
\min_{\Omega_f} & \quad J(\Omega_f) = J(u(\Omega_f)) \\
\{ & \quad \text{div} \left( \nu_{\Omega_f} (|\nabla u|) \nabla u \right) = F \quad \text{in } \Omega \\
& \quad u = 0 \quad \text{on } \partial \Omega
\end{align*}
\]

iron/air only via \( \nu \)

\( \hat{\nu} \ll \nu_0 \)

where \( \nu_{\Omega_f} (|\nabla u|) = \chi_{\Omega_f} \hat{\nu}(|\nabla u|) + \chi_{\Omega_{\text{air}}} \nu_0 \).

Idea: For each triangle \( T_k \) compute sensitivity w.r.t. change of \( \nu_k \) (reluctivity in \( T_k \))

\[
\frac{dJ}{d\nu_k}
\]

and switch elements from iron to air (i.e. from “ON” to “OFF”) where sensitivity (most) negative.
Sensitivity-Based Top. Opt.: On/Off Method

\[
\begin{align*}
\min_{\Omega_f} \mathcal{J}(\Omega_f) &= \mathcal{J}(u(\Omega_f)) \\
\left\{ \begin{array}{l}
-\text{div} \left( \nu_{\Omega_f}(|\nabla u|) \nabla u \right) = F & \text{in } \Omega \\
u &= 0 & \text{on } \partial\Omega
\end{array} \right.
\end{align*}
\]

where \( \nu_{\Omega_f}(|\nabla u|) = \chi_{\Omega_f} \hat{\nu}(|\nabla u|) + \chi_{\Omega_{air}} \nu_0 \).

**Idea:** For each triangle \( T_k \) compute sensitivity w.r.t. change of \( \nu_k \) (reluctivity in \( T_k \))

\[
\frac{d\mathcal{J}}{d\nu_k}
\]

and switch elements from **iron** to **air** (i.e. from “ON” to “OFF”) where sensitivity (most) negative.

\[
\mathcal{J}_{\text{init}} = 1.71055 \times 10^{-3}
\]
\[
\mathcal{J}_{\text{opt}} = 0.98557 \times 10^{-3}
\]
Sensitivity-Based Top. Opt.: On/Off Method

\[
\min_{\Omega_f} \mathcal{J}(\Omega_f) = \mathcal{J}(u(\Omega_f))
\]

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\begin{aligned}
-\text{div} \left( \nu_{\Omega_f}(|\nabla u|)\nabla u \right) &= F \quad \text{in } \Omega \\
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**Idea:** For each triangle \( T_k \) compute sensitivity w.r.t. change of \( \nu_k \) (reluctivity in \( T_k \))

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\mathcal{J}_{\text{init}} &= 1.71055 \times 10^{-3} \\
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\]

Heuristic method!
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4 A Locally Modified Finite Element Method

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6 Conclusion & Outlook
Outline

2 Topology Optimization

- Overview
- On/Off Method by Takahashi et al.
- Topological Derivative for Nonlinear Magnetostatics
  - Preliminaries
  - Variations at scale $\varepsilon$
  - Topological Asymptotic Expansion (1)
  - Variations at scale 1
  - Topological Asymptotic Expansion (2)
- Comparison with On/Off Sensitivity
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Overview: Topological Derivative

Idea:

Sensitivity of $J = J(\Omega) = J(u(\Omega))$ w.r.t. insertion of hole $\omega_\varepsilon = x_0 + \varepsilon \omega$ ($\omega \ldots$ unit disk (e.g.))
Overview: Topological Derivative

**Idea:**

Sensitivity of $J = J(\Omega) = J(u(\Omega))$ w.r.t. insertion of hole $\omega_\varepsilon = x_0 + \varepsilon \omega$ ($\omega \ldots$ unit disk (e.g.))
Overview: Topological Derivative

Idea:
Sensitivity of $J = J(\Omega) = J(u(\Omega))$ w.r.t. insertion of hole $\omega_\varepsilon = x_0 + \varepsilon \omega$
($\omega \ldots$ unit disk (e.g.))

Topological asymptotic expansion

$$J(u(\Omega_\varepsilon)) - J(u(\Omega)) = \varepsilon^N G(x_0) + o(\varepsilon^N)$$

$N \ldots$ space dimension (here: $N = 2$)

$G(x_0) \ldots$ topological derivative of $J$ at $x_0$
Overview: Topological Derivative

Idea:

Sensitivity of $\mathcal{J} = \mathcal{J}(\Omega) = \mathcal{J}(u(\Omega))$ w.r.t. insertion of hole $\omega_\varepsilon = x_0 + \varepsilon \omega$
($\omega$...unit disk (e.g.))

Topological asymptotic expansion

$\mathcal{J}(u(\Omega_\varepsilon)) - \mathcal{J}(u(\Omega)) = \varepsilon^N G(x_0) + o(\varepsilon^N)$

$N$... space dimension (here: $N = 2$)

$G(x_0)$... topological derivative of $\mathcal{J}$ at $x_0$

$G(x_0) < 0 \implies \mathcal{J}(u(\Omega_\varepsilon)) < \mathcal{J}(u(\Omega))$ for $\varepsilon$ small enough
Topological Derivative for Nonlinear Problem

Collaboration with Samuel Amstutz (Univ. Avignon, France)

References


- S. Amstutz, A. Bonnafé: Topological asymptotic analysis for a class of quasilinear elliptic equations (accepted in *J. Math. Pures Appl.*
Assumption (A1)

- \( \mathcal{J}(u_\varepsilon) - \mathcal{J}(u_0) = \langle \frac{\partial \mathcal{J}}{\partial u}(u_0), u_\varepsilon - u_0 \rangle + o(\|u_\varepsilon - u_0\|^2) \) or
- \( \mathcal{J}(u) = \mathcal{J}(u|_{\Omega \setminus \Omega_d}) \) where \( \Omega_d \) design subdomain

We make the following additional assumptions for \( \hat{\nu} \):

1. \( \hat{\nu} \in C^2(\mathbb{R}_0^+) \).
2. \( \hat{\nu}'(0) = 0 \).
3. There exist non-negative constants \( \tilde{c}_7, \tilde{c}_8, \tilde{c}', \tilde{c}'' \) such that the nonlinear function \( \hat{\nu} \) satisfies the relations

\[
|2 \hat{\nu}'(|\varphi|) + \hat{\nu}''(|\varphi|)| \varphi | | \leq \tilde{c}_7 \quad \forall \varphi \in \mathbb{R}^2,
\]
\[
|3 \hat{\nu}''(|\varphi|) + \hat{\nu}'''(|\varphi|)| \varphi | | \leq \tilde{c}_8 \quad \forall \varphi \in \mathbb{R}^2.
\]

\[
|\hat{\nu}'(|\varphi|)| \leq \tilde{c}' \quad \forall \varphi \in \mathbb{R}^2,
\]
\[
|\hat{\nu}''(|\varphi|)| \leq \tilde{c}'' \quad \forall \varphi \in \mathbb{R}^2.
\]

4. \( \delta := \inf_{s > 0} (\hat{\nu}'(s)s)/\hat{\nu}(s) > -\frac{1}{3} \)

Note: Assumptions are fulfilled for the realistic set of data used in all numerical experiments.
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2 Topology Optimization

- Overview
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**Task:** Compute topological asymptotic expansion for problem

\[
\min \mathcal{J}(u(\Omega))
\]

s.t. \( \int_{\Omega} \nu(|\nabla u|) \nabla u \cdot \nabla \eta = \langle F, \eta \rangle \quad \forall \eta \in H^1_0(\Omega) \)

\[
\nu(|\nabla u|) := \begin{cases} 
\hat{\nu}(|\nabla u|) & \text{in } \Omega_f \\
\nu_0 & \text{in } \Omega_{\text{air}}
\end{cases}
\]
**Task:** Compute topological asymptotic expansion for problem

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\min \mathcal{J}(u(\Omega))
\]

s.t. \( \int_{\Omega} \nu(|\nabla u|) \nabla u \cdot \nabla \eta = \langle F, \eta \rangle \quad \forall \eta \in H^1_0(\Omega) \)

**Unperturbed state problem**

Find \( u_0 \in H^1_0(\Omega) \) such that

\[
\int_{\Omega} T_0(\nabla u_0) \cdot \nabla \eta = \langle F, \eta \rangle
\]

for all \( \eta \in H^1_0(\Omega) \)

with \( T_\varepsilon(q) = \nu_\varepsilon(|q|)q \) (\( \varepsilon \geq 0 \)), where

\[
\nu(|\nabla u|) := \begin{cases} 
\hat{\nu}(|\nabla u|) & \text{in } \Omega_f \\
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Unperturbed state problem

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\]

for all \( \eta \in H^1_0(\Omega) \)

Perturbed state problem

Find \( u_\varepsilon \in H^1_0(\Omega) \)

\[
\int_{\Omega} T_\varepsilon(\nabla u_\varepsilon) \cdot \nabla \eta = \langle F, \eta \rangle
\]

for all \( \eta \in H^1_0(\Omega) \)

with \( T_\varepsilon(q) = \nu_\varepsilon(|q|) q \ (\varepsilon \geq 0) \), where

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\nu(|\nabla u|) := \begin{cases} 
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\[
\nu_\varepsilon(|\nabla u|) := \begin{cases} 
\nu_0 & \text{in } \omega_\varepsilon \\
\hat{\nu}(|\nabla u|) & \text{in } \Omega_f \setminus \omega_\varepsilon \\
\nu_0 & \text{in } \Omega_{\text{air}}
\end{cases}
\]
**Task:** Compute topological asymptotic expansion for problem

\[
\min J(u(\Omega)) \\
\text{s.t. } \int_{\Omega} \nu(|\nabla u|) \nabla u \cdot \nabla \eta = \langle F, \eta \rangle \quad \forall \eta \in H^1_0(\Omega)
\]

**Unperturbed state problem**

Find \( u_0 \in H^1_0(\Omega) \) such that

\[
\int_{\Omega} T_0(\nabla u_0) \cdot \nabla \eta = \langle F, \eta \rangle
\]

for all \( \eta \in H^1_0(\Omega) \)

**Perturbed state problem**

Find \( u_\varepsilon \in H^1_0(\Omega) \)

\[
\int_{\Omega} T_\varepsilon(\nabla u_\varepsilon) \cdot \nabla \eta = \langle F, \eta \rangle
\]

for all \( \eta \in H^1_0(\Omega) \)

with \( T_\varepsilon(q) = \nu_\varepsilon(|q|)q \ (\varepsilon \geq 0) \), where

\[
\nu(|\nabla u|) := \begin{cases} 
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\nu_0 & \text{in } \Omega_{air}
\end{cases}
\]

\[
\nu_\varepsilon(|\nabla u|) := \begin{cases} 
\nu_0 & \text{in } \omega_\varepsilon \\
\hat{\nu}(|\nabla u|) & \text{in } \Omega_f \setminus \overline{\omega_\varepsilon} \\
\nu_0 & \text{in } \Omega_{air}
\end{cases}
\]

We are interested in variation of direct state: \( u_\varepsilon - u_0 \)
Variation of Direct State \( u_\varepsilon - u_0 \)

\[
0 = \int_{\Omega} T_\varepsilon (\nabla u_\varepsilon) \cdot \nabla \eta - \int_{\Omega} T_0 (\nabla u_0) \cdot \nabla \eta \quad \Leftrightarrow
\]
Variation of Direct State \( u_\varepsilon - u_0 \)

\[
0 = \int_\Omega T_\varepsilon (\nabla u_\varepsilon) \cdot \nabla \eta - \int_\Omega T_0 (\nabla u_0) \cdot \nabla \eta \quad \Leftrightarrow 
\]

Variation of direct state at scale \( \varepsilon \)

Find \( \tilde{u}_\varepsilon := u_\varepsilon - u_0 \in V_0 \) such that

\[
\int_\Omega (T_\varepsilon (\nabla u_0 + \nabla \tilde{u}_\varepsilon) - T_\varepsilon (\nabla u_0)) \cdot \nabla \eta = -\int_{\omega_\varepsilon} (\nu_0 - \hat{\nu}(|\nabla u_0|)) \nabla u_0 \cdot \nabla \eta \quad \forall \eta \in V_0 \quad (*)
\]

where \( T_\varepsilon (q) := \nu_\varepsilon(|q|)q \) for \( q \in \mathbb{R}^2 \).
Variation of Direct State, \( u_\varepsilon - u_0 \), and Adjoint State \( v_\varepsilon \)

\[
0 = \int_\Omega T_\varepsilon (\nabla u_\varepsilon) \cdot \nabla \eta - \int_\Omega T_0 (\nabla u_0) \cdot \nabla \eta \quad \Leftrightarrow
\]

**Variation of direct state at scale \( \varepsilon \)**

Find \( \tilde{u}_\varepsilon := u_\varepsilon - u_0 \in V_0 \) such that

\[
\int_\Omega \left( T_\varepsilon (\nabla u_0 + \nabla \tilde{u}_\varepsilon) - T_\varepsilon (\nabla u_0) \right) \cdot \nabla \eta = -\int_{\omega_\varepsilon} \left( \nu_0 - \nu(|\nabla u_0|) \right) \nabla u_0 \cdot \nabla \eta \quad \forall \eta \in V_0 \quad (*)
\]

where \( T_\varepsilon (q) := \nu_\varepsilon (|q|) q \) for \( q \in \mathbb{R}^2 \).

**Adjoint state at scale \( \varepsilon \)**

For \( \varepsilon \geq 0 \), find \( v_\varepsilon \in V_0 \) such that

\[
\int_\Omega (DT_\varepsilon (\nabla u_0) \nabla \eta) \cdot \nabla v_\varepsilon = -\int_\Omega \frac{\partial J}{\partial u} (u_0) \eta \quad \forall \eta \in V_0 \quad (**)
\]
Variation of Direct State, $u_\varepsilon - u_0$, and of Adjoint State $v_\varepsilon - v_0$

$$0 = \int_\Omega T_\varepsilon(\nabla u_\varepsilon) \cdot \nabla \eta - \int_\Omega T_0(\nabla u_0) \cdot \nabla \eta \quad \iff$$

Variation of direct state at scale $\varepsilon$

Find $\tilde{u}_\varepsilon := u_\varepsilon - u_0 \in V_0$ such that

$$\int_\Omega (T_\varepsilon(\nabla u_0 + \nabla \tilde{u}_\varepsilon) - T_\varepsilon(\nabla u_0)) \cdot \nabla \eta = -\int_{\omega_\varepsilon} (v_0 - \hat{v}(|\nabla u_0|)) \nabla u_0 \cdot \nabla \eta \quad \forall \eta \in V_0 \quad (*)$$

where $T_\varepsilon(q) := \nu_\varepsilon(|q|)q$ for $q \in \mathbb{R}^2$.

Adjoint state at scale $\varepsilon$

For $\varepsilon \geq 0$, find $v_\varepsilon \in V_0$ such that

$$\int_\Omega (DT_\varepsilon(\nabla u_0) \nabla \eta) \cdot \nabla v_\varepsilon = -\int_\Omega \frac{\partial J}{\partial u}(u_0) \eta \quad \forall \eta \in V_0 \quad (**)$$

Variation of adjoint state at scale $\varepsilon$

Find $\tilde{v}_\varepsilon := v_\varepsilon - v_0 \in V_0$ such that

$$\int_\Omega (DT_\varepsilon(\nabla u_0) \nabla \eta) \cdot \nabla \tilde{v}_\varepsilon = -\int_{\omega_\varepsilon} ((v_0 I - DT_0(\nabla u_0)) \nabla \eta) \cdot \nabla v_0 \quad \forall \eta \in V_0 \quad (***)$$
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- Comparison with On/Off Sensitivity
- Computational Issues
- Numerical Experiments
Topological Asymptotic Expansion

We have

$$\mathcal{J}(u_\varepsilon) - \mathcal{J}(u_0) = \left\langle \frac{\partial \mathcal{J}}{\partial u}(u_0), u_\varepsilon - u_0 \right\rangle + o(\varepsilon^N)$$
Topological Asymptotic Expansion

We have

\[ J(u_\varepsilon) - J(u_0) = \left\langle \frac{\partial J}{\partial u}(u_0), u_\varepsilon - u_0 \right\rangle + o(\varepsilon^N) \]

\[ \overset{(**)}{=} - \int_{\Omega} DT_\varepsilon(\nabla u_0) \nabla \tilde{u}_\varepsilon \cdot \nabla v_\varepsilon + o(\varepsilon^N) \]
Topological Asymptotic Expansion

We have

\[ J(u_\varepsilon) - J(u_0) = \langle \frac{\partial J}{\partial u}(u_0), u_\varepsilon - u_0 \rangle + o(\varepsilon^N) \]

\[ \text{(**) } - \int_\Omega DT_\varepsilon(\nabla u_0) \nabla \tilde{u}_\varepsilon \cdot \nabla v_\varepsilon + o(\varepsilon^N) \]

\[ \text{(***) } - \int_\Omega DT_\varepsilon(\nabla u_0) \nabla \tilde{u}_\varepsilon \cdot \nabla v_0 + \int_{\Omega'} (\nu_0 I - DT_0(\nabla u_0)) \nabla \tilde{u}_\varepsilon \cdot \nabla v_0 \]

\[ + o(\varepsilon^N) \]
Topological Asymptotic Expansion

We have

\[ J(u_\varepsilon) - J(u_0) = \langle \frac{\partial J}{\partial u}(u_0), u_\varepsilon - u_0 \rangle + o(\varepsilon^N) \]

\[(**\quad) - \int_\Omega DT_\varepsilon(\nabla u_0) \nabla \tilde{u}_\varepsilon \cdot \nabla v_\varepsilon + o(\varepsilon^N) \]

\[(***) - \int_\Omega DT_\varepsilon(\nabla u_0) \nabla \tilde{u}_\varepsilon \cdot \nabla v_0 + \int_{\omega_\varepsilon} (\nu_0 I - DT_0(\nabla u_0)) \nabla \tilde{u}_\varepsilon \cdot \nabla v_0 \]

\[+ \int_\Omega (T_\varepsilon(\nabla u_0 + \nabla \tilde{u}_\varepsilon) - T_\varepsilon(\nabla u_0)) \cdot \nabla v_0 \]

\[+ \int_{\omega_\varepsilon} (\nu_0 - \hat{\nu}(|\nabla u_0|)) \nabla u_0 \cdot \nabla v_0 \]

\[+ o(\varepsilon^N) \]
Topological Asymptotic Expansion

We have

\[ J(u_\varepsilon) - J(u_0) = \left\langle \frac{\partial J}{\partial u}(u_0), u_\varepsilon - u_0 \right\rangle + o(\varepsilon^N) \]

\[
(\ast\ast) \quad - \int_\Omega DT_\varepsilon (\nabla u_0) \nabla \tilde{u}_\varepsilon \cdot \nabla \nu_\varepsilon + o(\varepsilon^N)
\]

\[
(\ast\ast\ast) \quad - \int_\Omega DT_\varepsilon (\nabla u_0) \nabla \tilde{u}_\varepsilon \cdot \nabla \nu_0 + \int_{\omega_\varepsilon} (\nu_0 I - DT_0(\nabla u_0)) \nabla \tilde{u}_\varepsilon \cdot \nabla \nu_0
\]

\[
+ \int_\Omega (T_\varepsilon (\nabla u_0 + \nabla \tilde{u}_\varepsilon) - T_\varepsilon (\nabla u_0)) \cdot \nabla \nu_0
\]

\[
+ \int_{\omega_\varepsilon} (\nu_0 - \hat{\nu}(|\nabla u_0|)) \nabla u_0 \cdot \nabla \nu_0 + \int_{\omega_\varepsilon} (\nu_0 - \hat{\nu}(|\nabla u_0|)) \nabla u_0 \cdot \nabla \nu_\varepsilon
\]

\[
- \int_{\omega_\varepsilon} (\nu_0 - \hat{\nu}(|\nabla u_0|)) \nabla u_0 \cdot \nabla \nu_\varepsilon + o(\varepsilon^N)
\]
Topological Asymptotic Expansion

We have

\[ J(u_\varepsilon) - J(u_0) = \left\langle \frac{\partial J}{\partial u}(u_0), u_\varepsilon - u_0 \right\rangle + o(\varepsilon^N) \]

\( (**) \quad - \int_\Omega DT_\varepsilon (\nabla u_0) \nabla \tilde{u}_\varepsilon \cdot \nabla v_\varepsilon + o(\varepsilon^N) \)

\( (***) \quad - \int_\Omega DT_\varepsilon (\nabla u_0) \nabla \tilde{u}_\varepsilon \cdot \nabla v_0 + \int_{\omega_\varepsilon} (\nu_0 I - DT_0(\nabla u_0)) \nabla \tilde{u}_\varepsilon \cdot \nabla v_0 \]

\[ + \int_\Omega \left( T_\varepsilon (\nabla u_0 + \nabla \tilde{u}_\varepsilon) - T_\varepsilon (\nabla u_0) \right) \cdot \nabla v_0 \]

\[ + \int_{\omega_\varepsilon} (\nu_0 - \hat{\nu}(|\nabla u_0|)) \nabla u_0 \cdot \nabla v_0 + \int_{\omega_\varepsilon} (\nu_0 - \hat{\nu}(|\nabla u_0|)) \nabla u_0 \cdot \nabla v_\varepsilon \]

\[ - \int_{\omega_\varepsilon} (\nu_0 - \hat{\nu}(|\nabla u_0|)) \nabla u_0 \cdot \nabla v_\varepsilon + o(\varepsilon^N) \]
Topological Asymptotic Expansion

We have

\[ \mathcal{J}(u_\varepsilon) - \mathcal{J}(u_0) = ... = j_1(\varepsilon) + j_2(\varepsilon) + o(\varepsilon^N) \]

with

\[
\begin{align*}
j_1(\varepsilon) &= \int_{\omega_\varepsilon} (\nu_0 - \hat{\nu}(|\nabla u_0|)) \nabla u_0 \cdot (\nabla \nu_0 + \nabla \tilde{\nu}_\varepsilon) \\
j_2(\varepsilon) &= \int_{\Omega} \left( T_\varepsilon (\nabla u_0 + \nabla \tilde{u}_\varepsilon) - T_\varepsilon (\nabla u_0) - D T_\varepsilon (\nabla u_0) \nabla \tilde{u}_\varepsilon \right) \cdot (\nabla \nu_0 + \nabla \tilde{\nu}_\varepsilon) \\
&= : S_{\nabla u_0}^\varepsilon (\nabla \tilde{u}_\varepsilon)
\end{align*}
\]
Topological Asymptotic Expansion

We have

$$J(u_\varepsilon) - J(u_0) = ... = j_1(\varepsilon) + j_2(\varepsilon) + o(\varepsilon^N)$$

$$\equiv \varepsilon^N G(x_0) + o(\varepsilon^N)$$

with

$$j_1(\varepsilon) = \int_{\omega_\varepsilon} (\nu_0 - \hat{v}(|\nabla u_0|))\nabla u_0 \cdot (\nabla \nu_0 + \nabla \tilde{v}_\varepsilon)$$

$$j_2(\varepsilon) = \int_{\Omega} (T_\varepsilon (\nabla u_0 + \nabla \tilde{u}_\varepsilon) - T_\varepsilon (\nabla u_0) - DT_\varepsilon (\nabla u_0) \nabla \tilde{u}_\varepsilon) \cdot (\nabla \nu_0 + \nabla \tilde{v}_\varepsilon)$$

$$= : S^\varepsilon_{\nabla u_0} (\nabla \tilde{u}_\varepsilon)$$
Topological Asymptotic Expansion

We have

$$J(u_\varepsilon) - J(u_0) = ... = j_1(\varepsilon) + j_2(\varepsilon) + o(\varepsilon^N)$$

$$= \varepsilon^N G(x_0) + o(\varepsilon^N)$$

with

$$j_1(\varepsilon) = \int_{\omega_\varepsilon} (\nu_0 - \hat{\nu}(\vert \nabla u_0 \vert)) \nabla u_0 \cdot (\nabla \nu_0 + \nabla \tilde{\nu}_\varepsilon)$$

$$j_2(\varepsilon) = \int_{\Omega} \left( T_\varepsilon (\nabla u_0 + \nabla \tilde{u}_\varepsilon) - T_\varepsilon (\nabla u_0) - DT_\varepsilon (\nabla u_0) \nabla \tilde{u}_\varepsilon \right) \cdot (\nabla \nu_0 + \nabla \tilde{\nu}_\varepsilon)$$

$$= : S_\varepsilon^{\nabla u_0} (\nabla \tilde{u}_\varepsilon)$$

It remains to show that there exist $J_1, J_2$ independent of $\varepsilon$ such that

$$j_1(\varepsilon) = \varepsilon^N J_1 + o(\varepsilon^N)$$

and

$$j_2(\varepsilon) = \varepsilon^N J_2 + o(\varepsilon^N)$$
Outline

2 Topology Optimization

- Overview
- On/Off Method by Takahashi et al.
- Topological Derivative for Nonlinear Magnetostatics
  - Preliminaries
  - Variations at scale $\varepsilon$
  - Topological Asymptotic Expansion (1)
  - Variations at scale 1
    - Topological Asymptotic Expansion (2)
- Comparison with On/Off Sensitivity
- Computational Issues
- Numerical Experiments
Variation of Direct State: Approximations

Variation of direct state at scale $\varepsilon$

Find $\tilde{u}_\varepsilon := u_\varepsilon - u_0 \in V_0$ such that

$$\int_{\Omega} (T_\varepsilon (\nabla u_0 + \nabla \tilde{u}_\varepsilon) - T_\varepsilon (\nabla u_0)) \cdot \nabla \eta = - \int_{\omega_\varepsilon} (\nu_0 - \hat{\nu}(|\nabla u_0|)) \nabla u_0 \cdot \nabla \eta \quad \forall \eta \in V_0$$
Variation of Direct State: Approximations

Find $\tilde{u}_\varepsilon := u_\varepsilon - u_0 \in V_0$ such that

$$\int_{\Omega} (T_\varepsilon(\nabla u_0 + \nabla \tilde{u}_\varepsilon) - T_\varepsilon(\nabla u_0)) \cdot \nabla \eta = -\int_{\omega_\varepsilon} (\nu_0 - \hat{\nu}(|\nabla u_0|)) \nabla u_0 \cdot \nabla \eta \quad \forall \eta \in V_0$$

Approximation 1: replace function $\nabla u_0$ by constant $U_0 := \nabla u_0(x_0)$
Variation of Direct State: Approximations

Variation of direct state at scale $\varepsilon$

Find $\tilde{u}_\varepsilon := u_\varepsilon - u_0 \in V_0$ such that

$$
\int_\Omega \left( T_\varepsilon (\nabla u_0 + \nabla \tilde{u}_\varepsilon) - T_\varepsilon (\nabla u_0) \right) \cdot \nabla \eta = - \int_{\omega_\varepsilon} (\nu_0 - \hat{\nu}(|\nabla u_0|)) \nabla u_0 \cdot \nabla \eta \quad \forall \eta \in V_0
$$

Approximation 1: replace function $\nabla u_0$ by constant $U_0 := \nabla u_0(x_0)$

Approximation 2: Change of scale
Transform domain with hole of size $\varepsilon$
Transform domain with hole of size $\varepsilon$

to large domain with hole of unit size
Transform domain with hole of size $\varepsilon$

$\Omega_\varepsilon$

$\omega_\varepsilon$

to large domain with hole of unit size

$B(0,1)$

$\mathbb{R}^2$

and send outer boundary to infinity (Approximation!)
Variation of Direct State: Approximations

Variation of direct state at scale $\varepsilon$

Find $\tilde{u}_\varepsilon := u_\varepsilon - u_0 \in V_0$ such that

$$
\int_{\Omega} \left( T_\varepsilon (\nabla u_0 + \nabla \tilde{u}_\varepsilon) - T_\varepsilon (\nabla u_0) \right) \cdot \nabla \eta = - \int_{\omega_\varepsilon} (\nu_0 - \hat{\nu}(|\nabla u_0|)) \nabla u_0 \cdot \nabla \eta \quad \forall \eta \in V_0
$$

Approximation 1: replace function $\nabla u_0$ by constant $U_0 := \nabla u_0(x_0)$

Approximation 2: Change of scale
Variation of Direct State: Approximations

Variation of direct state at scale \( \varepsilon \)

Find \( \tilde{u}_\varepsilon := u_\varepsilon - u_0 \in V_0 \) such that

\[
\int_{\Omega} \left( T_\varepsilon (\nabla u_0 + \nabla \tilde{u}_\varepsilon) - T_\varepsilon (\nabla u_0) \right) \cdot \nabla \eta = - \int_{\omega_\varepsilon} (\nu_0 - \hat{\nu}(|\nabla u_0|)) \nabla u_0 \cdot \nabla \eta \quad \forall \eta \in V_0
\]

Approximation 1: replace function \( \nabla u_0 \) by constant \( U_0 := \nabla u_0(x_0) \)

Approximation 2: Change of scale

Variation of direct state at scale 1

Find \( H \in \mathcal{H}(\mathbb{R}^N) \) such that

\[
\int_{\mathbb{R}^N} \left( \tilde{T}(U_0 + \nabla H) - \tilde{T}(U_0) \right) \cdot \nabla \eta = - \int_{\omega} (\nu_0 - \hat{\nu}(|U_0|)) U_0 \cdot \nabla \eta \quad \forall \eta \in \mathcal{H}(\mathbb{R}^N)
\]
Variation of Direct State at Scale 1

Find $H \in \mathcal{H}(\mathbb{R}^N)$ such that

$$
\int_{\mathbb{R}^N} \left( \tilde{T}(U_0 + \nabla H) - \tilde{T}(U_0) \right) \cdot \nabla \eta = - \int_{\omega} (\nu_0 - \hat{\nu}(|U_0|)) U_0 \cdot \nabla \eta \quad \forall \eta \in \mathcal{H}(\mathbb{R}^N)
$$

(2)

where $\omega = B(0, 1)$, $\mathcal{H}(\mathbb{R}^N) := \{ u \in \mathcal{D}'(\mathbb{R}^N) : w_2 u \in L^2(\mathbb{R}^N), \nabla u \in \mathbb{R}^N \}/\mathbb{R}$ and

$$
\tilde{T}(W) = \begin{cases} 
\nu_0 W & \text{in } \omega \\
\hat{\nu}(|W|) W & \text{in } \mathbb{R}^N \setminus \bar{\omega}
\end{cases}
$$
Variation of Direct State at Scale 1

Find $H \in \mathcal{H}(\mathbb{R}^N)$ such that

$$
\int_{\mathbb{R}^N} \left( \tilde{T}(U_0 + \nabla H) - \tilde{T}(U_0) \right) \cdot \nabla \eta = -\int_{\omega} (\nu_0 - \hat{\nu}(|U_0|)) U_0 \cdot \nabla \eta \quad \forall \eta \in \mathcal{H}(\mathbb{R}^N)
$$

(2)

where $\omega = B(0, 1)$, $\mathcal{H}(\mathbb{R}^N) := \{ u \in \mathcal{D}'(\mathbb{R}^N) : w_2 u \in L^2(\mathbb{R}^N), \nabla u \in \mathbb{R}^N \}/\mathbb{R}$ and

$$
\tilde{T}(W) = \begin{cases} 
\nu_0 W & \text{in } \omega \\
\hat{\nu}(|W|) W & \text{in } \mathbb{R}^N \setminus \omega 
\end{cases}
$$

Theorem

There exists a unique solution $H \in \mathcal{H}(\mathbb{R}^N)$ to problem (2).

Proof: Minty-Browder Theorem
Variation of Direct State at Scale 1

Find $H \in \mathcal{H}(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} \left( \tilde{T}(U_0 + \nabla H) - \tilde{T}(U_0) \right) \cdot \nabla \eta = -\int_{\omega} \left( \nu_0 - \hat{\nu}(|U_0|) \right) U_0 \cdot \nabla \eta \quad \forall \eta \in \mathcal{H}(\mathbb{R}^N) \quad (2)$$

where $\omega = B(0, 1)$, $\mathcal{H}(\mathbb{R}^N) := \{u \in \mathcal{D}'(\mathbb{R}^N) : w_2 u \in L^2(\mathbb{R}^N), \nabla u \in \mathbb{R}^N \}/\mathbb{R}$ and

$$\tilde{T}(W) = \begin{cases} \nu_0 W & \text{in } \omega \\ \hat{\nu}(|W|) W & \text{in } \mathbb{R}^N \setminus \bar{\omega} \end{cases}$$

**Proposition**

Let $N = 2$ and $\omega = B(0, 1)$. Then there exists $\tilde{H}$ of the class $H \in \mathcal{H}(\mathbb{R}^N)$ and $\tau > \frac{N}{2} - 1$

$$\tilde{H}(y) = O \left( |y|^{-\tau} \right) \text{ as } |y| \to \infty$$

**Sketch of proof:** $H$ is defined via $QH = 0$

- Show that a function $P$ with the desired behavior at infinity is a supersolution and $0$ is a subsolution, i.e., $\langle Q \ 0, \eta \rangle \leq 0$ and $0 \leq \langle Q \ P, \eta \rangle \ \forall \eta \in \mathcal{V}(\mathbb{R}^N)$, $\text{supp}(\eta) \subset \mathbb{R}_+^N$, $\eta \geq 0$ a.e.
- Show that this implies that $0 \leq \tilde{H}(x) \leq P(x)$ for almost all $x \in \mathbb{R}_+^N$
### Approximations for Adjoint State

- $\nabla u_0 \quad \leadsto \quad U_0$
- $\nabla v_0 \quad \leadsto \quad V_0 := \nabla v_0(x_0)$
- Change of scale

#### Variation of adjoint state at scale 1

Find $K \in \mathcal{H}(\mathbb{R}^N)$ such that

$$
\int_{\mathbb{R}^N} (D\tilde{T}(U_0)\nabla \eta) \cdot \nabla K = -\int_{\omega} ((\nu_0 I - DT_0(U_0))\nabla \eta) \cdot V_0 \quad \forall \eta \in \mathcal{H}(\mathbb{R}^N)
$$

(3)
Approximations for Adjoint State

- \( \nabla u_0 \rightsquigarrow U_0 \)
- \( \nabla v_0 \rightsquigarrow V_0 := \nabla v_0(x_0) \)
- Change of scale

Variation of adjoint state at scale 1

Find \( K \in \mathcal{H}(\mathbb{R}^N) \) such that

\[
\int_{\mathbb{R}^N} (D \tilde{T}(U_0) \nabla \eta) \cdot \nabla K = - \int_{\omega} ((\nu_0 I - DT_0(U_0)) \nabla \eta) \cdot V_0 \quad \forall \eta \in \mathcal{H}(\mathbb{R}^N)
\]  

Theorem

There exists a unique solution \( K \in \mathcal{H}(\mathbb{R}^N) \) to problem (3).
Approximations for Adjoint State

- $\nabla u_0 \rightsquigarrow U_0$
- $\nabla v_0 \rightsquigarrow V_0 := \nabla v_0(x_0)$
- Change of scale

Variation of adjoint state at scale 1

Find $K \in \mathcal{H}(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} (D\tilde{T}(U_0) \nabla \eta) \cdot \nabla K = -\int_\omega (\nu_0 I - DT_0(U_0)) \nabla \eta \cdot V_0 \quad \forall \eta \in \mathcal{H}(\mathbb{R}^N) \quad (3)$$

Theorem

*There exists a unique solution $K \in \mathcal{H}(\mathbb{R}^N)$ to problem (3).*

Proposition

$$\tilde{K}(y) = O(|y|^{1-N}) \text{ as } |y| \to \infty$$
Outline

2 Topology Optimization

- Overview
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- Numerical Experiments
Expansion of linear Term $j_1(\varepsilon)$

Recall: $\mathcal{J}(\Omega_\varepsilon) - \mathcal{J}(\Omega) = j_1(\varepsilon) + j_2(\varepsilon) + o(\varepsilon^N) \overset{!}{=} \varepsilon^N G(x_0) + o(\varepsilon^N)$

Following the approximation steps, we define

$$J_1 := (\nu_0 - \hat{\nu}(|U_0|)) U_0 \cdot \int_\omega V_0 + \nabla K$$
Recall: \( J(\Omega_\varepsilon) - J(\Omega) = j_1(\varepsilon) + j_2(\varepsilon) + o(\varepsilon^N) = \varepsilon^N G(x_0) + o(\varepsilon^N) \)

Following the approximation steps, we define

\[
J_1 := (\nu_0 - \hat{\nu}(|U_0|)) U_0 \cdot \int_\omega V_0 + \nabla K
\]

Proposition

\[
j_1(\varepsilon) = \varepsilon^N J_1 + o(\varepsilon^N)
\]
Recall: \( J(\Omega_\varepsilon) - J(\Omega) = j_1(\varepsilon) + j_2(\varepsilon) + o(\varepsilon^N) \overset{!}{=} \varepsilon^N G(x_0) + o(\varepsilon^N) \)

Following the approximation steps, we define

\[
J_1 := (\nu_0 - \hat{\nu}(|U_0|)) U_0 \cdot \int_\omega V_0 + \nabla K
\]

**Proposition**

\[
j_1(\varepsilon) = \varepsilon^N J_1 + o(\varepsilon^N)
\]
Expansion of linear Term $j_1(\varepsilon)$

Recall: $J(\Omega_\varepsilon) - J(\Omega) = j_1(\varepsilon) + j_2(\varepsilon) + o(\varepsilon^N) \triangleq \varepsilon^N G(x_0) + o(\varepsilon^N)$

Remark

Because of the linearity of the equation defining $K$, also the mapping

$$V_0 \mapsto (\nu_0 - \hat{\nu}(|U_0|)) \int_\omega V_0 + \nabla K$$

is linear, thus there exists a polarization matrix $\mathcal{P} = \mathcal{P}(\omega, U_0)$ s.t.

$$J_1 = U_0^T \mathcal{P}(\omega, U_0) V_0$$
Expansion of Nonlinear Term $j_2(\varepsilon)$

Recall: $J(\Omega_{\varepsilon}) - J(\Omega) = j_1(\varepsilon) + j_2(\varepsilon) + o(\varepsilon^N) \overset{!}{=} \varepsilon^N G(x_0) + o(\varepsilon^N)$
Expansion of Nonlinear Term $j_2(\varepsilon)$

Recall: $J(\Omega_\varepsilon) - J(\Omega) = j_1(\varepsilon) + j_2(\varepsilon) + o(\varepsilon^N) \overset{!}{=} \varepsilon^N G(x_0) + o(\varepsilon^N)$

Following the approximation steps, we define

$$J_2 := \int_{\mathbb{R}^N} S_{U_0}(\nabla H) \cdot (V_0 + \nabla K)$$

where $S_{U_0}(\nabla H) := \tilde{T}(U_0 + \nabla H) - \tilde{T}(U_0) - D\tilde{T}(U_0)\nabla H$
Recall: \( J(\Omega_\varepsilon) - J(\Omega) \equiv j_1(\varepsilon) + j_2(\varepsilon) + o(\varepsilon^N) \equiv \varepsilon^N G(x_0) + o(\varepsilon^N) \)

Following the approximation steps, we define

\[ J_2 := \int_{\mathbb{R}^N} S_{U_0}(\nabla H) \cdot (V_0 + \nabla K) \]

where \( S_{U_0}(\nabla H) := \tilde{T}(U_0 + \nabla H) - \tilde{T}(U_0) - D\tilde{T}(U_0) \nabla H \)

**Proposition**

\[ j_2(\varepsilon) = \varepsilon^N J_2 + o(\varepsilon^N) \]
Expansion of Nonlinear Term $j_2(\varepsilon)$

Recall: $\mathcal{J}(\Omega_\varepsilon) - \mathcal{J}(\Omega) = \sqrt{j_1(\varepsilon)} + \sqrt{j_2(\varepsilon)} + o(\varepsilon^N) = \varepsilon^N G(x_0) + o(\varepsilon^N)$

Following the approximation steps, we define

$$J_2 := \int_{\mathbb{R}^N} S_{U_0}(\nabla H) \cdot (V_0 + \nabla K)$$

where $S_{U_0}(\nabla H) := \tilde{T}(U_0 + \nabla H) - \tilde{T}(U_0) - D\tilde{T}(U_0)\nabla H$

**Proposition**

$$j_2(\varepsilon) = \varepsilon^N J_2 + o(\varepsilon^N)$$
Final Result

Recall: \( J(\Omega_{\varepsilon}) - J(\Omega) = j_1(\varepsilon) + j_2(\varepsilon) + o(\varepsilon^N) = \varepsilon^N G(x_0) + o(\varepsilon^N) \)

Theorem (Amstutz, G., 2016)

Let \( J \) and \( \hat{\nu} \) satisfy Assumption (A1). Then the topological derivative for our optimization problem reads

\[
G(x_0) = U_0^T P V_0 + \int_{\mathbb{R}^N} S_{U_0}(\nabla H) \cdot (V_0 + \nabla K)
\]
Final Result

Recall: $J(\Omega_\varepsilon) - J(\Omega) = j_1(\varepsilon) + j_2(\varepsilon) + o(\varepsilon^N) = \varepsilon^N G(x_0) + o(\varepsilon^N)$

Theorem (Amstutz, G., 2016)

Let $J$ and $\hat{\nu}$ satisfy Assumption (A1). Then the topological derivative for our optimization problem reads

$$G(x_0) = U_0^T \mathcal{P} V_0 + \int_{\mathbb{R}^N} S_{U_0}(\nabla H) \cdot (V_0 + \nabla K)$$

Note:

- All the steps taken were for the case where we introduce linear material (air; $\nu = \nu_0$) inside nonlinear (iron; $\nu = \hat{\nu}(|\nabla u|)$),

  $$G = G^{f\rightarrow air}$$
Final Result

Recall: \( J(\Omega_\varepsilon) - J(\Omega) = j_1(\varepsilon) + j_2(\varepsilon) + o(\varepsilon^N) = \varepsilon^N G(x_0) + o(\varepsilon^N) \)

**Theorem (Amstutz, G., 2016)**

Let \( J \) and \( \hat{\nu} \) satisfy Assumption (A1). Then the topological derivative for our optimization problem reads

\[
G(x_0) = U_0^T \mathcal{P} V_0 + \int_{\mathbb{R}^N} S_{U_0}(\nabla H) \cdot (V_0 + \nabla K)
\]

**Note:**

- All the steps taken were for the case where we introduce linear material (air; \( \nu = \nu_0 \)) inside nonlinear (iron; \( \nu = \hat{\nu}(|\nabla u|) \)),

\[
G = G^{f \rightarrow air}
\]

- The same steps have to be conducted for the reverse scenario (introducing iron inside air region), to obtain

\[
G = G^{air \rightarrow f}
\]
Outline

1 Motivation and Problem Description

2 Topology Optimization
   - Overview
   - On/Off Method by Takahashi et al.
   - Topological Derivative for Nonlinear Magnetostatics
   - Comparison with On/Off Sensitivity
   - Computational Issues
   - Numerical Experiments

3 Shape Optimization

4 A Locally Modified Finite Element Method

5 Application to Electric Motor

6 Conclusion & Outlook
On/Off Sensitivity vs. Topological Derivative

- On/Off sens: What happens to $J$ when $\nu$ is perturbed a little bit
- Top. Der.: What happens to $J$ when hole is introduced ($\hat{\nu} \leadsto \nu_0$)
On/Off Sensitivity vs. Topological Derivative

- On/Off sens: What happens to $J$ when $\nu$ is perturbed a little bit
- Top. Der.: What happens to $J$ when hole is introduced ($\hat{\nu} \sim \nu_0$)

Comparison:

<table>
<thead>
<tr>
<th></th>
<th>$\frac{1}{T_k} \frac{dJ}{d\nu_k}$ (On/Off)</th>
<th>$G(x_0)$ (Top. Der.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>linear ($\hat{\nu} = \nu_1$)</td>
<td>$U_0^T V_0$</td>
<td>$C U_0^T V_0$</td>
</tr>
<tr>
<td>nonlinear</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note:

$$C^{f\rightarrow\text{air}} \neq C^{\text{air}\rightarrow f}$$  

(actually, $C^{f\rightarrow\text{air}} = -\frac{\nu_1}{\nu_0} C^{\text{air}\rightarrow f}$)

Difference is not captured in “On/Off sensitivity”
On/Off Sensitivity vs. Topological Derivative

- On/Off sens: What happens to $\mathcal{J}$ when $\nu$ is perturbed a little bit
- Top. Der.: What happens to $\mathcal{J}$ when hole is introduced ($\hat{\nu} \rightsquigarrow \nu_0$)

Comparison:

|           | $\frac{1}{|T_k|} \left| \frac{d}{d \nu_k} \mathcal{J} \right|$ (On/Off) | $G(x_0)$ (Top. Der.) |
|-----------|-------------------------------------------------|---------------------|
| Linear ($\hat{\nu} = \nu_1$) | $U_0^T V_0$ | $C U_0^T V_0$ |
| Nonlinear | $U_0^T V_0$ | $U_0^T \mathcal{P}(U_0) V_0$ $+ \int_{\mathbb{R}^2} S_{U_0}(\nabla H) \cdot (V_0 + \nabla K)$ |

Note:

$C_{\text{f$\rightarrow$air}} \neq C_{\text{air$\rightarrow$f}}$  
(actually, $C_{\text{f$\rightarrow$air}} = -\frac{\nu_1}{\nu_0} C_{\text{air$\rightarrow$f}}$)

$\mathcal{P}_{\text{f$\rightarrow$air}} \neq \mathcal{P}_{\text{air$\rightarrow$f}}$

Difference is not captured in “On/Off sensitivity”
On/Off Sensitivity vs. Topological Derivative

- $\mathcal{P}$ gives **right scaling** between introduction and removal of material

*Note:* $G^{f\rightarrow \text{air}} \neq G^{\text{air}\rightarrow f}$, difference cannot be captured by $\frac{d \mathcal{J}}{d \nu_k}$!

Initial design, $\mathcal{J}(u) = 0.450$

$\chi_{\Omega_f} G^{f\rightarrow \text{air}} - \chi_{\Omega_{\text{air}}} G^{\text{air}\rightarrow f}$

$\mathcal{J}(u) = 0.436$

$\mathcal{J}(u) = 0.406$
Outline

1 Motivation and Problem Description

2 Topology Optimization
   ■ Overview
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   ■ Numerical Experiments

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5 Application to Electric Motor

6 Conclusion & Outlook
Derivation of Polarization Matrix

\[
V_0 \mapsto (\nu_0 - \hat{\nu}(|U_0|)) \int\limits_{\omega} V_0 + \nabla K \overset{1}{=} \mathcal{P}V_0
\]

where \(K \in \mathcal{H}(\mathbb{R}^N)\) is the solution to:

\[
\int_{\mathbb{R}^N} DT_0(U_0) \nabla K \cdot \nabla \eta = -\int_{\omega} (\nu_0 I - DT_0(U_0))(V_0 + \nabla K) \nabla \eta \quad \forall \eta \in \mathcal{H}(\mathbb{R}^N)
\]
Derivation of Polarization Matrix

\[ V_0 \leftrightarrow (\nu_0 - \hat{\nu}(|U_0|)) \int_{\omega} V_0 + \nabla K = \mathcal{P} V_0 \]

where \( K \in \mathcal{H}(\mathbb{R}^N) \) is the solution to:

\[ \int_{\mathbb{R}^N} DT_0(U_0) \nabla K \cdot \nabla \eta = - \int_{\omega} (\nu_0 I - DT_0(U_0))(V_0 + \nabla K) \nabla \eta \quad \forall \eta \in \mathcal{H}(\mathbb{R}^N) \]

Solve transmission problem by a

- coordinate transformation \( y = DT_0(U_0)^{1/2} x \)
- special ansatz in elliptic coordinates

for \( V_0 = (1, 0)^T \mapsto K_{(1,0)} \) and

\( V_0 = (0, 1)^T \mapsto K_{(0,1)} \) and insert

\[
\begin{pmatrix}
1 \\
0
\end{pmatrix}
\leftrightarrow (\nu_0 - \hat{\nu}(|U_0|)) \int_{\omega} \begin{pmatrix}
1 \\
0
\end{pmatrix} + \nabla K_{(1,0)} =: 
\begin{pmatrix}
p_{11} \\
p_{21}
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 \\
1
\end{pmatrix}
\leftrightarrow (\nu_0 - \hat{\nu}(|U_0|)) \int_{\omega} \begin{pmatrix}
0 \\
1
\end{pmatrix} + \nabla K_{(0,1)} =: 
\begin{pmatrix}
p_{12} \\
p_{22}
\end{pmatrix}
\]
Derivation of Polarization Matrix

\[ V_0 \leftrightarrow (\nu_0 - \hat{\nu}(|U_0|)) \int_\omega V_0 + \nabla K = \mathcal{P} V_0 \]

**Proposition**

Let \( U_0 = R_\varphi(|U_0|, 0)^T \) and denote \( \alpha := \hat{\nu}(|U_0|) \) and \( \beta = \hat{\nu}'(|U_0|)|U_0| \). Then it holds that

\[ J_1 = U_0^T \mathcal{P} V_0 \]

with

\[
\mathcal{P} = \mathcal{P}^{f \rightarrow \text{air}}(\omega, DT_0(U_0)) = (\nu_0 - \alpha)|\omega| R_\varphi \begin{pmatrix}
\frac{\alpha + \beta + \sqrt{\alpha(\alpha + \beta)}}{\nu_0 + \sqrt{\alpha(\alpha + \beta)}} & 0 \\
0 & \frac{\alpha + \sqrt{\alpha(\alpha + \beta)}}{\nu_0 + \sqrt{\alpha(\alpha + \beta)}}
\end{pmatrix} R_\varphi^T
\]
Derivation of Polarization Matrix

\[ V_0 \leftrightarrow (\nu_0 - \hat{\nu}(|U_0|)) \int_{\omega} V_0 + \nabla K = \mathcal{P} V_0 \]

**Proposition**

Let \( U_0 = R_\varphi(|U_0|, 0)^T \) and denote \( \alpha := \hat{\nu}(|U_0|) \) and \( \beta = \hat{\nu}'(|U_0|)|U_0| \). Then it holds that

\[ J_1 = U_0^T \mathcal{P} V_0 \]

with

\[
\mathcal{P} = \mathcal{P}_{\text{air}}^f (\omega, D T_0(U_0)) = (\nu_0 - \alpha)|\omega| R_\varphi \left( \begin{array}{cc}
\frac{\alpha + \beta + \sqrt{\alpha (\alpha + \beta)}}{\nu_0 + \sqrt{\alpha (\alpha + \beta)}} & 0 \\
0 & \frac{\alpha + \sqrt{\alpha (\alpha + \beta)}}{\nu_0 + \sqrt{\alpha (\alpha + \beta)}}
\end{array} \right) R_\varphi^T
\]

**Remark**

In Case II, we have

\[
\mathcal{P} = \mathcal{P}_{\text{air}}^f (\omega, D T_0(U_0)) = (\alpha - \nu_0) |\omega| R_\varphi \left( \begin{array}{cc}
\frac{2\nu_0}{\nu_0 + \alpha + \beta} & 0 \\
0 & \frac{2\nu_0}{\nu_0 + \alpha}
\end{array} \right) R_\varphi^T.
\]
Computational Issues

Topological derivative

\[ G(x_0) = U_0^T \mathcal{P} V_0 + \int_{\mathbb{R}^N} S_{U_0}(\nabla H) \cdot (V_0 + \nabla K) \]

Problems:

- \( H = H(U_0) = H(\nabla u(x_0)) \) solution of nonlinear problem on \( \mathbb{R}^N \)
- want to have \( G(x_0) \) for all points \( x_0 \) in design domain
  \( \implies \) need to solve nonlinear problem for \( H \) for every possible \( U_0 = \nabla u(x_0) \)
Computational Issues

\[ J_2 = J_2(U_0, V_0) = \int_{\mathbb{R}^N} S U_0(\nabla H(U_0)) \cdot (V_0 + \nabla K(U_0, V_0)) \]
Computational Issues

\[ J_2 = J_2(U_0, V_0) = \int_{\mathbb{R}^N} S_{U_0}(\nabla H(U_0)) \cdot (V_0 + \nabla K(U_0, V_0)) \]

- \( J_2(RU_0, RV_0) = J_2(U_0, V_0) \) for rotation matrix \( R \)
- \( J_2 \) linear in \( V_0 \)
\[ J_2 = J_2(U_0, V_0) = \int_{\mathbb{R}^N} S_{U_0}(\nabla H(U_0)) \cdot (V_0 + \nabla K(U_0, V_0)) \]

- \( J_2(R \cdot U_0, R \cdot V_0) = J_2(U_0, V_0) \) for \( R \) ... rotation matrix
- \( J_2 \) linear in \( V_0 \)

\[ J_2(U_0, V_0) = J_2(t \cdot R_\theta \cdot e_1, s \cdot R_\varphi \cdot e_1) \] 

for \( U_0 = t \cdot R_\theta \cdot e_1 \) and \( V_0 = s \cdot R_\varphi \cdot e_1 \) we have
Computational Issues

\[ J_2 = J_2(U_0, V_0) = \int_{\mathbb{R}^N} S_{U_0}(\nabla H(U_0)) \cdot (V_0 + \nabla K(U_0, V_0)) \]

- \( J_2(RU_0, RV_0) = J_2(U_0, V_0) \) for \( R \) … rotation matrix
- \( J_2 \) linear in \( V_0 \)

\[
\Longrightarrow \text{for } U_0 = t R_\theta e_1 \text{ and } V_0 = s R_\varphi e_1 \text{ we have}
\]
\[
J_2(U_0, V_0) = J_2(t R_\theta e_1, s R_\varphi e_1) = J_2(t e_1, s R_\varphi - \theta e_1)
\]
Computational Issues

\[ J_2 = J_2(U_0, V_0) = \int_{\mathbb{R}^N} S_{U_0}(\nabla H(U_0)) \cdot (V_0 + \nabla K(U_0, V_0)) \]

- \( J_2(R \ U_0, R \ V_0) = J_2(U_0, V_0) \) for \( R \) rotation matrix
- \( J_2 \) linear in \( V_0 \)

\[ \text{for } U_0 = t \ R_\theta \ e_1 \text{ and } V_0 = s \ R_\varphi \ e_1 \text{ we have} \]

\[ J_2(U_0, V_0) = J_2(t \ R_\theta \ e_1, s \ R_\varphi \ e_1) \]
\[ = J_2(t \ e_1, s \ R_\varphi - \theta \ e_1) \]
\[ = J_2(t \ e_1, s \ \cos(\varphi - \theta) \ e_1 + s \ \sin(\varphi - \theta) \ e_2) \]
Computational Issues

\[ J_2 = J_2(U_0, V_0) = \int_{\mathbb{R}^N} S_{U_0}(\nabla H(U_0)) \cdot (V_0 + \nabla K(U_0, V_0)) \]

- \( J_2(RU_0, RV_0) = J_2(U_0, V_0) \) for \( R \ldots \) rotation matrix
- \( J_2 \) linear in \( V_0 \)

\[ \Rightarrow \text{for } U_0 = t\, R_\theta \, e_1 \text{ and } V_0 = s\, R_\varphi \, e_1 \text{ we have} \]

\[ J_2(U_0, V_0) = J_2(t\, R_\theta \, e_1, s\, R_\varphi \, e_1) \]
\[ = J_2(t\, e_1, s\, R_\varphi - \theta \, e_1) \]
\[ = J_2(t\, e_1, s\, \cos(\varphi - \theta)\, e_1 + s\, \sin(\varphi - \theta)\, e_2) \]
\[ = s\, \cos(\varphi - \theta)\, J_2(t\, e_1, e_1) + s\, \sin(\varphi - \theta)\, J_2(t\, e_1, e_2) \]
Computational Issues

\[ J_2 = J_2(U_0, V_0) = \int_{\mathbb{R}^N} S_{U_0}(\nabla H(U_0)) \cdot (V_0 + \nabla K(U_0, V_0)) \]

- \( J_2(R U_0, R V_0) = J_2(U_0, V_0) \) for \( R \ldots \) rotation matrix
- \( J_2 \) linear in \( V_0 \)

\[ \Rightarrow \text{for } U_0 = t R_{\theta} e_1 \text{ and } V_0 = s R_{\varphi} e_1 \text{ we have} \]

\[ J_2(U_0, V_0) = J_2(t R_{\theta} e_1, s R_{\varphi} e_1) \]
\[ = J_2(t e_1, s R_{\varphi-\theta} e_1) \]
\[ = J_2(t e_1, s \cos(\varphi - \theta) e_1 + s \sin(\varphi - \theta) e_2) \]
\[ = s \cos(\varphi - \theta) J_2(t e_1, e_1) + s \sin(\varphi - \theta) J_2(t e_1, e_2) \]

\[ \Rightarrow \text{Pre-compute } J_2(t e_1, e_i) \text{ for typical values of } t = |U_0| \text{ and interpolate to evaluate} \]
\[ J_2(U_0, V_0) \]
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Computation of Topological Derivative

- Approximate unbounded domain by disk of radius 1000 with homogeneous Dirichlet BCs

Figure: $H_h$ for $U_0 = (0.1, 0)^T$ in case I
Approximate unbounded domain by disk of radius 1000 with homogeneous Dirichlet BCs

Figure: $K_{h,10}$ for $U_0 = (0.1, 0)^T$ in case I
Computation of Topological Derivative

- Approximate unbounded domain by disk of radius 1000 with homogeneous Dirichlet BCs

Figure: $K_{h,01}$ for $U_0 = (0.1, 0)^T$ in case I
Computation of Topological Derivative

- Approximate unbounded domain by disk of radius 1000 with homogeneous Dirichlet BCs

Figure: $H_h$ for $U_0 = (0.1, 0)^T$ in case II
Computation of Topological Derivative

- Approximate unbounded domain by disk of radius 1000 with homogeneous Dirichlet BCs

**Figure:** $K_{h,10}$ for $U_0 = (0.1, 0)^T$ in case II
Approximate unbounded domain by disk of radius 1000 with homogeneous Dirichlet BCs

Figure: $K_{h,01}$ for $U_0 = (0.1, 0)^T$ in case II
Computation of Topological Derivative

Figure: $J_1(\|U_0\|, \varphi - \theta)$ in case I, order of magnitude: $10^6$
Computation of Topological Derivative

Figure: $J_2(|U_0|, \varphi - \theta)$ in case I, order of magnitude: 1

$\Longrightarrow J_2$ negligible compared to $J_1$ in this application
Computation of Topological Derivative

Figure: $J_1(|U_0|, \varphi - \theta)$ in case II, order of magnitude: $10^7$
Figure: $J_2(|U_0|, \varphi - \theta)$ in case II, order of magnitude: 1

$\implies J_2$ negligible compared to $J_1$ in this application
Algorithm

Represent design by level set function $\psi$:

$$
\psi(x) > 0 \Leftrightarrow x \in \Omega_f \\
\psi(x) < 0 \Leftrightarrow x \in \Omega_{air}
$$

Generalized topological derivative:

$$
\tilde{G}_\psi(x) := \begin{cases} 
G^{f \rightarrow air}(x) & x \in \Omega_f \\
-G^{air \rightarrow f}(x) & x \in \Omega_{air}
\end{cases}
$$

Sufficient optimality condition: $\psi = \tilde{G}_\psi$
Algorithm

Represent design by level set function $\psi$:

$$\begin{align*}
\psi(x) > 0 & \iff x \in \Omega_f \\
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\end{align*}$$

Generalized topological derivative:

$$\tilde{G}_\psi(x) := \begin{cases} 
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\end{cases}$$

Sufficient optimality condition: $\psi = \tilde{G}_\psi$

Proof:

Let $\hat{x} \in \Omega_f$. Then

$$0 < \psi(\hat{x}) = \tilde{G}_\psi(\hat{x}) = G^{f \rightarrow \text{air}}(\hat{x}),$$

thus, introducing air at $\hat{x}$ will yield an increase of $J$.

Analogous argument for $\hat{x} \in \Omega_{\text{air}}$. 

$\square$
Algorithm

Represent design by level set function $\psi$:

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Sufficient optimality condition: $\psi = \tilde{G}_\psi$

Level Set Algorithm

(i) Initialization: Choose $\psi_0$ with $\|\psi_0\| = 1$, compute $\tilde{G}_{\psi_0}$ and set $k = 0$

(ii) Set $\theta_k = \arccos \left( \psi_k, \frac{\tilde{G}_{\psi_k}}{\|\tilde{G}_{\psi_k}\|} \right)$ and

$$\psi_{k+1} = \frac{1}{\sin \theta_k} \left[ \sin((1 - \kappa_k)\theta_k) \psi_k + \sin(\kappa_k \theta_k) \frac{\tilde{G}_{\psi_k}}{\|\tilde{G}_{\psi_k}\|} \right]$$

where $\kappa_k = \max\{1, 1/2, 1/4 \ldots\}$ such that $\mathcal{J}(\psi_{k+1}) < \mathcal{J}(\psi_k)$

(iii) Compute $\tilde{G}_{\psi_{k+1}}$

(iv) If $\tilde{G}_{\psi_{k+1}} = \psi_{k+1}$ then stop, else set $k \leftarrow k + 1$ and go to (ii)
**Algorithm**

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where $\kappa_k = \max\{1, 1/2, 1/4 \ldots \}$ such that $\mathcal{J}(\psi_{k+1}) < \mathcal{J}(\psi_k)$

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Algorithm

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Generalized topological derivative:

\[ \tilde{G}_\psi(x) := \begin{cases} G^{f \to \text{air}}(x) & x \in \Omega_f \\ -G^{\text{air} \to f}(x) & x \in \Omega_{air} \end{cases} \]

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where $\kappa_k = \max\{1, 1/2, 1/4 \ldots \}$ such that $J(\psi_{k+1}) < J(\psi_k)$

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\]

**Sufficient optimality condition**: $\psi = \tilde{G}_\psi$

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**Level Set Algorithm**

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where $\kappa_k = \max\{1, 1/2, 1/4, \ldots\}$ such that $\mathcal{J}(\psi_{k+1}) < \mathcal{J}(\psi_k)$

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Sufficient optimality condition: $\psi = \tilde{G}_\psi$

**Level Set Algorithm**

(i) **Initialization**: Choose $\psi_0$ with $\|\psi_0\| = 1$, compute $\tilde{G}_{\psi_0}$ and set $k = 0$

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**Algorithm**

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Generalized topological derivative:

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\tilde{G}_{\psi}(x) := \begin{cases} 
G_{f \rightarrow air}^f(x) & x \in \Omega_f \\
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\end{cases}
$$

**Sufficient optimality condition:** $\psi = \tilde{G}_{\psi}$

---

**Level Set Algorithm**

(i) **Initialization:** Choose $\psi_0$ with $\|\psi_0\| = 1$, compute $\tilde{G}_{\psi_0}$ and set $k = 0$

(ii) Set $\theta_k = \arccos\left(\psi_k, \frac{\tilde{G}_{\psi_k}}{\|\tilde{G}_{\psi_k}\|}\right)$ and

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\psi_{k+1} = \frac{1}{\sin\theta_k} \left[ \sin((1 - \kappa_k)\theta_k) \psi_k + \sin(\kappa_k\theta_k) \frac{\tilde{G}_{\psi_k}}{\|\tilde{G}_{\psi_k}\|} \right]
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where $\kappa_k = \max\{1, 1/2, 1/4 \ldots\}$ such that $J(\psi_{k+1}) < J(\psi_k)$

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---

Goal: Filter out higher harmonics of radial component of magnetic flux density along air gap while keeping high first harmonic

\[
\min J(u(\Omega)) = \frac{THD(B_r(u(\Omega)))}{A_1(B_r(u(\Omega)))}
\]

s.t. \[
\int_\Omega \nu(|\nabla u|) \nabla u \cdot \nabla \eta = \langle F, \eta \rangle \quad \forall \eta \in H^1_0(\Omega)
\]
Numerical Results

**Goal:** Filter out higher harmonics of radial component of magnetic flux density along air gap while keeping high first harmonic

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\min J(u(\Omega)) = \frac{THD(B_r(u(\Omega)))}{A_1(B_r(u(\Omega)))}
\]

s.t. \[\int_{\Omega} \nu(|\nabla u|) \nabla u \cdot \nabla \eta = \langle F, \eta \rangle \quad \forall \eta \in H^1_0(\Omega)\]

Design at iteration 1, \(J(u) = 0.1582\)
**Goal:** Filter out higher harmonics of radial component of magnetic flux density along air gap while keeping high first harmonic

\[
\min J(u(\Omega)) = \frac{THD(B_r(u(\Omega)))}{A_1(B_r(u(\Omega)))}
\]

subject to
\[
\int_\Omega \nu(|\nabla u|) \nabla u \cdot \nabla \eta = \langle F, \eta \rangle \quad \forall \eta \in H_0^1(\Omega)
\]

Design at iteration 2, \( J(u) = 0.1473 \)
**Goal:** Filter out higher harmonics of radial component of magnetic flux density along air gap while keeping high first harmonic

\[
\min J(u(\Omega)) = \frac{THD(B_r(u(\Omega)))}{A_1(B_r(u(\Omega)))}
\]

s.t. \[
\int_\Omega \nu(|\nabla u|) \nabla u \cdot \nabla \eta = \langle F, \eta \rangle \quad \forall \eta \in H^1_0(\Omega)
\]

Design at iteration 3, \( J(u) = 0.1247 \)
**Goal:** Filter out higher harmonics of radial component of magnetic flux density along air gap while keeping high first harmonic

\[
\min \mathcal{J}(u(\Omega)) = \frac{THD(B_r(u(\Omega)))}{A_1(B_r(u(\Omega)))}
\]

subject to

\[
\int_{\Omega} \nu(|\nabla u|) \nabla u \cdot \nabla \eta = \langle F, \eta \rangle \quad \forall \eta \in H^1_0(\Omega)
\]

Design at iteration 4, \( \mathcal{J}(u) = 0.0993 \)
Goal: Filter out higher harmonics of radial component of magnetic flux density along air gap while keeping high first harmonic

\[
\min J(u(\Omega)) = \frac{THD(B_r(u(\Omega)))}{A_1(B_r(u(\Omega)))}
\]

s.t. \[
\int_\Omega \nu(|\nabla u|) \nabla u \cdot \nabla \eta = \langle F, \eta \rangle \quad \forall \eta \in H^1_0(\Omega)
\]

Design at iteration 5, \( J(u) = 0.0639 \)
**Goal:** Filter out higher harmonics of radial component of magnetic flux density along air gap while keeping high first harmonic

\[
\min \mathcal{J}(u(\Omega)) = \frac{THD(B_r(u(\Omega)))}{A_1(B_r(u(\Omega)))}
\]

s.t. \( \int_\Omega \nu(|\nabla u|) \nabla u \cdot \nabla \eta = \langle F, \eta \rangle \quad \forall \eta \in H^1_0(\Omega) \)

Design at iteration 6, \( \mathcal{J}(u) = 0.0594 \)
Goal: Filter out higher harmonics of radial component of magnetic flux density along air gap while keeping high first harmonic

\[
\min J(u(\Omega)) = \frac{THD(B_r(u(\Omega)))}{A_1(B_r(u(\Omega)))}
\]

\[
s.t. \int_\Omega \nu(|\nabla u|) \nabla u \cdot \nabla \eta = \langle F, \eta \rangle \quad \forall \eta \in H^1_0(\Omega)
\]

Design at iteration 7, \(J(u) = 0.0485\)
**Goal:** Filter out higher harmonics of radial component of magnetic flux density along air gap while keeping high first harmonic

\[
\min \mathcal{J}(u(\Omega)) = \frac{THD(B_r(u(\Omega)))}{A_1(B_r(u(\Omega)))}
\]

s.t. \[ \int_{\Omega} \nu(|\nabla u|) \nabla u \cdot \nabla \eta = \langle F, \eta \rangle \quad \forall \eta \in H^1_0(\Omega) \]

Design at iteration 8, \[ \mathcal{J}(u) = 0.0431 \]
**Goal:** Filter out higher harmonics of radial component of magnetic flux density along air gap while keeping high first harmonic

\[
\min J(u(\Omega)) = \frac{THD(B_r(u(\Omega)))}{A_1(B_r(u(\Omega)))}
\]

\[
\text{s.t. } \int_{\Omega} \nu(|\nabla u|) \nabla u \cdot \nabla \eta = \langle F, \eta \rangle \quad \forall \eta \in H^1_0(\Omega)
\]

Design at iteration 9, \( J(u) = 0.0384 \)
**Goal:** Filter out higher harmonics of radial component of magnetic flux density along air gap while keeping high first harmonic

\[
\min J(u(\Omega)) = \frac{THD(B_r(u(\Omega)))}{A_1(B_r(u(\Omega)))}
\]

\[
s.t. \int_{\Omega} \nu(|\nabla u|) \nabla u \cdot \nabla \eta = \langle F, \eta \rangle \quad \forall \eta \in H_0^1(\Omega)
\]

Design at iteration 10, \( J(u) = 0.0356 \)
**Goal:** Filter out higher harmonics of radial component of magnetic flux density along air gap while keeping high first harmonic

\[
\min J(u(\Omega)) = \frac{THD(B_r(u(\Omega)))}{A_1(B_r(u(\Omega)))}
\]

s.t. \[
\int_{\Omega} \nu(|\nabla u|) \nabla u \cdot \nabla \eta = \langle F, \eta \rangle \quad \forall \eta \in H_0^1(\Omega)
\]

**Design at iteration 20,** $J(u) = 0.030541$
**Goal:** Filter out higher harmonics of radial component of magnetic flux density along air gap while keeping high first harmonic

\[
\min J(u(\Omega)) = \frac{THD(B_r(u(\Omega)))}{A_1(B_r(u(\Omega)))}
\]

s.t. \( \int_{\Omega} \nu(|\nabla u|) \nabla u \cdot \nabla \eta = \langle F, \eta \rangle \quad \forall \eta \in H^1_0(\Omega) \)

Design at iteration 30, \( J(u) = 0.030136 \)
**Goal:** Filter out higher harmonics of radial component of magnetic flux density along air gap while keeping high first harmonic

\[
\min J(u(\Omega)) = \frac{THD(B_r(u(\Omega)))}{A_1(B_r(u(\Omega)))}
\]

\[
\text{s.t. } \int_{\Omega} \nu(\|\nabla u\|) \nabla u \cdot \nabla \eta = \langle F, \eta \rangle \quad \forall \eta \in H_0^1(\Omega)
\]

Design at iteration 40, \(J(u) = 0.030030\)
Goal: Filter out higher harmonics of radial component of magnetic flux density along air gap while keeping high first harmonic

\[
\min J(u(\Omega)) = \frac{THD(B_r(u(\Omega)))}{A_1(B_r(u(\Omega)))}
\]

s.t. \[\int_{\Omega} \nu(|\nabla u|) \nabla u \cdot \nabla \eta = \langle F, \eta \rangle \quad \forall \eta \in H^1_0(\Omega)\]

Final Design at iteration 43, \( J(u) = 0.030027 \)
**Goal:** Filter out higher harmonics of radial component of magnetic flux density along air gap while keeping high first harmonic

\[
\min J(u(\Omega)) = \frac{THD(B_r(u(\Omega)))}{A_1(B_r(u(\Omega)))}
\]

s.t. \[\int_\Omega \nu(|\nabla u|) \nabla u \cdot \nabla \eta = \langle F, \eta \rangle \quad \forall \eta \in H^1_0(\Omega)\]

Final Design at iteration 43, \(J(u) = 0.030027\)
References

K. Sturm
*Minimax Lagrangian Approach to the Differentiability of Nonlinear PDE-Constrained Shape Functions without Saddle Point Assumption*,

P. G., A. Laurain, U. Langer, H. Meftahi, K. Sturm,
*Shape Optimization of an Electric Motor Subject to Nonlinear Magnetostatics*,
Shape Optimization for Nonlinear Magnetostatics

Model Problem from 2D Magnetostatics

\[
\begin{align*}
\min_{\Omega_f \in \mathcal{O}} \quad & \mathcal{J}(u(\Omega_f)) \\
\text{s.t.} \quad & -\text{div} \left( \nu_{\Omega_f} (|\nabla u|) \nabla u \right) = F \quad \text{in } \Omega \\
& \quad u = 0 \quad \text{on } \partial\Omega 
\end{align*}
\]

(4)

where

\[ \mathcal{O} = \{ \Omega_f \subset \Omega_f^{\text{ref}}, \Omega_f \text{ open and Lipschitz with uniform constant } L_\mathcal{O} \}. \]
Model Problem from 2D Magnetostatics

\[
\begin{align*}
\min_{\Omega_f \in \mathcal{O}} \quad & \mathcal{J}(u(\Omega_f)) \\
\text{s.t.} \quad & \begin{cases}
-\text{div} \left( \nu_{\Omega_f} (|\nabla u|) \nabla u \right) = F & \text{in } \Omega \\
\quad u = 0 & \text{on } \partial\Omega
\end{cases}
\end{align*}
\]

(4)

where

\[
\mathcal{O} = \{ \Omega_f \subset \Omega_f^{\text{ref}}, \Omega_f \text{ open and Lipschitz with uniform constant } L_\mathcal{O} \}.
\]

Proposition [G., Langer, Laurain, Meftahi, Sturm, 2015]

There exists a solution \( \Omega_f \in \mathcal{O} \) to the shape optimization problem above.
Shape Optimization for Nonlinear Magnetostatics

**Definition**

Eulerian Semiderivative in direction $V \in C^{0,1}_{c}(\Omega, \mathbb{R}^{d})$

$$dJ(\Omega; V) := \lim_{t \to 0} \frac{J(\Omega_t) - J(\Omega)}{t}$$

If $V \mapsto dJ(\Omega; V)$ exists and is linear, then we call $J$ **shape differentiable** and $dJ$ the **shape derivative** of $J$
Definition

Eulerian Semiderivative in direction $V \in C^{0,1}_c(\Omega, \mathbb{R}^d)$

$$d\mathcal{J}(\Omega; V) := \lim_{t \to 0} \frac{\mathcal{J}(\Omega_t) - \mathcal{J}(\Omega)}{t}$$

If $V \mapsto d\mathcal{J}(\Omega; V)$ exists and is linear, then we call $\mathcal{J}$ shape differentiable and $d\mathcal{J}$ the shape derivative of $\mathcal{J}$.
Shape Optimization for Nonlinear Magnetostatics

### Definition

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$$dJ(\Omega; V) := \lim_{t \to 0} \frac{J(\Omega_t) - J(\Omega)}{t}$$

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Shape Optimization for Nonlinear Magnetostatics

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**Boundary expression** (Hadamard-Zolésio)

\[
dJ(\Omega; V) = \int_{\partial\Omega_f} g V \cdot \mathbf{n} ds
\]

**Domain expression**

\[
dJ(\Omega; V) = \int_{\Omega} F(V, DV) dx
\]
Shape Optimization for Nonlinear Magnetostatics

Definition

Eulerian Semiderivative in direction \( V \in C^{0,1}_c(\Omega, \mathbb{R}^d) \)

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dJ(\Omega; V) := \lim_{t \to 0} \frac{J(\Omega_t) - J(\Omega)}{t}
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If \( V \mapsto dJ(\Omega; V) \) exists and is linear, then we call \( J \) shape differentiable and \( dJ \) the shape derivative of \( J \)

Boundary expression (Hadamard-Zolésio)

\[
dJ(\Omega; V) = \int_{\partial \Omega_f} g V \cdot n ds
\]

- \( V = -g n \) is descent direction

Domain expression

\[
dJ(\Omega; V) = \int_\Omega F(V, D\Omega) dx
\]

- descent direction \( V \) via auxiliary BVP
Shape Optimization for Nonlinear Magnetostatics

Definition

Eulerian Semiderivative in direction $V \in \mathcal{C}^{0,1}_c(\Omega, \mathbb{R}^d)$

$$d\mathcal{J}(\Omega; V) := \lim_{t \to 0} \frac{\mathcal{J}(\Omega_t) - \mathcal{J}(\Omega)}{t}$$

If $V \mapsto d\mathcal{J}(\Omega; V)$ exists and is linear, then we call $\mathcal{J}$ shape differentiable and $d\mathcal{J}$ the shape derivative of $\mathcal{J}$.

Boundary expression (Hadamard-Zolésio)

$$d\mathcal{J}(\Omega; V) = \int_{\partial\Omega_f} g V \cdot n ds$$

- $V = -g n$ is descent direction
- Often needed in neighborhood of $\partial\Omega_f$

Domain expression

$$d\mathcal{J}(\Omega; V) = \int_{\Omega} F(V, DV) dx$$

- descent direction $V$ via auxiliary BVP
- More general
- Better accuracy in FE Setting

Theorem [G., Langer, Laurain, Meftahi, Sturm, 2015]

Let \( \nu \) satisfy the natural assumptions. Then \( \mathcal{J} \) is shape differentiable and

\[
d\mathcal{J}(\Omega; V) = -\int_{\Omega_{mag}} \mathbb{P}'(0) \nabla p \cdot M^\perp \, dx + \int_{D} \nu(x, |\nabla u|) \mathbb{Q}'(0) \nabla u \cdot \nabla p \, dx
- \int_{D} \frac{\partial \zeta \nu(x, |\nabla u|)}{|\nabla u|} (\nabla u \cdot \nabla p) \nabla u \cdot \nabla p \, dx
\]

where \( \mathbb{P}'(0) = (\text{div} \, V) I_2 - DV^T \), \( \mathbb{Q}'(0) = (\text{div} \, V) I_2 - DV^T - DV \), \( I_2 \in \mathbb{R}^{2,2} \) is the identity matrix, and \( u, p \in H^1_0(D) \) are state and adjoint variable, respectively.
Theorem [G., Langer, Laurain, Meftahi, Sturm, 2015]

Let $\nu$ satisfy the natural assumptions. Then $\mathcal{J}$ is shape differentiable and

$$d\mathcal{J}(\Omega; V) = -\int_{\Omega_{mag}} \mathbb{P}'(0) \nabla p \cdot M^\perp \, dx + \int_{D} \nu(x, |\nabla u|) \mathbb{Q}'(0) \nabla u \cdot \nabla p \, dx$$

$$- \int_{D} \frac{\partial \xi \nu(x, |\nabla u|)}{|\nabla u|} (DV^T \nabla u \cdot \nabla u) (\nabla u \cdot \nabla p) \, dx$$

where $\mathbb{P}'(0) = (\text{div } V) I_2 - DV^T$, $\mathbb{Q}'(0) = (\text{div } V) I_2 - DV^T - DV$, $I_2 \in \mathbb{R}^{2 \times 2}$ is the identity matrix, and $u, p \in H^1_0(D)$ are state and adjoint variable, respectively.

- Want to find $V$ such that $d\mathcal{J}(\Omega; V) < 0$. 

P. Gangl (JKU Linz, LCM)
Theorem [G., Langer, Laurain, Meftahi, Sturm, 2015]

Let $\nu$ satisfy the natural assumptions. Then $J$ is shape differentiable and

$$
\begin{align*}
\frac{dJ(\Omega; V)}{d\Omega} &= -\int_{\Omega_{\text{mag}}} \mathbb{P}'(0) \nabla p \cdot M^\perp \, dx + \int_D \nu(x, |\nabla u|) \mathbb{Q}'(0) \nabla u \cdot \nabla p \, dx \\
&\quad - \int_D \frac{\partial \zeta \nu(x, |\nabla u|)}{|\nabla u|} \left( DV^T \nabla u \cdot \nabla u \right)(\nabla u \cdot \nabla p) \, dx
\end{align*}
$$

where $\mathbb{P}'(0) = (\text{div } V) I_2 - DV^T$, $\mathbb{Q}'(0) = (\text{div } V) I_2 - DV^T - DV$, $I_2 \in \mathbb{R}^{2 \times 2}$ is the identity matrix, and $u, p \in H^1_0(D)$ are state and adjoint variable, respectively.

- Want to find $V$ such that $\frac{dJ(\Omega; V)}{d\Omega} < 0$.
- Trick to obtain a descent direction $V$:
  
  Choose symm. pos. def. bilinear form $b : H^1_0(D) \times H^1_0(D) \to \mathbb{R}$ and solve

$$(b(V, W) = -\frac{dJ(\Omega; W)}{d\Omega} \quad \forall W \in H^1_0(D))$$
Theorem [G., Langer, Laurain, Meftahi, Sturm, 2015]

Let \( \nu \) satisfy the natural assumptions. Then \( J \) is shape differentiable and

\[
\begin{align*}
    dJ(\Omega; V) &= -\int_{\Omega_{mag}} \mathbb{P}'(0) \nabla p \cdot M^\perp \, dx + \int_D \nu(x, |\nabla u|) Q'(0) \nabla u \cdot \nabla p \, dx \\
    &\quad - \int_D \frac{\partial \zeta \nu(x, |\nabla u|)}{|\nabla u|} (DV^T \nabla u \cdot \nabla u)(\nabla u \cdot \nabla p) \, dx
\end{align*}
\]

where \( \mathbb{P}'(0) = (\text{div} V) I_2 - DV^T \), \( Q'(0) = (\text{div} V) I_2 - DV^T - DV \), \( I_2 \in \mathbb{R}^{2 \times 2} \) is the identity matrix, and \( u, p \in H^1_0(D) \) are state and adjoint variable, respectively.

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- Trick to obtain a descent direction \( V \):
  
  Choose symm., pos. def. bilinear form \( b : H^1_0(D) \times H^1_0(D) \rightarrow \mathbb{R} \) and solve
  
  \[
  b(V, W) = -dJ(\Omega; W) \quad \forall \ W \in H^1_0(D)
  \]

  \[ \implies dJ(\Omega; V) = -b(V, V) < 0 \]
Theorem [G., Langer, Laurain, Meftahi, Sturm, 2015]

Let $\nu$ satisfy the natural assumptions. Then $\mathcal{J}$ is shape differentiable and

$$d\mathcal{J}(\Omega; V) = -\int_{\Omega_{\text{mag}}} \mathbb{P}'(0) \nabla p \cdot M^\perp \; dx + \int_D \nu(x, |\nabla u|) \mathcal{Q}'(0) \nabla u \cdot \nabla p \; dx$$

$$- \int_D \frac{\partial \zeta}{|\nabla u|} (DV^T \nabla u \cdot \nabla u) (\nabla u \cdot \nabla p) \; dx$$

where $\mathbb{P}'(0) = (\text{div} V) I_2 - DV^T$, $\mathcal{Q}'(0) = (\text{div} V) I_2 - DV^T - DV$, $I_2 \in \mathbb{R}^{2 \times 2}$ is the identity matrix, and $u, p \in H^1_0(D)$ are state and adjoint variable, respectively.

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  Choose symm., pos. def. bilinear form $b : H^1_0(D) \times H^1_0(D) \to \mathbb{R}$ and solve

  $$b(V, W) = -d\mathcal{J}(\Omega; W) \quad \forall W \in H^1_0(D)$$

  $$\implies d\mathcal{J}(\Omega; V) = -b(V, V) < 0$$

  $$\implies \mathcal{J}(\Omega_t) < \mathcal{J}(\Omega) \text{ for small } t$$
iter=0, $\mathcal{J}(u) = 1.684 \times 10^{-3}$
Shape Optimization

Numerical Results

\[ \text{iter}=0, \ J(u) = 1.684 \times 10^{-3} \]
Numerical Results

$$\text{iter} = 15, \mathcal{J}(u) = 1.431 \times 10^{-3}$$
Shape Optimization

Numerical Results

iter=15, $\mathcal{J}(u) = 1.431 \times 10^{-3}$
iter=30, $\mathcal{J}(u) = 1.169 \times 10^{-3}$
Numerical Results

\[ \text{iter}=30, \quad J(u) = 1.169 \times 10^{-3} \]
Shape Optimization

Numerical Results

iter=45, $J(u) = 1.043 \times 10^{-3}$
Shape Optimization

Numerical Results

\begin{align*}
\text{iter}=45, \quad J(u) & = 1.043 \times 10^{-3} \\
\end{align*}
Numerical Results

\[ \text{iter}=60, \mathcal{J}(u) = 0.992 \times 10^{-3} \]
iter=60, $\mathcal{J}(u) = 0.992 \, \text{e-3}$
Shape Optimization

Numerical Results

\[ \text{iter}=70, \ J(u) = 0.985 \times 10^{-3} \]
Numerical Results

\[ \text{iter}=70, \mathcal{J}(u) = 0.985 \times 10^{-3} \]
Numerical Results

**Question:** How to resolve interface?
Outline

1. Motivation and Problem Description

2. Topology Optimization

3. Shape Optimization

4. A Locally Modified Finite Element Method
   - The Method
   - Numerical Example

5. Application to Electric Motor

6. Conclusion & Outlook
Outline

1 Motivation and Problem Description
2 Topology Optimization
3 Shape Optimization
4 A Locally Modified Finite Element Method
   ■ The Method
   ■ Numerical Example
5 Application to Electric Motor
6 Conclusion & Outlook
Problem: Evolving interface is not aligned with mesh!
FEM for Interface Problem

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Naive approach: Assign element to $\Omega_1$ or $\Omega_2$ based on position of centroid
Problem: Evolving interface is not aligned with mesh!

Naive approach: Assign element to $\Omega_1$ or $\Omega_2$ based on position of centroid

Drawbacks:
- Jagged interface
- $\|\nabla (u - u_h)\|_{L^2(\Omega)} = O(h^{1/2})$

Loss of accuracy
FEM for Interface Problem

Problem: Evolving interface is not aligned with mesh!

Naive approach: Assign element to $\Omega_1$ or $\Omega_2$ based on position of centroid

Drawbacks:

- Jagged interface
- $\|\nabla(u - u_h)\|_{L^2(\Omega)} = O(h^{1/2})$

Possible remedies:
FEM for Interface Problem

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Drawbacks:
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Possible remedies:
- Remeshing
  - Expensive
Problem: Evolving interface is not aligned with mesh!

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- Jagged interface
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Possible remedies:
- Remeshing
  - Expensive
- Move mesh points along descent vector field
  - No Topology Changes possible
FEM for Interface Problem

Problem: Evolving interface is not aligned with mesh!

Naive approach: Assign element to $\Omega_1$ or $\Omega_2$ based on position of centroid

Drawbacks:
- Jagged interface
- $\|\nabla (u - u_h)\|_{L^2(\Omega)} = O(h^{1/2})$
  Loss of accuracy

Possible remedies:
- Remeshing
  - Expensive
- Move mesh points along descent vector field
  - No Topology Changes possible
- Existing methods: XFEM, unfitted Nitsche method, ...
  - Variable connectivity of system matrix
  - Variable number of unknowns
A Locally Modified Finite Element Method for Interface Problems

**Idea:** Local Modification of FE Mesh such that interface is resolved accurately
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Based on approach for quadrilateral mesh:

A Locally Modified Finite Element Method for Interface Problems

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Requirement: Hierarchical structure of mesh
A Locally Modified Finite Element Method for Interface Problems

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A Locally Modified Finite Element Method for Interface Problems

**Idea:** Local Modification of FE Mesh such that interface is resolved accurately

Based on approach for quadrilateral mesh:


**Requirement:** Hierarchical structure of mesh

\[
P_4 = P_1 + s(P_2 - P_1) \\
P_6 = P_1 + r(P_3 - P_1)
\]
A Locally Modified Finite Element Method for Interface Problems

Idea: Local Modification of FE Mesh such that interface is resolved accurately

Based on approach for quadrilateral mesh:


Requirement: Hierarchical structure of mesh

\[ P_4 = P_1 + s(P_2 - P_1) \]
\[ P_6 = P_1 + r(P_3 - P_1) \]
A Locally Modified Finite Element Method for Interface Problems

**Idea:** Local Modification of FE Mesh such that interface is resolved accurately

Based on approach for quadrilateral mesh:


Requirement: Hierarchical structure of mesh

Choose $P_4, P_5, P_6$ such that
- interface is resolved accurately

\[
\begin{align*}
P_4 &= P_1 + s(P_2 - P_1) \\
P_6 &= P_1 + r(P_3 - P_1)
\end{align*}
\]
A Locally Modified FEM

The Method

A Locally Modified Finite Element Method for Interface Problems

Idea: Local Modification of FE Mesh such that interface is resolved accurately

Based on approach for quadrilateral mesh:


Requirement: Hierarchical structure of mesh

Choose $P_4, P_5, P_6$ such that

- interface is resolved accurately
- all interior angles are bounded away from $180^\circ$

\[
P_4 = P_1 + s(P_2 - P_1) \\
P_6 = P_1 + r(P_3 - P_1)
\]
A Locally Modified Finite Element Method for Interface Problems

4 different configurations:

- **Configuration A**: $r, s < 1/2$

\[ P_4 = (1 - s)P_1 + sP_2 \]
\[ P_5 = \frac{1}{2} (P_2 + P_3) \]
\[ P_6 = (1 - r)P_1 + rP_3 \]
A Locally Modified Finite Element Method for Interface Problems

4 different configurations:

- Configuration A: \( r, s < 1/2 \)
- Configuration B: \( r, s > 1/2 \)

\[
P_4 = (1 - s)P_1 + sP_2 \\
P_5 = (1 - s)P_3 + sP_2 \\
P_6 = (1 - r)P_1 + rP_3
\]
A Locally Modified Finite Element Method for Interface Problems

4 different configurations:

- Configuration A: $r, s < 1/2$
- Configuration B: $r, s > 1/2$
- Configuration C: $r > 1/2, s < 1/2$ (or vice versa)

\[ P_4 = (1 - s)P_1 + sP_2 \]
\[ P_5 = \frac{1}{2}(P_2 + P_3) \]
\[ P_6 = (1 - r)P_1 + rP_3 \]
A Locally Modified Finite Element Method for Interface Problems

4 different configurations:

- Configuration A: \( r, s < 1/2 \)
- Configuration B: \( r, s > 1/2 \)
- Configuration C: \( r > 1/2, s < 1/2 \) (or vice versa)
- Configuration D: \( r \in (0, 1/2), s = 1 \)

\[
P_4 = (1 - r)P_1 + rP_2
\]
\[
P_5 = \frac{1}{2}(P_2 + P_3)
\]
\[
P_6 = (1 - r)P_1 + rP_3
\]
A Locally Modified Finite Element Method for Interface Problems

**Lemma (G., 2016)**

*All interior angles in configurations A, B, C, D are bounded by $180^\circ - \delta/2$ independent of $r$, $s$, where $\delta$ is the minimal angle in the makro mesh.*
A Locally Modified FEM

The Method

Configuration A: \( r < 1/2, \ s < 1/2 \)

Configuration B: \( r > 1/2, \ s > 1/2 \)

Configuration C: \( r < 1/2, \ s > 1/2 \)

Configuration D: \( 0 < r < 1, \ s = 1 \)
A Locally Modified Finite Element Method for Interface Problems

Configuration A  Configuration B  Configuration C  Configuration D

Lemma (G., 2016)

All interior angles in configurations A, B, C, D are bounded by $180^\circ - \delta/2$ independent of $r, s$, where $\delta$ is the minimal angle in the makro mesh.
A Locally Modified Finite Element Method for Interface Problems

Configuration A  
Configuration B  
Configuration C  
Configuration D

Lemma (G., 2016)

All interior angles in configurations A, B, C, D are bounded by $180^\circ - \delta/2$ independent of $r, s$, where $\delta$ is the minimal angle in the makro mesh.

Theorem (Frei & Richter 2014)

Let $\Omega \subset \mathbb{R}^2$ be a domain with convex polygonal boundary, split into $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$, where $\Gamma$ is a smooth interface with $C^2$-parametrization. We assume that $\Gamma$ divides $\Omega$ in such a way that the solution $u \in H^1_0(\Omega)$ satisfies the stability estimate

$$u \in H^1_0(\Omega) \cap H^2(\Omega_1 \cup \Omega_2), \quad \|u\|_{H^2(\Omega_1 \cup \Omega_2)} \leq c_s \|f\|.$$ 

For the corresponding modified finite element solution $u_h \in V_h$ it holds that

$$\|\nabla (u - u_h)\|_{L^2(\Omega)} \leq C h \|f\|, \quad \|u - u_h\|_{L^2(\Omega)} \leq C h^2 \|f\|.$$
Outline

1 Motivation and Problem Description

2 Topology Optimization

3 Shape Optimization

4 A Locally Modified Finite Element Method
   - The Method
   - Numerical Example

5 Application to Electric Motor

6 Conclusion & Outlook
Numerical Example

Let $\Omega = \Omega_1 \cup \Gamma_I \cup \Omega_2 \subset \mathbb{R}^2$ with the interface $\Gamma_I = \overline{\Omega}_1 \cap \overline{\Omega}_2$.

$$-\nabla \cdot (\kappa_1 \nabla u) = f \text{ in } \Omega_1$$

$$-\nabla \cdot (\kappa_2 \nabla u) = f \text{ in } \Omega_2$$

$$\left[ u \right] = 0 \text{ on } \Gamma_I$$

$$\left[ \kappa \partial_n u \right] = 0 \text{ on } \Gamma_I$$

$$u = g_D \text{ on } \partial \Omega$$

Let $\Omega_1 = B_R(x_m)$ and $\Omega_2 = \Omega \setminus \overline{\Omega}_1$ and choose $f$ and $g_D$ such that

$$u(x) = \begin{cases} -4\kappa_1\kappa_2^2 R^2 \| x - x_m \|^2 + 2R^4 \kappa_2 (2\kappa_2 \kappa_1 - 1) & x \in \Omega_1 \\ -2\kappa_2 \| x - x_m \|^4 & x \in \Omega_2. \end{cases}$$
Numerical Example

$k$ without Interface Technique

$k$ with Interface Technique

Convergence without interface treatment

<table>
<thead>
<tr>
<th>nVerts</th>
<th>$h$</th>
<th>$|u - u_h|_{L^2}$</th>
<th>rate $L_2$</th>
<th>$|\nabla(u - u_h)|_{L^2}$</th>
<th>rate $H_1$</th>
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## Numerical Example

### $\kappa$ without Interface Technique

### $\kappa$ with Interface Technique

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Condition of the Problem

angle of interface: 30 deg
Condition of the Problem

angle of interface: 15 deg
Condition of the Problem

angle of interface: 7.5 deg
Condition of the Problem

angle of interface: 3.75 deg
Condition of the Problem

angle of interface: 1.875 deg
Condition of the Problem

angle of interface: 0.9375 deg
Condition of the Problem

angle of interface: 0.46875 deg
Condition of the Problem

angle of interface: 0.23438 deg
Condition of the Problem

angle of interface: 0.11719 deg
Condition of the Problem

angle of interface: 0.058594 deg
Condition of the Problem

angle of interface: 0.029297 deg
# Condition of the Problem

## No Scaling

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## Jacobi Preconditioned

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Outline

1 Motivation and Problem Description
2 Topology Optimization
3 Shape Optimization
4 A Locally Modified Finite Element Method
5 Application to Electric Motor
6 Conclusion & Outlook
Numerical Example

Synchronous Reluctance Motor

- No magnets
- Find optimal shape/topology of rotor
- Maximize torque

Source: www.quintecgmbh.com  
Source: www.elektrotechnik.vogel.de
Numerical Example

Synchronous Reluctance Motor

\[
\max \mathcal{J}(u) = \int_{\Gamma_0} \nabla u^T Q(x) \nabla u \, ds
\]

\[\text{s.t. } \left\{ \begin{array}{l}
-\text{div} (\nu_{\Omega_f} (|\nabla u|) \nabla u) = F \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial \Omega
\end{array} \right.\]

Source: www.quintecgmbh.com

Source: www.elektrotechnik.vogel.de
**Numerical Example**

**Synchronous Reluctance Motor**

\[
\max \mathcal{J}(u) = \int_{\Gamma_0} \nabla u^T Q(x) \nabla u \, ds
\]

s.t.
\[
\begin{aligned}
- \text{div} \left( \nu_{\Omega_f} (|\nabla u|) \nabla u \right) &= F \quad \text{in } \Omega \\
\nabla u &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]
Optimization Pipeline

1. Topology Optimization
   - Level set algorithm based on topological derivative
   - Resolve interface

2. Shape Optimization
   - Level set algorithm to allow for topology changes
   - Resolve interface
Optimization Pipeline

1. Topology Optimization
   - Level set algorithm based on topological derivative
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2. Shape Optimization
   - Level set algorithm to allow for topology changes
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Stage I: Topology Optimization

\[ \text{iter} = 0, \mathcal{J}(u) = 0.053 \]
Stage I: Topology Optimization

iter = 1, $J(u) = 1.198$
Stage I: Topology Optimization

iter = 2, $\mathcal{J}(u) = 6.024$
Stage I: Topology Optimization

iter = 3, $\mathcal{J}(u) = 11.143$
Stage I: Topology Optimization

iter = 4, $\mathcal{J}(u) = 11.877$
Stage I: Topology Optimization

\[ \text{iter} = 5, \quad J(u) = 12.036 \]
Stage I: Topology Optimization

iter = 6, $\mathcal{J}(u) = 12.858$
Stage I: Topology Optimization

iter = 7, $\mathcal{J}(u) = 13.162$
Stage I: Topology Optimization

\[ \text{iter} = 8, \ J(u) = 13.449 \]
Stage I: Topology Optimization

iter = 9, $J(u) = 13.717$
Stage I: Topology Optimization

iter = 10, $\mathcal{J}(u) = 13.833$
Stage I: Topology Optimization

iter = 11, $\mathcal{J}(u) = 13.973$
Stage I: Topology Optimization

iter = 12, $\mathcal{J}(u) = 14.157$
Stage I: Topology Optimization

iter = 13, $\mathcal{J}(u) = 14.205$
Stage I: Topology Optimization

iter = 14, $J(u) = 14.579$
Stage I: Topology Optimization

iter = 15, $\mathcal{J}(u) = 14.833$
Stage I: Topology Optimization

\[ \text{iter} = 25, \ J(u) = 15.325 \]
Stage I: Topology Optimization

iter = 35, $J(u) = 15.659$
Stage I: Topology Optimization

\[ \text{iter} = 45, \quad \mathcal{J}(u) = 15.744 \]
Stage I: Topology Optimization

iter = 55, $\mathcal{J}(u) = 15.798$
Optimization Pipeline

1. **Topology Optimization**
   - Level set algorithm based on topological derivative
   - Resolve interface

2. **Shape Optimization**
   - Level set algorithm to allow for topology changes
   - Resolve interface
Optimization Pipeline

1. Topology Optimization
   - Level set algorithm based on topological derivative
   - Resolve interface

2. Shape Optimization
   - Level set algorithm to allow for topology changes
   - Resolve interface
Stage II: Shape Optimization

\[ \text{iter} = 0, \ J(u) = 15.800 \]
Stage II: Shape Optimization

\[ \text{iter} = 1, \ J(u) = 15.931 \]
Stage II: Shape Optimization

iter = 2, $\mathcal{J}(u) = 16.107$
Stage II: Shape Optimization

\[ \text{iter} = 3, \ J(u) = 16.127 \]
Outline

1. Motivation and Problem Description
2. Topology Optimization
3. Shape Optimization
4. A Locally Modified Finite Element Method
5. Application to Electric Motor
6. Conclusion & Outlook
Conclusion

- Derivation of Topological Derivative
- Derivation of Shape Derivative
- Modified FE Method for accurately resolving interfaces
- Combination of Topology and Shape Optimization

Outlook:

- Application to rotating machines
- 3-dimensional model: $H^1(\Omega) \rightsquigarrow H(\text{curl}; \Omega)$
- Couple with Isogeometric Analysis in 2D and 3D
- Eddy current problems for starting phase of motor: Space-Time Methods?
Conclusion

- Derivation of Topological Derivative
- Derivation of Shape Derivative
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Outlook:

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- Couple with Isogeometric Analysis in 2D and 3D
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Thank you for your attention!