ROLE OF SENSORS AND ACTUATORS IN DISTRIBUTED PARAMETER SYSTEMS

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Thanks to...

AFOSR, NSERC, SharcNet

IMA
Control system components

- plant (system being controlled)
- controller
- actuator
- sensor
Any system modelled by a set of linear ordinary differential equations can be written in the following form:

\[ \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \]
\[ y(t) = Cx(t) + Eu(t). \quad (1) \]

where \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \), \( C \in \mathbb{R}^{p \times n} \) and \( E \in \mathbb{R}^{p \times m} \). Roughly, \( A \) describes the internal dynamics, \( B \) the effect of the controlled input on the state; and \( C \), \( E \) describe the sensors.
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Balance of an Inverted Stick

Linearize $\theta$ about the vertical, define

$$z(t) = \begin{bmatrix} d(t) \\ d(t) + 0.5L\dot{\theta}(t) \\ \dot{d}(t) \\ \dot{d}(t) + 0.5L\ddot{\theta}(t) \end{bmatrix}$$

$$\dot{z}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{m_p g}{0.5m_c L} & -\frac{m_p g}{0.5m_c L} & 0 & 0 \\ -\frac{0.5}{0.5L} & \frac{0.5}{0.5L} & 0 & 0 \end{bmatrix} z(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} F(t)$$
Effect of measurement point on balancing

Measure position at distance $\alpha L$ along the stick,

$$y(t) = d(t) + \alpha L \theta(t)$$

$$\hat{y}(s) = G(s)\hat{u}(s),$$

$$G(s) = \frac{(0.5 - \alpha)Ls^2 - g}{s^2(0.5m_cLs^2 - (m_c + m_p)g)}.$$ 

Eigenvalues/Poles: $0(2\times), \quad \lambda_{1,2} = \pm \sqrt{\frac{(1+\frac{m_p}{m_c})g}{0.5L}}. \quad \lambda_1 > 0$
Effect of measurement point on balancing

Measure position at distance $\alpha L$ along the stick,

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Zeros: $z_{1,2} = \pm \sqrt{\frac{g}{(0.5 - \alpha)L}}$. If $\alpha < 0.5$, $z_1 > 0$.

This restricts performance, and the ability to design a closed loop system that will cope with uncertainty.
Control of Acoustic Noise (v1)

\[
\begin{align*}
\frac{\partial p(x, t)}{\partial t} &= -\frac{\partial v(x, t)}{\partial x}, \\
\frac{\partial v(x, t)}{\partial t} &= -\frac{\partial p(x, t)}{\partial x},
\end{align*}
\]

\[v(0, t) = u(t), \quad p(L, t) = 0\]
Control of Acoustic Noise (v2)  

$$\frac{\partial p(x, t)}{\partial t} = -\frac{\partial v(x, t)}{\partial x},$$

$$\frac{\partial v(x, t)}{\partial t} = -\frac{\partial p(x, t)}{\partial x},$$

$$A\dot{c}(t) = \pi a^2 v(0, t),$$

$$m\ddot{c}(t) + d\dot{c}(t) + kc(t) = Bu(t) - Ap(0, t).$$

Better model includes loudspeaker dynamics
Experiment vs Model (no actuator dynamics)

(mid-point pressure) / (input velocity) frequency response

Measured (- - - -) and calculated (——)
Experiment vs Model (with actuator dynamics)

(mid-point pressure) / (input velocity) frequency response

Measured (---) and calculated (—)
Sensors for Second-order systems

\[ \ddot{w}(t) + A_0 w(t) + D\dot{w}(t) = B_0 u(t) \]

Measure:
- position \( w(x_0, t) \)
- velocity \( \dot{w}(x_0, t) \)
- acceleration \( \ddot{w}(x_0, t) \)
Accelerometers

- unless very strong assumptions are made on stiffness $A_o$ and damping $D$, acceleration measurements are not well-posed
- sensors used to measure acceleration are micro-electro-mechanical systems (MEMS)
- commonly, a mass is suspended between two capacitors and the measured voltage is proportional to the mass position
- Letting $F(t)$ be force applied by structure to accelerometer,

\[ m\ddot{a}(t) + k\dot{a}(t) + d\dot{a}(t) = F(t). \]
Model for second-order system with accelerometer

Using Hamilton’s principle to obtain a description for the dynamics of a structure coupled to an accelerometer,

\[
\begin{align*}
    m\ddot{a}(t) + k(a(t) - C_0 w(t)) + d(\dot{a}(t) - C_0 \dot{w}(t)) &= 0, \\
    \rho\ddot{w}(t) + A_0 w(t) + D\dot{w}(t) + kC_0^*(C_0 w(t) - a(t)) + dC_0^*(C_0 \dot{w}(t) - \dot{a}(t)) &= B_0 u(t)
\end{align*}
\]
Model for second-order system with accelerometer

Using Hamilton’s principle to obtain a description for the dynamics of a structure coupled to an accelerometer,

\[
\begin{align*}
    m\dddot{a}(t) + k(a(t) - C_ow(t)) + d(\dot{a}(t) - C_ow(t)) &= 0, \\
    \rho\dddot{w}(t) + A_ow(t) + Dw(t) + kC_ow^*(C_ow(t) - a(t)) \\
    + dC_ow^*(C_ow\dot{w}(t) - \dot{a}(t)) &= B_ou(t)
\end{align*}
\]

\[y(t) = \alpha(C_ow(t) - a(t))\],

where \(C_o\) indicates position measurement at a point and \(\alpha\) is a parameter.
Model for second-order system with accelerometer

Using Hamilton’s principle to obtain a description for the dynamics of a structure coupled to an accelerometer,

\[
\begin{align*}
    m\ddot{a}(t) + k(a(t) - C_0w(t)) + d(\dot{a}(t) - C_0\dot{w}(t)) &= 0, \\
    \rho\ddot{w}(t) + A_0w(t) + D\dot{w}(t) + kC_0^*(C_0w(t) - a(t)) \\
    + dC_0^*(C_0\dot{w}(t) - \dot{a}(t)) &= B_0u(t)
\end{align*}
\]

\[y(t) = \alpha(C_0w(t) - a(t)),\]

where \(C_0\) indicates position measurement at a point and \(\alpha\) is a parameter.

Observation bounded on natural (energy) state-space.
Distributed parameter systems

Acoustic Noise in Duct

Plate (Demetriou, WPI)

Lake Huron
Actuator/sensor location

- optimal filtering/estimation in 1970’s: Bensoussan, Curtain, Ichikawa
- Clark & Fuller, 1992

Results from this study indicate that optimization of control actuators and error sensors provides a method for realizing adaptive structures for active structural acoustic control, rivalling in importance the performance increases gained when acoustic control is achieved with microphone error sensors and multiple control actuators.

- Fahroo & Ito 1996, optimal damping design
- Fahroo & Demetriou 2000
  - noise reduction in a cavity is 43dB vs 25dB
  - better performance with static output feedback, optimally placed, than with full state feedback at another location
Experiment: Cantilevered Beam

- 70 × 7 × .85cm beam
- 2 actuators, each 7 × 7cm
- optimal locations: 1 and 2
- 4 patches attached; only 2 activated at a time
- 2 laser sensors
Tip Displacement of Controlled Beam
Optimal Actuator Location

\[ \dot{z}(t) = Az(t) + B(r)u(t), \quad z(0) = z_0 \]

- \( A \) generates a strongly continuous semigroup \( S(t) \) on \( \mathcal{Z} \)
- \( B(r) \in \mathcal{L}(\mathcal{U}, \mathcal{Z}) \)
- \( M \) actuators with locations in some closed and bounded set \( \Omega \subset \mathbb{R}^N \)
- location \( r \) is a vector of length \( M \), \( r \in \Omega^M \)

Optimal Actuator Location Problem

\[ \min_{r \in \Omega^M} \hat{\mu} [r] \]

for some measure of performance \( \hat{\mu} \).
Typical Performance Objectives

- controllability
- linear-quadratic cost (response to initial condition)
- $H_2$-cost (known disturbance)
- $H_\infty$-cost (unknown disturbance)
Typical Performance Objectives

- controllability \( \leftrightarrow \) MOST COMMON
- linear-quadratic cost (response to initial condition)
- \( H_2 \)-cost (known disturbance)
- \( H_\infty \)-cost (unknown disturbance)
Controllability

\((A, B(r))\) controllable \iff solution \(L_c(r)\) to

\[ AL_c(r) + L_c(r)A^* + B(r)B(r)^* = 0 \]

is positive definite.

- Minimum energy to steer \(z(0) = 0 \rightarrow z_f\) is \(z_f^*L_c(r)^{-1}z_f\).
- The most energy required over all targets \(z_f\) is

\[
\sup_{\|z_f\| \leq 1} \|z_f^*L_c(r)^{-1}z_f\| = \lambda_{\text{max}}L_c(r)^{-1}
\]

Maximize \(\lambda_{\text{min}}(L_c(r))\)
Controllability

\((A, B(r))\) controllable \(\iff\) solution \(L_c(r)\) to

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\sup_{\|z_f\| \leq 1} \|z_f^*L_c(r)^{-1}z_f\| = \lambda_{max}L_c(r)^{-1}
\]

\[
\text{Maximize } \lambda_{min}(L_c(r))
\]

Most common criterion in engineering literature
Illustrative Example

\[ \frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} + c_v \frac{\partial w}{\partial t} + c_d \frac{\partial^5 w}{\partial t x^4} = b_r(x) u(t), \quad 0 < x < 1, \]

\[ b_r(x) = \begin{cases} 
1/\delta, & |x - r| < \frac{\delta}{2} \\
0, & |x - r| \geq \frac{\delta}{2} 
\end{cases} \]
Controllability of Beam
Controllability of Beam

5 modes (−) and 10 modes (−−)
Controllability vs Approximation Order

The figure shows a scatter plot with the x-axis representing the number of modes and the y-axis representing the smallest eigenvalue. The points on the graph suggest a trend where as the number of modes increases, the smallest eigenvalue decreases.
Approximate and Exact Controllability

\((A, B(r))\) controllable \(\iff\) solution \(L_c(r)\) to

\[\begin{align*}
AL_c(r) + L_c(r)A^* + B(r)B(r)^* &= 0.
\end{align*}\]

is positive definite.

- Exact controllability: Can steer to any point in \(\mathcal{Z}\). Equivalent to \(L_c^{-1}\) bounded.
- Approximate controllability: Can steer arbitrarily close to any point. Equivalent to \(L_c^{-1}\) exists; i.e.

\[
L_c = \begin{bmatrix}
1 & 0 & \ldots & \\
0 & \frac{1}{2} & \ldots & \\
0 & 0 & \frac{1}{3} & \\
\vdots & 0 & 0 & \frac{1}{4} & \ldots
\end{bmatrix}.
\]

- For an approximately controllable system, \(\lambda_n(L_c) > 0\) but

\[
\lim_{n \to \infty} \lambda_n(L_c) \to 0.
\]
Comparision of optimally controllable and LQ-optimal actuator locations

Tip position

Best LQ controller at

- - - optimally controllable location

- - - LQ-optimal location

Control force
Design Objectives

- controllability \(\leftarrow\) MOST COMMON
Design Objectives

- controllability \[\leftrightarrow\] NUMERICALLY UNRELIABLE
Design Objectives

- controllability
- linear-quadratic cost (response to initial condition)
- $H_2$-cost (known disturbance)
- $H_\infty$-cost (unknown disturbance)
Design Objectives

- controllability
- linear-quadratic cost (response to initial condition)
- $H_2$-cost (known disturbance)
- $H_\infty$-cost (unknown disturbance)

mechatronic approach
locate actuators as part of controller design
Linear Quadratic (LQ) Control

\[ \inf_{u \in L_2(0, \infty; \mathcal{U})} \int_0^\infty \langle Cz(t), Cz(t) \rangle + \langle u(t), u(t) \rangle \, dt \]

\[ J(u, z_0) \]

**Theorem**

If \((A, B)\) is stabilizable then there exists a unique \(\Pi \geq 0\) such that for all \(z \in D(A)\),

\[ (\Pi A + A^* \Pi + C^* C - \Pi BB^* \Pi)z = 0, \]

- Optimal cost \(\inf_{u \in L_2(0, \infty; \mathcal{U})} J(u, z_0) = \langle z_0, \Pi z_0 \rangle\)
- Optimal control \(u(t) = -Kz(t)\) where \(K = B^* \Pi\)
Calculation of Linear Quadratic Regulator

Operator ARE

\[ A^* \Pi + \Pi A - \Pi BB^* \Pi + C^* C = 0 \]

- Need to approximate solution
- Approximate \( A, B, C \) by \( A_n, B_n, C_n \)
- Solve finite-dimensional ARE for \( \Pi_n \)
- Approximation \( K_n = R^{-1} B_n^* \Pi_n \) used to control original system
Convergence of $\Pi_n$

Assume that for each $z \in \mathcal{H}$, $u \in \mathcal{U}$, $y \in \mathcal{Y}$,

\[(A1i) \quad \| S_n(t)Pnz - S(t)z \| \to 0 \text{ (uniformly on } [0, T]) \]

\[(A2i) \quad \| B_nu - Bu \| \to 0, \quad \| CnPnz - Cz \| \to 0, \]
Convergence of $\Pi_n$

**Standard Assumptions for Controller Design**

Assume that for each $z \in \mathcal{H}$, $u \in \mathcal{U}$, $y \in \mathcal{Y}$,

(A1i) $\|S_n(t)P_nz - S(t)z\| \to 0$ (uniformly on $[0, T]$)

(A1ii) $\|S^*_n(t)P_nz - S^*(t)z\| \to 0$ (same)

(A2i) $\|B_nu - Bu\| \to 0$, $\|C_nP_nz - Cz\| \to 0$,

(A2ii) $\|B^*_nz - B^*z\| \to 0$, $\|C^*_ny - C^*y\| \to 0$
Convergence of $\Pi_n$

Standard Assumptions for Controller Design

Assume that for each $z \in \mathcal{H}$, $u \in \mathcal{U}$, $y \in \mathcal{Y}$,

(A1i) $\|S_n(t)P_n z - S(t)z\| \to 0$ (uniformly on $[0, T]$)

(A1ii) $\|S_n^*(t)P_n z - S^*(t)z\| \to 0$ (same)

(A2i) $\|B_n u - Bu\| \to 0$, $\|C_n P_n z - C z\| \to 0$,

(A2ii) $\|B_n^* z - B^* z\| \to 0$, $\|C_n^* y - C^* y\| \to 0$

(A3i) $(A_n, B_n)$ is uniformly exponentially stabilizable:

$\exists K_n \in \mathcal{L}(\mathcal{H}_n, \mathcal{U})$, $\|K_n\| \leq M$,

$\|e^{(A_n - B_n K_n) t} P_n z\| \leq M_1 e^{-\omega_1 t} |z|$

(A3ii) $(A_n, C_n)$ is uniformly exponentially detectable:

$\exists F_n \in \mathcal{L}(\mathcal{Y}, \mathcal{H}_n)$, $\|F_n\| \leq M$,

$\|e^{(A_n - F_n C_n) t} P_n\| \leq M_2 e^{-\omega_2 t}$
**Theorem 1**

*If the standard assumptions for controller design hold, then for all* $z \in \mathcal{H}$,*

- $\|\Pi_n P_n z - \Pi z\| \to 0$
- *there exists constants* $M_2 \geq 1$, $\alpha_2 > 0$, *independent of* $n$, *such that*

  $$\|e^{(A_n - B_n K_n)t}\| \leq M_2 e^{-\alpha_2 t}.$$  

- *For sufficiently large* $n$, *semigroups generated by* $A - BK_n$ *are uniformly exponentially stable*

- *Cost with feedback* $K_n$ *converges to optimal:*

  $$J(-K_n z(t), z_0) \to \langle \Pi z_0, z_0 \rangle.$$
Theorem 1

If the standard assumptions for controller design hold, then for all $z \in \mathcal{H}$,

- $\|\Pi_n P_n z - \Pi z\| \to 0$
- there exists constants $M_2 \geq 1$, $\alpha_2 > 0$, independent of $n$, such that
  \[ \|e^{(A_n - B_n K_n) t}\| \leq M_2 e^{-\alpha_2 t}. \]
- For sufficiently large $n$, semigroups generated by $A - BK_n$ are uniformly exponentially stable
- Cost with feedback $K_n$ converges to optimal:
  \[ J(-K_n z(t), z_0) \to \langle \Pi z_0, z_0 \rangle. \]

Performance arbitrarily close to optimal can be achieved with $K_n$. 
Optimal Actuator Location Problem

\[ \dot{z}(t) = Az(t) + B(r)u(t), \quad z(0) = z_0 \]

- \( A \) generates a strongly continuous semigroup \( S(t) \) on \( \mathcal{Z} \), \( B(r) \in \mathcal{L}(\mathcal{U}, \mathcal{Z}) \)
- Consider \( m \) actuators with locations in some closed and bounded set \( \Omega \subset \mathbb{R}^N \).
- Location \( r \) is a vector of length \( m \), \( r_i \in \Omega \subset \mathbb{R}^N \)
- Choose \( r \) to minimize performance criterion
- Joint controller design/actuator location
LQ-optimal actuator location

\[
\inf_{u \in L_2(0, \infty; \mathcal{U})} \int_0^\infty \left\langle Cz(t), Cz(t) \right\rangle + \left\langle u(t), u(t) \right\rangle dt
\]

for each \( r \), optimal cost is \( \left\langle \Pi(r)z_0, z_0 \right\rangle \) where \( \Pi(r) \) solves ARE.

minimize response to the worst \( z(0) \)

\[
\max_{z_0 \in \mathcal{H}} \left\langle \Pi(r)z_0, z_0 \right\rangle = \| \Pi(r) \|
\]

\[
\hat{\mu} = \inf_{r \in \Omega^m} \| \Pi(r) \|
\]
LQ-optimal actuator location

\[
\inf_{u \in L_2(0, \infty; \mathcal{U})} \int_0^\infty \langle Cz(t), Cz(t) \rangle + \langle u(t), u(t) \rangle \, dt
\]

for each \( r \), optimal cost is \( \langle \Pi(r)z_0, z_0 \rangle \) where \( \Pi(r) \) solves ARE.

- minimize response to the worst \( z(0) \)

\[
\max_{z_0 \in \mathcal{H}, \|z_0\|=1} \langle \Pi(r)z_0, z_0 \rangle = \|\Pi(r)\|
\]

\[
\hat{\mu} = \inf_{r \in \Omega_m} \|\Pi(r)\|
\]

could also regard \( z(0) \) as random variable; if variance is \( V \), cost is

\[
\text{trace} V^{\frac{1}{2}} \Pi(r) V^{\frac{1}{2}}.
\]
Well-posedness of LQ-optimal actuator location

**Theorem 2**

If

- for any \( r_0 \), \( \lim_{r \to r_0} \| B(r) - B(r_0) \| = 0 \),
- \((A, B(r))\) are all stabilizable, \((A, C)\) is detectable
- \( B \) is a compact operator,

then

\[ \lim_{r \to r_0} \| \Pi(r) - \Pi(r_0) \| = 0. \]

Also, there exists \( \hat{r} \) such that

\[ \| \Pi(\hat{r}) \| = \inf_{r \in \Omega^m} \| \Pi(r) \| = \hat{\mu}. \]
Well-posedness of LQ-optimal actuator location

**Theorem 2**

_If_

- for any \( r_0 \), \( \lim_{r \to r_0} \| B(r) - B(r_0) \| = 0 \),
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_then_

\[
\lim_{r \to r_0} \| \Pi(r) - \Pi(r_0) \| = 0.
\]

Also, there exists \( \hat{r} \) such that

\[
\| \Pi(\hat{r}) \| = \inf_{r \in \Omega^m} \| \Pi(r) \| = \hat{\mu}.
\]
Optimal actuator location for pinned beam with viscous damping

\[
\frac{\partial^2 w}{\partial t^2} + c_d \frac{\partial w}{\partial t} + \frac{\partial^4 w}{\partial x^4} = b(x)u(t), \quad t \geq 0, 0 < x < 1,
\]

\[
w(0, t) = 0, \quad w_{xx}(0, t) = 0, \quad w(1, t) = 0, \quad w_{xx}(1, t) = 0.
\]

\[
b(r) = \begin{cases} 
1/\delta, & |x - r| < \frac{\delta}{2} \\
0, & |x - r| \geq \frac{\delta}{2}
\end{cases}
\]

- reduce state uniformly: \( C = I \)
- eigenfunction approximations
- satisfy Standard Assumptions for control
Optimal performance ($\|\Pi_n\|$), $C = I$
Optimal actuator location ($\|\Pi_n\|$), $C = I$
Solution to ARE

\[ \dot{z}(t) = Az(t) + Bu(t) \]

- \( A^* = -A \) (as in the undamped wave and beam equations).
- \( C = B^* \)

implies \( \Pi = I \) is a solution to the ARE

\[ A^*\Pi + \Pi A - \Pi BR^{-1}B^*\Pi + C^*C = 0 \]

Solution \( \Pi \) is not compact.

\( \Pi \) must be compact for uniform approximation by finite rank \( \Pi_n \).
Solution to ARE

\[ \dot{z}(t) = Az(t) + Bu(t) \]

- \( A^* = -A \) (as in the undamped wave and beam equations).
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Solution \( \Pi \) is not compact.

\( \Pi \) must be compact for uniform approximation by finite rank \( \Pi_n \).
**Theorem 3**

*If $B$, $C$ are compact operators, and $(A, B, C)$ stabilizable and detectable, then $\Pi$ is a compact operator.*

Proof: Use representation

$$\Pi = \int_0^\infty S_K(t)^* [C^* C + K^* R K] S_K(t) dt$$

where $K = R^{-1} B^* \Pi$. 
Conditions for compact solution to ARE

**Theorem 3**

*If $B$, $C$ are compact operators, and $(A, B, C)$ stabilizable and detectable, then $\Pi$ is a compact operator.*

Proof: Use representation

$$\Pi = \int_0^\infty S_K(t)^* [C^* C + K^* R K] S_K(t) dt$$

where $K = R^{-1} B^* \Pi$. 
Convergence of $LQ$–Optimal Actuator Location

**Theorem 4**

In addition to (*) assume that

- $(A_n, B_n(r), C_n)$ satisfies standard assumptions on approximations for controller design
- $C$ is a compact operator.

Then

$$\hat{\mu} = \lim_{n \to \infty} \hat{\mu}_n,$$

and there exists a subsequence $\{\hat{r}_m\}$ of $\{\hat{r}_n\}$ such that

$$\hat{\mu} = \lim_{m \to \infty} \|\Pi(\hat{r}_m)\|.$$
Viscously damped beam, state weight is $C = [I \ 0]$
Solution to ARE for analytic semigroups

Theorem

Consider a smooth bounded domain $\Omega \subset \mathbb{R}^N$. Let $A$ be an elliptic operator on $L^2(\Omega)$ of order $2m$, subject to appropriate boundary conditions. Then $A$ generates an analytic semigroup on $L^2(\Omega)$ and the solution $\Pi$ to the ARE is a compact operator.

- Uniform convergence of $\Pi_n$ to $\Pi$ occurs for typical approximation schemes
- Applies to unbounded control operators $B$ and observation $C$
- Convergence of optimal actuator locations, even for non-compact weight on state
Example: Simply supported beam with Kelvin-voigt damping
$H_2 - cost$

$$\dot{z}(t) = Az(t) + B(r)u(t) + Dd(t), \quad z(0) = 0$$

- $A$ generates a strongly continuous semigroup $S(t)$ on $\mathcal{Z}$
- $B(r) \in \mathcal{L}(U, \mathcal{Z}), \ D \in \mathcal{L}(V, \mathcal{Z})$
- full information: input to the controller is
  $$y_2(t) = \begin{bmatrix} z(t) \\ d(t) \end{bmatrix}$$
- known disturbance $d$; usually white noise

Cost

$$\int_0^\infty \|y_1(t)\|^2 dt, \quad y(t) = CZ(t) + Eu(t)$$ (2)

- If $E^*C = 0$ and $E^*E = I$, cost is identical to LQ cost with $Q = C^*C$ and $R = E^*E = I$. 
$H_2$ — cost

\[ \dot{z}(t) = A z(t) + B(r) u(t) + D d(t), \quad z(0) = 0 \]

- $A$ generates a strongly continuous semigroup $S(t)$ on $\mathcal{Z}$
- $B(r) \in \mathcal{L}(U, \mathcal{Z})$, $D \in \mathcal{L}(V, \mathcal{Z})$
- full information: input to the controller is

\[ y_2(t) = \begin{bmatrix} z(t) \\ d(t) \end{bmatrix} \]

- known disturbance $d$; usually white noise

Cost

\[ \int_0^\infty \|y_1(t)\|^2 dt, \quad y(t) = C z(t) + Eu(t) \]  \hspace{1cm} (2)

- If $E^* C = 0$ and $E^* E = I$, cost is identical to LQ cost with

\[ Q = C^* C \text{ and } R = E^* E = I. \]
$H_2$ cost

\[ \dot{z}(t) = Az(t) + B(r)u(t) + Dd(t), \quad z(0) = 0 \]

- $A$ generates a strongly continuous semigroup $S(t)$ on $\mathbb{Z}$
- $B(r) \in \mathcal{L}(U, \mathbb{Z})$, $D \in \mathcal{L}(V, \mathbb{Z})$
- full information: input to the controller is
  \[ y_2(t) = \begin{bmatrix} z(t) \\ d(t) \end{bmatrix} \]
- known disturbance $d$; usually white noise

Cost

\[ \int_0^\infty \|y_1(t)\|^2 dt, \quad y(t) = Cz(t) + Eu(t) \quad (2) \]

- If $E^*C = 0$ and $E^*E = I$, cost is identical to LQ cost with $Q = C^*C$ and $R = E^*E = I$. 
**Theorem**

The $H_2$-optimal control is the state feedback

$$u(t) = -B^*(r)\Pi(r)z(t)$$

where $\Pi(r)$ solves an ARE which yields optimal cost

$$Tr(B_1^*\Pi(r)B)$$

and the optimal norm of the closed loop transfer function is

$$\sqrt{Tr(D^*\Pi(r)D)}.$$
Well-posedness of $H_2$-optimal actuator location

**Theorem**

Assume for any $r \in \Omega^M$,

$$\lim_{s \to r} \|B(s) - B(r)\| = 0.$$  

and the standard approximation assumptions are satisfied for each $(A_n, B_n(r), C_n)$, and that $D$ is compact. Then the approximating optimal costs converge to the exact optimal cost, that is

$$\inf_{r \in \Omega^M} Tr(D^* \Pi(r) D) = \lim_{n \to \infty} \inf_{r \in \Omega^M} Tr(D_n^* \Pi_n(r) D_n),$$

Also there is a subsequence of approximating actuator locations $\hat{r}_n$ so

$$\hat{\mu} = \lim_{n \to \infty} Tr(D_n^* \Pi(\hat{r}_n) D_n);$$
Consider placing the actuator to minimizing the $H_2$-cost over all possible disturbance shapes:

$$\inf_{r \in \Omega^M} \sup_{b_1 \in \mathcal{H}} \langle b_1, \Pi(r)b_1 \rangle = \inf_{r \in \Omega^M} \|\Pi(r)\|.$$ 

Same cost as the optimal LQ cost, and the conditions for an approximation to be used are the same.

Worst disturbance shape is eigenfunction of $\lambda_{max} \Pi(r)$. 
Cantilevered Beam

<table>
<thead>
<tr>
<th>max of $b_1(\xi)$</th>
<th>opt. location</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25L</td>
<td>0.5091L</td>
</tr>
<tr>
<td>0.50L</td>
<td>0.9950L</td>
</tr>
<tr>
<td>0.75L</td>
<td>0.9950L</td>
</tr>
<tr>
<td>1 L</td>
<td>0.9950L</td>
</tr>
<tr>
<td>0.471L</td>
<td>0.4475L</td>
</tr>
<tr>
<td>L</td>
<td>0.9950L</td>
</tr>
</tbody>
</table>

Table: Optimal actuator location for various spatial distributions with $C = C_{tip}$
Diffusion

\[ \Omega \]

Diffusivity \( \kappa(x_1, x_2) \)
Diffusion

\[
\frac{\partial z}{\partial t}(x_1, x_2, t) = \nabla \cdot (\kappa(x_1, x_2) \nabla z(x_1, x_2, t)) + b(x_1, x_2)u(t) + v(t),
\]

\[z(x_1, x_2, \cdot) = 0 \text{ on } \partial \Omega,\]

\[y(t) = \int \int_{\Omega} z(x_1 x_2, t) \, dx,\]

where

\[b(x_1, x_2) = \begin{cases} 
\frac{1}{\delta}, & (x_1, x_2) \in \Box(r_1, r_2), \\
0, & \text{otherwise},
\end{cases}\]

with \(\Box(r_1, r_2, \delta)\) is a square centered at \((r_1, r_2)\) and side \(\delta = 0.2\).
Disturbance location and optimal actuator locations, cost $C = I$
Disturbance location and optimal actuator locations, cost $C = I$
Comparision of Different Criteria: Simply Supported Beam

\[
\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} + c_v \frac{\partial w}{\partial t} + c_d \frac{\partial^5 w}{\partial t \partial x^4} = b_r(x)u(t) + b_{0.3}(x)d(t), \quad 0 < x < 1,
\]

\[
b_r(x) = \begin{cases} 
1/\delta, & |x - r| < \frac{\delta}{2} \\
0, & |x - r| \geq \frac{\delta}{2}
\end{cases}
\]

\(d\) is disturbance; assumed here white noise.
$H_2$-optimal actuator locations
$H_2$-optimal actuator locations

The diagram shows the $H_2$ cost (relative) for different values of $Q$ and $R$, with $Q = I$ and $R = 100$, $R = 1$, $R = 0.01$, and $R = 0.0001$. The actuator location is plotted on the x-axis, and the cost is plotted on the y-axis.
Control Signal - $H_2$ cost

$L_2$-norm

$L_\infty$-norm
Design Objectives

- controllability $\Leftarrow$ MOST COMMON
- linear-quadratic cost (response to initial condition)
- $H_2$-cost (known disturbance)
- $H_\infty$-cost (unknown disturbance)
Design Objectives

- controllability $\iff$ NUMERICALLY UNRELIABLE
- linear-quadratic cost (response to initial condition)
- $H_2$-cost (known disturbance)
- $H_\infty$-cost (unknown disturbance)
Design Objectives

- controllability
- linear-quadratic cost (response to initial condition)
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Performance depends on actuator location
Design Objectives

- controllability
- linear-quadratic cost (response to initial condition)
- $H_2$-cost (known disturbance)
- $H_\infty$-cost (unknown disturbance)

Performance depends on actuator location

Optimal location depends on objective
Optimal Spatial Distribution of Damping

\[ \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 w}{\partial x^2} + a(x) \frac{\partial w}{\partial t} = 0, \quad \omega \subset [0, 1] \]

\[ w(0, t) = 0, w(1, t) = 0. \]

For domain \( \omega \) of total length \( \ell \), what is best choice of \( a(x) \)?

Different ways to measure “best”

- minimize energy of the system
- decay rate
Optimal Spatial Distribution of Damping

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\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 w}{\partial x^2} + a(x) \frac{\partial w}{\partial t} = 0, \quad \omega \subset [0, 1]
\]

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\]

For domain \(\omega\) of total length \(\ell\), what is best choice of \(a(x)\)?

Different ways to measure “best”

- minimize energy of the system (Fahroo & Ito, 1996)
  - requires solving Lyapunov eqn
  - minimize some norm of the solution different criteria \(\Rightarrow\) different designs
  - over set with bounded variation and mass there is an optimal solution
  - non-convergence of approximating solutions

- decay rate
Optimal Spatial Distribution of Damping

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\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 w}{\partial x^2} + a(x) \frac{\partial w}{\partial t} = 0, \quad \omega \subset [0, 1]
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w(0, t) = 0, \quad w(1, t) = 0.
\]

For domain \(\omega\) of total length \(\ell\), what is best choice of \(a(x)\)?

Different ways to measure “best”

- minimize energy of the system
- decay rate
- for small mass of damping, constant damping best (Cox & Zuazua 1991)
- \(a(x) = k \chi_{\omega}(x)\), small \(k\) (Hebrard & Henrot, 2003,2005)
  - without constraint on \# intervals, no optimum exists (unless \(\ell = 0, 1, \frac{1}{2}\))
  - optimum for \(N\) modes is at node of \(N + 1\)st and is bad choice
Optimal Spatial Distribution of Damping

\[
\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 w}{\partial x^2} + a(x) \frac{\partial w}{\partial t} = 0, \quad \omega \subset [0, 1]
\]

\[
w(0, t) = 0, \, w(1, t) = 0.
\]

For domain \(\omega\) of total length \(\ell\), what is best choice of \(a(x)\)?

Different ways to measure “best”

- minimize energy of the system
- decay rate
- approximation theory with either energy or decay rate as objective function is open question
Sensor Location

\[
\dot{z}(t) = Az(t) + Bu(t) + B_1 d(t)
\]
\[
y_{sen}(t) = C_2(s)z(t) + E_{21} d(t)
\]

- \( M \) sensors with locations in some compact set \( \Omega \subset \mathbb{R}^n \).
- Estimation is dual to control.
- Maximizing observability has the same issues as maximizing controllability in actuator location.
- Kalman filter minimizes trace of error.
- Research in 1970's on finite-time optimal estimation (Bensoussan, Curtain, Ichikawa ...).
- Recent work for time-varying systems (Wu&Jacob, Rautenberg).
- Can also place where feedback functional gain is large (Batten, Borggaard, Burns).
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Estimation: linear one-d channel flow

\[ c^2 v_{xx} - c_d v_t + \beta v_{xxtt} + d(x)w_1(t), \]

Top line is optimal sensor location - minimum variance
Controlled Boussinesq equation
Kalman filtering for linearized Boussinesq equation

Minimum variance location

\[ \xi_{\text{cov}}^* = (0.01666, 0.06666) \]

Arbitrary location

\[ \xi_{\text{rd}}^* = (0.08333, 0.48333) \]
Summary

- Locate actuators as part of controller design
- Compactness is critical for well-posedness of problem and for validity of calculations
- Numerics an issue: even if controllers designed with fixed actuator location converge, actuator locations may not
- Problem needs to be properly formulated
- Optimal location depends on cost criterion
- Optimal location not always intuitive
Some Open Problems

- Efficient procedure for calculating $\mathcal{H}_\infty$-optimal locations
- Effect of diffusivity, other parameters
- Weakly damped systems
- Incorporating statistical information about hardware, model
- Interaction of sensors and actuators in output feedback
- Moving sensors
- Optimal shape
- Robustness
- Nonlinearities