

A Weak Convergence Approach to Inventory Control Using a Long-term Average Criterion

Chao Zhu

(Joint with K.L Helmes and R.H. Stockbridge)

University of Wisconsin - Milwaukee

May, 2018

IMA Workshop on Stochastic Control, Computational Methods,
and Applications

Outline

Introduction

Model and Problem Formulation

(s, S) -Policy

Optimality Via Weak Convergence

Examples

Summary

Formulation: The Inventory Process

- ▶ **The inventory process** (in the absence of orders):

$$dX_0(t) = \mu(X_0(t))dt + \sigma(X_0(t))dW(t), \quad X(0) = x_0 \in \mathcal{I} = (a, b),$$

where $-\infty \leq a < b \leq +\infty$.

- ▶ a is an *attracting* point:
 a may be a regular, exit or natural boundary point.
- ▶ b is a *non-attracting* point:
 b may be a natural or entrance point.

Inventory Management

The manager can decide *WHEN* and *HOW MUCH* to restock the inventory:

$$\begin{array}{cccccccc} \text{WHEN:} & 0 & \leq & \tau_1 & \leq & \tau_2 & \leq & \tau_3 & \leq & \dots \\ & & & \downarrow & & \downarrow & & \downarrow & & \\ \text{HOW MUCH:} & & & Y_1 & & Y_2 & & Y_3 & & \dots \end{array}$$

For each $k = 1, 2, \dots$,

- ▶ τ_k , the k th order time, is an $\{\mathcal{F}_t\}$ -stopping time, and
- ▶ Y_k , the k th order size, is an $\{\mathcal{F}_{\tau_k}\}$ -measurable nonnegative random variable.
- ▶ $X(\tau_k-)$: the inventory level just before the k th order,
- ▶ $X(\tau_k)$: the inventory level at the k th order,

$$X(\tau_k) = X(\tau_k-) + Y_k \geq X(\tau_k-)$$

Inventory Management

- ▶ The controlled inventory

$$X(t) = X(0-) + \int_0^t \mu(X(s))ds + \int_0^t \sigma(X(s))dW(s) + \sum_{k=1}^{\infty} I_{\{\tau_k \leq t\}} Y_k,$$

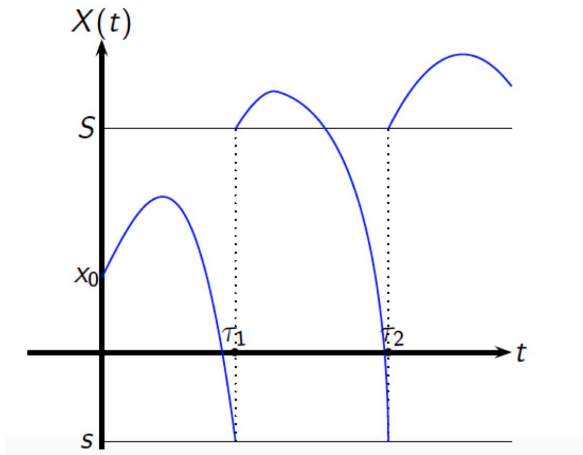
$$X(t) \in \mathcal{E},$$

- ▶ The state space \mathcal{E} :

$$\mathcal{E} = \begin{cases} (a, b), & \text{if } a \text{ and } b \text{ are natural boundaries,} \\ [a, b), & \text{if } a \text{ is attainable and } b \text{ is natural,} \\ (a, b], & \text{if } a \text{ is natural and } b \text{ is entrance,} \\ [a, b], & \text{if } a \text{ is attainable and } b \text{ is entrance.} \end{cases}$$

- ▶ Define $\mathcal{R} = \{(y, z) \in \mathcal{E}^2 : y < z\}$,
 y is the **pre-order** and z the **post-order** inventory levels, resp.
- ▶ Also denote $\overline{\mathcal{R}} = \{(y, z) \in \mathcal{E}^2 : y \leq z\}$

The (s, S) Policy



Question: Is the (s, S) -policy optimal? In what sense?

Long-term Average Cost

- ▶ $c_0 : \mathcal{I} \rightarrow \mathbb{R}^+$: holding/back-order cost rate,
- ▶ $c_1 : \overline{\mathcal{R}} \rightarrow \mathbb{R}^+$: ordering cost function, where

$$\mathcal{R} = \{(y, z) \in \mathcal{I}^2 : y < z\}, \quad \overline{\mathcal{R}} = \{(y, z) \in \mathcal{I}^2 : y \leq z\}.$$

- ▶ $\exists k_1 > 0$ s.t. $c_1 \geq k_1$; thus k_1 is the fixed cost for each order.

Long-term Average Cost:

$$J(\tau, Y) := \limsup_{t \rightarrow \infty} \frac{1}{t} \mathbb{E}_{x_0} \left[\int_0^t c_0(X(s)) ds + \sum_{k=1}^{\infty} I_{\{\tau_k \leq t\}} c_1(X(\tau_k -), X(\tau_k)) \right].$$

Selected Literature

Discounted Criterion

- ▶ **A. Bensoussan and J.L. Lions** (1984). *Impulse control and quasi-variational inequalities*. Gauthier-Villars, Montrouge; Heyden & Son, Inc., Philadelphia, PA.
- ▶ **L.H.R. Alvarez and J. Lempa** (2008). On the Optimal Stochastic Impulse Control of Linear Diffusions, *SIAM J. Control Optim.*, **47**, 703–742.
- ▶ **M. Egami** (2008). A Direct Solution Method for Stochastic Impulse Control Problems of One-Dimensional Diffusions, *SIAM J. Control Optim.*, **47**, 1191–1218.
- ▶ **K.J. Arrow, S. Karlin and H. Scarf** (1958). *Studies in the Mathematical Theory of Inventory and Production*, SUP, Stanford.
- ▶ **A. Bensoussan, R.H. Liu, and S.P. Sethi** (2005). Optimality of an (s, S) policy with compound Poisson and diffusion demands: a quasi-variational inequalities approach, *SIAM J. Control Optim.*, **44**, 1650–1676.
- ▶ **L. Benkherouf and A. Bensoussan** (2009). Optimality of an (s, S) policy with compound Poisson and diffusion demands: a quasi-variational inequalities approach, *SIAM J. Control Optim.*, **48**, 756–762.

Selected Literature

Long-term Average Criterion

- ▶ **A. Bensoussan** (2011). *Dynamic programming and inventory control*, Studies in Probability, Optimization and Statistics, IOS Press, Amsterdam.
- ▶ **J. G. Dai and D. Yao** (2013). Brownian inventory models with convex holding cost, part 1: Average-optimal controls. *Stoch. Syst.*, 3(2):442–499.
- ▶ **S. He, D. Yao and H. Zhang** (2017). Optimal Ordering Policy for Inventory Systems with Quantity-Dependent Setup Costs. *Math. Oper. Res.*, 42(4): 979–1006.
- ▶ **K. L. Helmes, R.H. Stockbridge, and C.Z.** (2017). Continuous inventory models of diffusion type: long-term average cost criterion. *Annals Appl. Probab.*, 27(3): 1831–1885.
- ▶ **M. Ormeci, J.G. Dai, and J.V. Vate** (2008). Impulse control of Brownian motion: the constrained average cost case, *Operations Research*, **56(3)**, 618–629.
- ▶ **D. Yao, X. Chao and J. Wu** (2015). Optimal control policy for a Brownian inventory system with concave ordering cost. *J. Appl. Probab.*, 52:909–925.

Long-term Average Control Problems

The Usual Approaches

- ▶ Vanishing discount method ([Robin, 1981, Lions and Perthame, 1986]):

- ▶ Discount problem

$$\alpha v^\alpha(x) + F(x, Dv^\alpha(x), D^2v^\alpha(x)) = 0,$$

- ▶ the limiting behavior of $-\alpha v^\alpha$ and $v^\alpha(x) - v^\alpha(x_0)$ as $\alpha \downarrow 0$.

- ▶ Long-term average HJB equation [He et al., 2017, Dai and Yao, 2013, Jack and Zervos, 2006]:

$$F(x, Du(x), D^2u(x)) = \lambda.$$

- ▶ Often the state space needs to be bounded and/or controls need to be in a restricted class.

What We Intend to Do?

1. **Formulation:** general 1-dimensional diffusion model with general boundary behavior.
2. **Holding/backorder and ordering costs:** general and relaxed conditions.
3. **New approach:** weak convergence.
4. **Result:** the (s, S) policy is optimal in the admissible class of general impulse controls under very mild conditions.

Assumptions: Boundary Points

- (a) Both the *speed measure* M and the *scale function* S of the process X_0 are absolutely continuous with respect to Lebesgue measure.
- (b) The left boundary a is *attracting* and the right boundary b is *non-attracting*.
Moreover, when b is a natural boundary, $M[y, b) < \infty$ for each $y \in \mathcal{I}$.
The boundaries $a = -\infty$ and $b = \infty$ are required to be natural.

Remark

These conditions are more general than the negative drift condition commonly used in the literature.

Assumptions: Costs

- (a) The holding/back-order cost function $c_0 : \mathcal{I} \rightarrow \mathbb{R}^+$ is continuous. Moreover, at the boundaries

$$\lim_{x \rightarrow a} c_0(x) =: c_0(a) \in \overline{\mathbb{R}^+}, \quad \lim_{x \rightarrow b} c_0(x) =: c_0(b) \in \overline{\mathbb{R}^+};$$

we require $c_0(\pm\infty) = \infty$. Finally, for each $y \in \mathcal{I}$,

$$\int_y^b c_0(v) dM(v) < \infty. \quad (1)$$

- (b) The function $c_1 : \overline{\mathcal{R}} \rightarrow \overline{\mathbb{R}^+}$ is continuous with $c_1 \geq k_1 > 0$ for some constant k_1 .

Some examples:

$$c_1(y, z) = k_1 + k_2(z - y), \quad c_1(y, z) = k_1 + k_2\sqrt{z - y}.$$

Some Remarks Concerning the Costs

In the literature

([Jack and Zervos, 2006, He et al., 2017, Dai and Yao, 2013]),

- ▶ Usually the holding/back-order cost function is assumed to be convex, monotone, and/or with certain growth conditions.
- ▶ The ordering cost function usually takes the form of “fixed plus proportional cost” or is assumed to be concave.
- ▶ Compared with our previous work ([Helmes et al., 2017]), this work also removes many structural assumptions on c_0 and c_1 .

For example, the requirement that c_0 approaches ∞ at each boundary and the rather restrictive modularity condition of that paper on c_1 are unnecessary.

The Basic Strategy

1. First examine the inventory process under the (s, S) -policy with $s = y$ and $S = z$:
The cost of such a policy is given by a nonlinear function $F(y, z)$, $y < z \in \mathcal{E}$ and an optimal (s_*, S_*) -policy exists under certain conditions.
2. Next we construct a particular auxiliary function $G_0(x)$, $x \in \mathcal{I}$.
The function G_0 solves a system similar to **but different from** the long-term QVI.
3. Finally we establish optimality of the (s_*, S_*) ordering policy in the general class of admissible policies via **weak convergence** and **appropriate approximation** of G_0 .

The Inventory Process under the (s, S) -Policy

Let $(y, z) \in \mathcal{R}$. Define the ordering policy (τ, Y) by

$$\begin{cases} \tau_1 = \inf\{t \geq 0 : X(t-) = y\}, \\ \tau_k = \inf\{t > \tau_{k-1} : X(t-) = y\}, \quad k \geq 2, \end{cases} \quad \text{and} \quad Y_k = z - y, \quad k \geq 1.$$

- ▶ Thanks to [Sigman and Wolff, 1993], the process X has a unique stationary distribution, whose density π is

$$\pi(x) = \begin{cases} 0, & a < x \leq y, \\ 2\kappa m(x)S[y, x], & y \leq x \leq z, \\ 2\kappa m(x)S[y, z], & z \leq x < b, \end{cases}$$

in which $\kappa = (\int_y^z 2S[y, x]dM(x) + 2S[y, z]M[z, b])^{-1}$.

- ▶ The constant κ gives the expected frequency of orders.

The Inventory Process under the (s, S) -Policy (cont'd)

- ▶ The cost of such a policy is given by

$$J_0(\tau, Y) = \frac{c_1(y, z) + g_0(z) - g_0(y)}{\zeta(z) - \zeta(y)}, \quad (2)$$

where

$$g_0(x) := 2 \int_{x_0}^x \int_u^b c_0(v) dM(v) dS(u),$$
$$\zeta(x) := 2 \int_{x_0}^x M[u, b] dS(u),$$

and x_0 is the initial inventory.

- ▶ Note that for any $y < z \in \mathcal{E}$,

$$\mathbb{E}_z \left[\int_0^{\tau_y} c_0(X_0(s)) ds \right] = g_0(z) - g_0(y), \quad \mathbb{E}_z[\tau_y] = \zeta(z) - \zeta(y).$$

- ▶ Also $c_1(y, z)$ is the ordering cost.
- ▶ Thus the right-hand side of Eq. (2) has a natural interpretation:

$$\frac{c_1(y, z) + g_0(z) - g_0(y)}{\zeta(z) - \zeta(y)} = \frac{\text{expected total cost per cycle}}{\text{expected cycle length}}.$$

The Function F

Define

$$F(y, z) := \frac{c_1(y, z) + g_0(z) - g_0(y)}{\zeta(z) - \zeta(y)}, \quad (y, z) \in \mathcal{R},$$

Suppose

- (a) The boundary a is regular; or exit; or a is a natural boundary for which either (i) or (ii) hold:
 - (i) $c_0(a) = \infty$;
 - (ii) $c_0(a) < \infty$, the function $F_0(\cdot, z)$ is strictly decreasing in a neighborhood of a for each $z \in \mathcal{E}$ and there exists some $(\hat{y}, \hat{z}) \in \mathcal{R}$ such that $F_0(\hat{y}, \hat{z}) < c_0(a)$.
- (b) The boundary b is entrance; or b is natural for which either (i) or (ii) hold:
 - (i) $c_0(b) = \infty$;
 - (ii) $c_0(b) < \infty$, $F_0(y, \cdot)$ is strictly increasing in a neighborhood of b for every $y \in \mathcal{E}$ and there exists some $(\tilde{y}, \tilde{z}) \in \mathcal{R}$ such that $F_0(\tilde{y}, \tilde{z}) < c_0(b)$.

Nonlinear Optimization and Optimal (s, S) Policy

Under these conditions:

- ▶ There exists a pair $(y_*, z_*) \in \mathcal{R}$ such that

$$F(y_*, z_*) = F_* := \inf \{F(y, z) : (y, z) \in \mathcal{R}\}. \quad (3)$$

- ▶ The (y_*, z_*) -policy is optimal in the class of all (s, S) ordering policies

$$F_* = F(y_*, z_*) = J_0(\tau^*, Y^*).$$

The operators:

$$Af(x) = \frac{1}{2} \frac{d}{dM} \left[\frac{df(x)}{dS} \right], \quad \forall x \in \mathcal{I},$$

$$Bf(y, z) = f(z) - f(y), \quad \forall (y, z) \in \overline{\mathcal{R}}.$$

The Auxiliary Function G_0

Define

$$G_0(x) = g_0(x) - F_*\zeta(x), \quad x \in \mathcal{E}.$$

- ▶ $G_0 \in C(\mathcal{E}) \cap C^2(\mathcal{I})$ and G_0 extends uniquely to $\bar{\mathcal{E}}$.
- ▶ G_0 is a solution of the system

$$\begin{cases} Af(x) + c_0(x) - F_* = 0, & x \in \mathcal{I}, \\ Bf(y, z) + c_1(y, z) \geq 0, & (y, z) \in \bar{\mathcal{R}} \\ f(x_0) = 0, \\ Bf(y_0^*, z_0^*) + c_1(y_0^*, z_0^*) = 0. \end{cases} \quad (4)$$

Moreover, the first relation extends by continuity to $\bar{\mathcal{E}}$.

- ▶ The system (4) is similar to but different from the usual QVI for long-term average impulse control problem:

$$\min \left\{ Au(x) + c_0(x) - F_*, \min_{z \in \mathcal{E}} [Bu(x, z) + c_1(x, z)] \right\} = 0, \quad \forall x \in \mathcal{E}.$$

The Verification?

Let β_n be a localizing sequence. Then

$$\begin{aligned} & \mathbb{E}_{x_0}[G_0(X(t \wedge \beta_n))] - G_0(x_0) \\ &= \mathbb{E}_{x_0} \left[\int_0^{t \wedge \beta_n} A G_0(X(s)) ds + \sum_{k=1}^{\infty} I_{\{\tau_k \leq t \wedge \beta_n\}} B G_0(X(\tau_k -), X(\tau_k)) \right] \\ &\geq \mathbb{E}_{x_0} \left[\int_0^{t \wedge \beta_n} [F_* - c_0(X(s))] ds - \sum_{k=1}^{\infty} I_{\{\tau_k \leq t \wedge \beta_n\}} c_1(X(\tau_k -), X(\tau_k)) \right] \end{aligned}$$

and hence

$$\begin{aligned} & F_* - \liminf_{t \rightarrow \infty} \liminf_{n \rightarrow \infty} \frac{1}{t} \mathbb{E}_{x_0}[G_0(X(t \wedge \beta_n))] \\ &\leq \limsup_{t \rightarrow \infty} \frac{1}{t} \mathbb{E} \left[\int_0^t c_0(X(s)) ds + \sum_{k=1}^{\infty} I_{\{\tau_k \leq t\}} c_1(X(\tau_k -), X(\tau_k)) \right] = J_0(\tau, Y) \end{aligned}$$

NOT SO FAST if the state space is **unbounded**.

Previous Attempts

- ▶ Optimality in a smaller class:
Optimality in the restricted class of impulse controls so that the transversality condition is satisfied.
- ▶ Ad hoc comparison result:
For **specific models**, it is sufficient to consider policies whose order-to locations are uniformly bounded above.

Expected Occupation and Ordering Measures

Define

$$\mu_{0,t}(\Gamma_0) = \frac{1}{t} \mathbb{E} \left[\int_0^t I_{\Gamma_0}(X(s)) ds \right], \quad \Gamma_0 \in \mathcal{B}(\mathcal{E}),$$

$$\mu_{1,t}(\Gamma_1) = \frac{1}{t} \mathbb{E} \left[\sum_{k=1}^{\infty} I_{\{\tau_k \leq t\}} I_{\Gamma_1}(X(\tau_k-), X(\tau_k)) \right], \quad \Gamma_1 \in \mathcal{B}(\overline{\mathcal{R}}).$$

We have the following observations:

- ▶ Let $\{t_i : i \in \mathbb{N}\}$ be such that $\lim_{i \rightarrow \infty} t_i = \infty$. If $\{\mu_{0,t_i} : i \in \mathbb{N}\}$ is not tight, then $J_0(\tau, Y) = \infty$.
- ▶ If $J_0(\tau, Y) < \infty$, then $\{\mu_{0,t}\}$ is tight.
- ▶ But $J_0(\tau, Y) < \infty$ does not necessarily imply that $\{\mu_{1,t}\}$ is tight.
- ▶ For each μ_0 attained as a weak limit of some sequence $\{\mu_{0,t_j}\}$ as $t_j \rightarrow \infty$, we have $\int_{\overline{\mathcal{E}}} c_0(x) \mu_0(dx) \leq J_0(\tau, Y) < \infty$.

Approximation of the Function G_0

Define

$$G_n(x) = \frac{G_0(x)}{1 + \frac{1}{n}h(G_0(x))}, \quad x \in \mathcal{E}, \quad (5)$$

where

$$h(x) = \begin{cases} -\frac{1}{8}x^4 + \frac{3}{4}x^2 + \frac{3}{8}, & \text{for } |x| \leq 1, \\ |x|, & \text{for } |x| \geq 1. \end{cases}$$

Under some technical assumptions on G_0 , we can show that

$$\lim_{n \rightarrow \infty} AG_n(x) = AG_0(x) \quad \forall x \in \mathcal{I}$$

$$\lim_{n \rightarrow \infty} BG_n(y, z) = BG_0(y, z) \quad \forall (y, z) \in \overline{\mathcal{R}}.$$

Key Observations

Let $(\tau, Y) \in \mathcal{A}_0$ with $J_0(\tau, Y) < \infty$. Let $\{t_j : j \in \mathbb{N}\}$ be a sequence such that $\lim_{j \rightarrow \infty} t_j = \infty$ and

$$\begin{aligned} J_0(\tau, Y) &= \lim_{j \rightarrow \infty} \frac{1}{t_j} \mathbb{E} \left[\int_0^{t_j} c_0(X(s)) ds + \sum_{k=1}^{\infty} I_{\{\tau_k \leq t_j\}} c_1(X(\tau_k-), X(\tau_k)) \right] \\ &= \lim_{j \rightarrow \infty} \left(\int_{\mathcal{E}} c_0(x) \mu_{0,t_j}(dx) + \int_{\mathcal{R}} c_1(y, z) \mu_{1,t_j}(dy \times dz) \right). \end{aligned}$$

Then

$$0 = \lim_{j \rightarrow \infty} \left(\int_{\mathcal{E}} AG_n(x) \mu_0(dx) + \int_{\mathcal{R}} BG_n(y, z) \mu_{1,t_j}(dy \times dz) \right), \quad \forall n \in \mathbb{N}$$

and

$$\liminf_{n \rightarrow \infty} \liminf_{j \rightarrow \infty} \int_{\mathcal{R}} (BG_n(y, z) + c_1(y, z)) \mu_{1,t_j}(dy \times dz) \geq 0,$$

$$\liminf_{n \rightarrow \infty} \int_{\mathcal{E}} (AG_n(x) + c_0(x)) \mu_0(dx) \geq \int_{\mathcal{E}} (AG_0(x) + c_0(x)) \mu_0(dx) \geq F_*.$$

Optimality

$$\begin{aligned} J_0(\tau, Y) &= \liminf_{n \rightarrow \infty} \lim_{j \rightarrow \infty} \left(\int_{\bar{\mathcal{E}}} (AG_n(x) + c_0(x)) \mu_{0,t_j}(dx) \right. \\ &\quad \left. + \int_{\bar{\mathcal{R}}} (BG_n(y, z) + c_1(y, z)) \mu_{1,t_j}(dy \times dz) \right) \\ &\geq \liminf_{n \rightarrow \infty} \liminf_{j \rightarrow \infty} \int_{\bar{\mathcal{E}}} (AG_n(x) + c_0(x)) \mu_{0,t_j}(dx) \\ &\quad + \liminf_{n \rightarrow \infty} \liminf_{j \rightarrow \infty} \int_{\bar{\mathcal{R}}} (BG_n(x) + c_1(y)) \mu_{1,t_j}(dy \times dz) \\ &\geq \liminf_{n \rightarrow \infty} \int_{\bar{\mathcal{E}}} (AG_n(x) + c_0(x)) \mu_0(dx) \\ &\quad + \liminf_{n \rightarrow \infty} \liminf_{j \rightarrow \infty} \int_{\bar{\mathcal{R}}} (BG_n(x) + c_1(y)) \mu_{1,t_j}(dy \times dz) \\ &\geq F_0^*. \end{aligned}$$

The Main Result

Theorem

(a) *Let $(\tau, Y) \in \mathcal{A}_0$. Then*

$$J_0(\tau, Y) \geq F_*.$$

(b) *Moreover, the (s, S) -policy with $s = y_*$ and $S = z_*$ is an optimal impulse policy.*

Some Remarks

- ▶ The ergodicity of the inventory process under the (s, S) -policy gives $J(\tau^*, Y^*) = F_*$.
- ▶ To establish the optimality of the (s_*, S_*) -policy in the general admissible class \mathcal{A}_0 , the usual approach needs to solve the associated QVI and verify the transversality condition. These are not easy; in particular if the state space is unbounded.
- ▶ This work proposes a weak convergence approach together with an appropriate approximation of the function G_0 to establish the optimality.

Drifted Brownian motion inventory models

- ▶ The classical model drifted Brownian motion on $(-\infty, \infty)$

- ▶ State:

$$X_0(t) = x_0 - \mu t + \sigma W(t), \quad x_0 \in (-\infty, \infty)$$

- ▶ Costs:

$$c_0(x) = \begin{cases} -c_b x, & x < 0, \\ c_h x, & x \geq 0, \end{cases} \quad \text{and} \quad c_1(y, z) = k_1 + k_2(z - y)$$

in which $c_b, c_h, k_1, k_2 > 0$.

- ▶ Drifted Brownian motion with reflection at $\{0\}$

- ▶ State:

$$X_0(t) = x_0 - \mu t + \sigma W(t) + L_0(t), \quad x_0 \in [0, \infty)$$

- ▶ Costs:

$$c_0(x) = k_3 x + k_4 e^{-x}, \quad \text{and} \quad c_1(y, z) = k_1 + k_2 \sqrt{z - y},$$

- ▶ In both models, the (s, S) -policy is optimal.

A Counter-intuitive Example

- ▶ Inventory level (in absence of ordering) is $X_0(t) = W(t) - t$, for $t \geq 0$,
- ▶ The cost functions are specified

$$c_0(x) = 2|x|, \forall x \in \mathbb{R}, \text{ and } c_1(y, z) = k_1 + (z - y), \forall (y, z) \in \overline{\mathcal{R}}.$$

- ▶ A special ordering policy (τ, Y) :
 - ▶ It runs in cycles, each of which is composed of two phases.
 - ▶ For cycle $i = 1, 2, 3, \dots$,
 - ▶ Phase 1 consists of using the $(0, 1)$ -ordering policy a total of 2^{i-1} times; the length of each sub-cycle is a random variable having mean 1.
 - ▶ Phase 2 involves a single $(0, 2^{(i-1)/2})$ -ordering policy followed immediately by using the $(2^{(i-1)/2}, 2^{(i-1)/2})$ -ordering policy 2^{i-1} times.
- ▶ Then (a) $J_0(\tau, Y) < \infty$; (b) $\{\mu_{0,t} : t > 0\}$ is tight as $t \rightarrow \infty$; and (c) $\{\mu_{1,t} : t > 0\}$ is not tight.

Geometric Brownian Motion Inventory Model

- ▶ In the absence of ordering, the inventory process X_0 satisfies

$$dX_0(t) = -\mu X_0(t)dt + \sigma X_0(t)dW(t), \quad X(0) = x_0 \in \mathcal{I} = (0, \infty),$$

in which $\mu, \sigma > 0$ and W is a standard Brownian motion.

- ▶ The scale and speed measures are: for $[l, x] \subset \mathcal{I}$

$$S[l, x] = \frac{\sigma^2}{2\mu + \sigma^2} \left[x^{1+2\mu/\sigma^2} - l^{1+2\mu/\sigma^2} \right] \quad \text{and}$$
$$M[l, x] = \frac{1}{2\mu + \sigma^2} \left[l^{-1-2\mu/\sigma^2} - x^{-1-2\mu/\sigma^2} \right].$$

- ▶ Note that $S(0, x] < \infty$ and $S[x, \infty) = \infty$ and thus 0 is attracting and ∞ is non-attracting. Both boundaries are natural.

Geometric Brownian Motion Inventory Model

- ▶ State:

$$dX_0(t) = -\mu X_0(t) dt + \sigma X_0(t) dW(t), \quad X(0) = x_0 \in \mathcal{I} = (0, \infty),$$

in which $\mu, \sigma > 0$.

- ▶ Costs:

- ▶ Case 1

$$c_0(x) = k_3 x + k_4 x^\beta, \quad \text{and} \quad c_1(y, z) = k_1 + k_2 \sqrt{z - y},$$

- ▶ Case 2

$$c_0(x) = \begin{cases} k_4(1 - x), & 0 \leq x \leq 1, \\ k_3(x - 1), & x \geq 1 \end{cases}$$

$$c_1(y, z) = k_1 + \frac{k_2}{2}(y^{-1/2} - z^{-1/2}) + \frac{k_2}{2}(z - y)$$

- ▶ In both models, the (s, S) -policy is optimal.

Summary

In this work, we

- ▶ formulated an impulse inventory control problem for a general one-dimensional diffusion with general boundary conditions under the long-term average cost criterion,
- ▶ used a weak convergence approach together with an appropriate approximation of the function G_0 to establish the optimality of the (s_*, S_*) policy in the general admissible class of impulse controls.
- ▶ provided a nonlinear optimization approach to determine the optimal levels s_* and S_* ,
- ▶ studied geometric and drifted Brownian motion inventory examples for illustration.

Thank you!



Dai, J. G. and Yao, D. (2013).

Brownian inventory models with convex holding cost, Part 1:
Average-optimal controls.

Stoch. Syst., 3(2):442–499.



He, S., Yao, D., and Zhang, H. (2017).

Optimal ordering policy for inventory systems with
quantity-dependent setup costs.

Math. Oper. Res., 42(4):979–1006.



Helmes, K. L., Stockbridge, R. H., and Zhu, C. (2017).

Continuous inventory models of diffusion type: Long-term
average cost criterion.

Ann. Appl. Probab., 27(3):1831–1885.



Jack, A. and Zervos, M. (2006).

Impulse control of one-dimensional Itô diffusions with an
expected and a pathwise ergodic criterion.

Appl. Math. Optim., 54(1):71–93.



Lions, P.-L. and Perthame, B. (1986).

Quasivariational inequalities and ergodic impulse control.

SIAM J. Control Optim., 24(4):604–615.



Robin, M. (1981).

On some impulse control problems with long run average cost.

SIAM Journal on Control and Optimization, 19(3):333–358.



Sigman, K. and Wolff, R. W. (1993).

A review of regenerative processes.

SIAM Rev., 35(2):269–288.