Phaseless Sampling and Reconstruction in a
Shift-Invariant Space

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Our Problem

- Phase retrieval arises in various fields of science and engineering, such as X-ray crystallography, coherent diffractive imaging, ptychography and more.

\[ f \leftrightarrow |f(t)|, \; t \in Y = \mathbb{R}^d \text{ or } X + \mathbb{Z}^d, \text{ where} \]

\[ f \in V(\phi) := \left\{ \sum_{k \in \mathbb{Z}^d} c(k)\phi(\cdot - k) \right\} \]

- We say a signal \( f \) is phase retrievable if \( g = \pm f \) on \( \mathbb{R}^d \) satisfying \( |g| = |f| \).
Define Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} dx.$$ 

Write

$$\hat{f}(\xi) = |\hat{f}(\xi)| e^{i\theta(\xi)}, \xi \in \mathbb{R}.$$ 

**Phase Retrieval:** Is that possible to find $\theta(\xi)$ from $|\hat{f}(\xi)|$?

1) PR problem is ill-posed without constraint on $f$

2) $g = e^{i\alpha}f \implies |\hat{g}| = |e^{i\alpha}\hat{f}| = |\hat{f}|$

3) Solvable if $f$ is symmetric and compactly supported.

$\hat{f}(\xi)$ is real-valued and analytic (hence nonseparable).
Finite-dimensional setting

The phase retrieval is how to recover $x$ from $|\hat{x}(k)|$, $1 \leq k \leq M$.

General Phase Retrieval:
How about magnitude of frame measurements

$$|\langle x, g_i \rangle|, 1 \leq i \leq M.$$  

**Theorem**

Let $\{g_i\}_{i=1}^M$ be a frame for $\mathbb{R}^N$. Then $x \in \mathbb{R}^N$ is phase retrieval if and only if for every subset $I \subset \{1, \cdots, M\}$, either $\{g_i\}_{i \in I}$ spans $\mathbb{R}^N$ or $\{g_i\}_{i \in I^c}$ spans $\mathbb{R}^N$.

\textsuperscript{a} R. Balan, P. Casazza and D. Edidin, On signal reconstruction without phase, Applied and Computational Harmonic Analysis, 20(2006), 345-356
Spatial signal:

\[ f \leftrightarrow f(t), \, t \in \mathbb{R}^d \]

Phase Retrieval Problem:

\[ f \leftarrow |f(t)|, \, t \in \mathbb{R}^d \]
Spatial signal:

\[ f \leftrightarrow f(t), \quad t \in \mathbb{R}^d \]

Phase Retrieval Problem:

\[ f \leftarrow |f(t)|, \quad t \in \mathbb{R}^d \]

Figure: linear spline signal

\[ f = \sum_{-4 \leq k \leq 4} c(k) h(t - k) \]
Nonseparability

**Theorem**

Let $V$ be a linear space. Then a signal $f \in V$ can be determined, up to a global sign, from $|f(t)|$, $t \in \mathbb{R}^d$ if and only if there do not exist nonzero signals $f_1, f_2 \in V$ such that

$$f = f_1 + f_2 \quad \text{and} \quad f_1 f_2 = 0.$$  \hspace{1cm} (1)

We call those signals without decomposition of two signals in (1) as nonseparable signals.

**Proof :**

($\implies$) Suppose, on the contrary, that there exist nonzero signals $f_1, f_2 \in V$ such that

$$f = f_1 + f_2 \quad \text{and} \quad f_1 f_2 = 0.$$

Set $g = f_1 - f_2$. Then $|g| = |f| = |f_1| + |f_2|$, but $g \neq \pm f$. This is a contradiction.

($\impliedby$) Assume that $f$ is nonseparable and $g \in V$ satisfies $|g| = |f|$. Set

$$g_1 := (f + g)/2 \quad \text{and} \quad g_2 := (f - g)/2 \in V.$$

Then $f = g_1 + g_2$ and $g_1 g_2 = 0$. Since $f$ is nonseparable and by the definition, it implies that either $g_1 = 0$ or $g_2 = 0$. Hence $g = \pm f$. Therefore $f$ is phase retrievable.
Corollary

Any symmetric and compactly supported signal $f$ is determined, up to a sign, from magnitudes of its Fourier measurements.

\[
\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} dx = \int_{\mathbb{R}} f(x) \cos(x\xi) dx
\]

is analytic and real-valued

- If $\hat{f}$ is separable, i.e. $\hat{f} = \hat{f}_1 + \hat{f}_2$ and $\hat{f}_1 \hat{f}_2 = 0$ analytic
  \[\implies\] either $\hat{f}_1 = 0$ or $\hat{f}_2 = 0$

- This proves that all $\hat{f}$ is nonseparable.
Recall: Any signal in the Paley-Wiener space are nonseparable, and hence any bandlimited signal are determined from its magnitude.  \(^1\)

Paley-Wiener space is a shift-invariant space generated by sinc function,

\[
PW_2 = \left\{ \sum_k c(k) \text{sinc}(t - k); \sum_k |c(k)|^2 < \infty \right\}
\]

where \( \text{sinc}(t) = \frac{\sin(2\pi t)}{2\pi t} \).

Problem: How to characterize phase retrieval of signals in a shift-invariant space

\[ V(\phi) = \left\{ \sum_k c(k) \phi(t - k) \right\} \]

generated by a compactly supported function \( \phi \)?

Shift-invariant space arises in wavelet analysis and approximation. It is widely-used model other than the band-limited model in signal processing.

How about shift-invariant space generated by a spline $B_N$, $N \geq 1$, where $B_1 = \chi_{[0,1)}$ and $B_N = B_1 \ast B_{N-1}$ (convolution)

Property: Not all signals in a shift-invariant space are nonseparable.

$B_N$ are supported on $[0, N)$, and $B_N(t) \pm B_N(t - N)$ is separable, as the support of $B_N$ and $B_N(t - N)$ do not overlap.
Local linear independence

One-to-one correspondence between an amplitude vector $c$ and a signal $f$ in the shift-invariant space $V(\phi)$,

$$c := (c(k))_{k \in \mathbb{Z}^d} \mapsto \sum_{k \in \mathbb{Z}^d} c(k)\phi(\cdot - k) =: f \in V(\phi).$$  \hspace{1cm} (2)

Definition

Let $\phi$ be a continuous function with compact support and $A$ be an open set. We say that $\phi$ has local linear independence on $A$ if

$$\sum_{k \in \mathbb{Z}^d} c(k)\phi(x - k) = 0$$

for all $x \in A$ implies that $c(k) = 0$ for all $k \in \mathbb{Z}^d$ satisfying $\phi(x - k) \not\equiv 0$ on $A$.

The global linear independence of a compactly supported function $\phi$ can be interpreted as its local linear independence on $\mathbb{R}^d$. \(^3\)

Question: How to determine the nonseparability of a signal in a shift-invariant space?

Theorem

Let $\phi$ be a continuous function supported in $[0, N]$ with all $N \times N$ submatrices of

$$\Phi = (\phi(t_i + m))_{1 \leq i \leq 2N-1, 0 \leq m \leq N-1}$$

being nonsingular. Then the following statements are equivalent.

1) $f$ is nonseparable

2) $\sum_{l=0}^{N-2} |c(k + l)|^2 \neq 0$ for all $K_-(f) - N + 1 < k < K_+(f) + 1$, where $K_-(f) = \inf\{k, c(k) \neq 0\}$ and $K_+(f) = \sup\{k, c(k) \neq 0\}$.

3) $f$ can be reconstructed from $X + Z$, where $X = \{t_i, 1 \leq i \leq 2N - 1\}$. 
1). \( f \) is nonseparable \( \iff \) 3). \( f \) can be reconstructed from \( X + Z \), where 
\[ X = \{ t_i, 1 \leq i \leq 2N - 1 \} \].

Interpretation: \( f \) can be reconstructed from \( |f(t)|, t \in \mathbb{R} \) if and only if it can be reconstructed from \( X + Z \), where \( X = \{ t_i, 1 \leq i \leq 2N - 1 \} \).
1). $f$ is nonseparable $\iff$ 2). $\sum_{l=0}^{N-2} |c(k+l)|^2 \neq 0$ for all \( K_-(f) - N + 1 < k < K_+(f) + 1 \).

Interpretation: Nonseparable signal in $V(\phi)$ does not have consecutive $N - 1$ zeros in the amplitude vector $c$, where $\phi$ is supported in $[0, N]$. 
1). \( f \) is nonseparable \( \iff \sum_{l=0}^{N-2} |c(k + l)|^2 \neq 0 \) for all \( K_-(f) - N + 1 < k < K_+(f) + 1 \).

Interpretation: Nonseparable signal in \( V(\phi) \) does not have consecutive \( N - 1 \) zeros in the amplitude vector \( \mathbf{c} \), where \( \phi \) is supported in \([0, N]\).

**Figure:** linear spline signal \( f = \sum_{-10 \leq k \leq 10} c(k)h(t - k) \) with \( c(0) = 0 \)
To characterize the nonseparability of signals on $\mathbb{R}^d$, $d \geq 1$, we introduce an undirected graph for a signal in the shift-invariant space $V(\phi)$ generated by a real-valued continuous function $\phi$ with compact support.

**Definition**

For any $f(x) \in V(\phi)$, define an undirected graph

$$G_f := (V_f, E_f),$$

where the vertex set

$$V_f = \{k \in \mathbb{Z}^d : c(k) \neq 0\}$$

contains supports of the amplitude vector of the signal $f$, and

$$E_f = \{(k, k') \in V_f \times V_f : k \neq k' \text{ and } \phi(x-k)\phi(x-k') \neq 0 \text{ for some } x \in \mathbb{R}^d\}$$

is the edge set associated with the signal $f$. 
Let $\phi$ be a compactly supported continuous function on $\mathbb{R}^d$ with local linear independence on any open set, and $f$ be a signal in the shift-invariant space $V(\phi)$. The graph $G_f$ in (3) is connected if and only if $f$ is nonseparable.

For $d = 1$, we have

$$(k, k') \in E_f \text{ if and only if } |k - k'| \leq N - 1,$$ (4)

provided that the support of $\phi$ is $[0, N]$ for some $N \geq 1$. 

Phaseless sampling and reconstruction

Theorem

Let $\phi$ be a compactly supported continuous function and $V(\phi)$ be the shift-invariant space. Then there exists a discrete set $\Gamma \subset (0,1)^d$ such that any nonseparable signal $f \in V(\phi)$ is determined, up to a sign, by its phaseless samples on the set $\Gamma + \mathbb{Z}^d$ with finite sampling density.

Set $\Phi_{(0,1)^d}(x) := (\phi(x - k))_k$.

$$\#(\Gamma) \leq \dim \left( \text{span}\{ \Phi_{(0,1)^d}(x)(\Phi_{(0,1)^d}(x))^T \} \right), \ x \in (0,1)^d.$$
Model of the generator

The generator in our consideration is the box splines $M_{\Xi}$, which are defined by

$$\int_{\mathbb{R}^d} g(x) M_{\Xi}(x) \, dx = \int_{\mathbb{R}^s} g(\Xi y) \, dy, \quad g \in L^2(\mathbb{R}^d),$$

where $\Xi \in \mathbb{Z}^{d \times s}$ is a matrix with full rank $d$.

Figure: box spline function
Reconstruction Algorithm

- \( f(t) = \sum_{k \in \mathbb{Z}^d} c(k)\phi(t - k) \) with finite duration.

- Noisy phaseless samples data on \( \Gamma = X + \mathbb{Z}^d \),

\[
z_{\epsilon}(\gamma) = |f(\gamma)|^2 + \epsilon(\gamma), \quad \gamma \in \Gamma,
\]

where \( \epsilon(\gamma) \in [-\epsilon, \epsilon], \gamma \in \Gamma \), are randomly selected with noise level \( \epsilon > 0 \).

- Conventional way

\[
\min \sum_{\gamma \in \Gamma} \left| |g(\gamma)| - \sqrt{z_{\epsilon}(\gamma)} \right|^2 \text{ subject to } g \in V(\phi).
\]

- However, it is *infinite-dimensional* and infeasible.

- Implemented in a distributed manner (Local minimization, phase adjustment, sewing, hard thresholding).
The proposed algorithm may not recover a nonseparable signal in a shift-invariant space if the noise level $\varepsilon$ is not sufficiently small.

The success rate of the proposed algorithm can be improved if we have phaseless samples on a discrete set with high sampling rate. Presented in Figure 4 is the success rate in percentage and the average amplitude error after 1000 trials for different noisy levels $\varepsilon$ and $X_K + Z$, $7 \leq K \leq 15$, where

$$X_K = \left\{ \frac{1}{K+1}, \ldots, \frac{K}{K+1} \right\}.$$
Figure: Plotted on the left is the success rate against noisy level $-\log_{10} |\epsilon|$ to recover a nonseparable cubic spline $f$ by the algorithm for 1000 trails, with $c(k), k \in \mathbb{Z}$, randomly selected and odd integers $7 \leq K \leq 15$. On the right is the average error $\log_{10} e(\epsilon)$ against noisy level $-\log_{10} |\epsilon|$ in the logarithmic scale for a nonseparable cubic spline $f$ running our algorithm for 1000 trails, where the error is counted in the average only when phases are saved successfully.
Thank You!