Optimal shape and location of actuators or sensors in PDE models

Yannick Privat, Emmanuel Trélat and Enrique Zuazua

CNRS, LJLL, Univ. Paris 6

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Motivations

What is the best shape and placement of sensors?

- Reduce the cost of instruments.
- Maximize the efficiency of reconstruction and estimations.
Outlines of this talk

1. Introduction and motivation
2. Modeling of the problem: a randomized criterion
3. Optimal observability for wave and heat equations
N-D wave/heat equation

\( \Omega \) open bounded connected subset of \( \mathbb{R}^n \) such that \( \partial \Omega \neq \emptyset \)
\( T > 0 \) fixed
\( \omega \subset \Omega \) subset of positive measure

<table>
<thead>
<tr>
<th>N-D wave equation</th>
<th>N-D heat equation</th>
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<tbody>
<tr>
<td>( \partial_{tt} y - \nabla y = 0 ) in ( (0, T) \times \Omega )</td>
<td>( \partial_t y - \nabla y = 0 ) in ( (0, T) \times \Omega )</td>
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<td>_{\partial \Omega} = 0 )</td>
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<td>( y(0, \cdot) = y^0, \ \partial_t y(0, \cdot) = y^1 ) in ( \Omega )</td>
<td>( y(0, \cdot) = y^0 ) in ( \Omega )</td>
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Generalization to many other boundary conditions (Neumann or mixed Dirichlet-Neumann or Robin on \( \partial \Omega \))

Observable variable (\( \omega \subset \Omega \) of positive measure)

\[
\chi_\omega(x)y(t, x) = \begin{cases} 
  y(t, x) & \text{if } x \in \omega \\
  0 & \text{else.}
\end{cases}
\]
Observability of the N-D wave/heat equation

Observability inequality (wave equation)

The time $T$ being chosen large enough, how to choose $\omega \subset \Omega$ to ensure that

$$\exists C > 0 \mid \forall (y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega), \quad C \|(y^0, y^1)\|_{L^2 \times H^{-1}}^2 \leq \int_0^T \int_\omega y(t, x)^2 \, dx \, dt$$

- **Microlocal Analysis.** Bardos, Lebeau and Rauch proved that, roughly in the class of $C^\infty$ domains, the observability inequality holds iff $(\omega, T)$ satisfies the GCC.

- **Observability constant:**

  $$C_T^{\text{wave}} = \inf_{y \text{ solution of the wave eq.} \quad (y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)} \frac{\int_0^T \int_\omega y(t, x)^2 \, dx \, dt}{\|(y^0, y^1)\|_{L^2(\Omega) \times H^{-1}(\Omega)}^2}.$$
Observability of the N-D wave/heat equation

Observability inequality (heat equation)
The time $T$ being fixed, how to choose $\omega \subset \Omega$ to ensure that

$$\exists C > 0 \mid C \|y(T, \cdot)\|_{L^2(\Omega)}^2 \leq \int_0^T \int_\omega y(t, x)^2 \, dx \, dt,$$

for every solution of the heat equation such that $y(0, \cdot) \in H^1_0 \cap H^2(\Omega)$?

$\rightarrow$ this ineq. holds for every open subset $\omega$ of $\Omega$;

$\rightarrow$ related to the inverse problem of recovering its final data from the $L^2$-observation of its solution on the set $\omega$ during a time $T$.

$$C_T^{\text{heat}}(\chi_\omega) = \inf_{y \text{ solution of the heat eq. } y(0, \cdot) \in H^1_0 \cap H^2(\Omega)} \frac{\int_0^T \int_\omega |y(t, x)|^2 \, dx \, dt}{\|y(T, \cdot)\|_{L^2(\Omega)}^2}.$$
Randomized observability constant
The randomization procedure

A first idea

It is \textbf{a priori} natural to model the problem as:

\[
\sup_{\omega \subset \Omega} \frac{C_T(\omega)}{\abs{\omega} = L \abs{\Omega}}
\]

BUT:

1. Theoretical difficulty due to crossed terms in the spectral expansion (cf Ingham inequalities).

2. In practice: many experiments, many measures. This deterministic constant is \textbf{pessimistic}: it gives an account for the \textbf{worst case}.

\[\rightarrow\] optimize shape and location of sensors \textbf{in average}, over a large number of measurements

\[\rightarrow\] define an \textbf{averaged} observability inequality
Randomized observability constant

The randomization procedure

Averaging over random initial data:

Randomized observability inequality (wave equation)

\[
C_{T, \text{rand}}(\omega) \| (y(0, \cdot), y_t(0, \cdot)) \|_{L^2 \times H^{-1}}^2 \leq \mathbb{E} \left( \int_0^T \int_\omega |y_\nu(t, x)|^2 \, dx \, dt \right)
\]

where

\[
y_\nu(t, x) = \sum_{j=1}^{+\infty} \left( \beta_{1,j}^\nu a_j e^{i\lambda_j t} + \beta_{2,j}^\nu b_j e^{-i\lambda_j t} \right) \phi_j(x)
\]

with \(\beta_{1,j}^\nu, \beta_{2,j}^\nu\) i.i.d. random variables (e.g., Bernoulli, Gaussian) of mean 0


with \((\phi_j)_{j \in \mathbb{N}^*}\) Hilbert basis of eigenfunctions

Randomization

- generates a full measure set of initial data
- does not regularize
Randomized observability constant

The randomization procedure

Recall that $(\phi_j)_{j \in \mathbb{N}^*}$ a fixed Hilbert basis of eigenfunctions of $\triangle$.

**Theorem**

$$C_{T, \text{rand}}(\chi_\omega) = T \inf_{j \in \mathbb{N}^*} \gamma_j \int_{\omega} \phi_j(x)^2 \, dx$$

with

$$\gamma_j = \begin{cases} 
1/2 & \text{for the wave equation} \\
1 & \text{for the Schrödinger equation} \\
\frac{e^{2\lambda_j^2 T} - 1}{2\lambda_j^2} & \text{for the heat equation}
\end{cases}$$

**Remark**

There holds $C_{T, \text{rand}}(\chi_\omega) \geq C_T(\chi_\omega)$.

For the wave equation, the randomized observability constant is a spectral quantity ignoring the rays’ contribution.

$$(\rightarrow \text{spectral criterion = half of the truth!})$$

There are examples where the inequality is strict:
- in 1D: $\Omega = (0, \pi)$, $T \neq k\pi$.
- in multi-D: $\Omega$ stadium-shaped, $\omega$ containing the wings.
What is the “best possible” observation domain \( \omega \) of given measure?

Optimal design problem (energy concentration criterion)

We investigate the problem of maximizing

\[
\frac{C_{T,\text{rand}}(\chi_\omega)}{T} = \inf_{j \in \mathbb{N}^*} \gamma_j \int_\omega \phi_j(x)^2 \, dx.
\]

over all possible subset \( \omega \subset \Omega \) of Lebesgue measure \( L|\Omega| \).
Related problems

Optimal design for control/stabilization problems

1. What is the "best domain" for achieving HUM optimal control?

\[ y_{tt} - \Delta y = \chi_\omega u \]

2. What is the "best domain" domain for stabilization (with localized damping)?

\[ y_{tt} - \Delta y = -k\chi_\omega y_t \]

See works by
- P. Hébrard, A. Henrot : theoretical and numerical results in 1D for optimal stabilization (for all initial data).
- S. Cox, P. Freitas, F. Fahroo, K. Ito, ... : variational formulations and numerics.
- M.I. Frecker, C.S. Kubrusly, H. Malebranche, S. Kumar, J.H. Seinfeld, ... : numerical investigations (among a finite number of possible initial data).
- K. Morris, S.L. Padula, O. Sigmund, M. Van de Wal, ... : numerical investigations for actuator placements (predefined set of possible candidates), Riccati approaches.
- and many others...
Solving of the optimal design problem

Generalities

General optimal design problem

\[
\sup_{\omega \subset \Omega, |\omega| = L|\Omega|} J(\chi_{\omega}) := \sup_{\omega \subset \Omega, |\omega| = L|\Omega|} \inf_{j \in \mathbb{N}^*} \gamma_j \int_{\Omega} \chi_{\omega}(x) \phi_j(x)^2 \, dx
\]

Admissible set for this problem:

\[
\mathcal{U}_L = \{ \chi_\omega \mid \omega \text{ is a measurable subset of } \Omega \text{ of measure } L|\Omega| \}.
\]
Optimal observability for wave and heat equations

$$\sup_{\omega \subset \Omega} \inf_{|\omega|=L|\Omega|} \gamma_j \int_\omega \phi_j(x)^2 \, dx$$

To solve the problem, we distinguish between:

- parabolic equations (e.g., heat, Stokes)
- wave or Schrödinger equations

Remarks

- requires some knowledge on the asymptotic behavior of $\phi_j^2$
- $\mu_j = \phi_j^2 \, dx$ is a probability measure
  \[ \Rightarrow \text{strong difference between } \gamma_j \sim e^{\lambda_j T} \text{ (parabolic)} \text{ and } \gamma_j = 1 \text{ (hyperbolic)} \]
Solving of the optimal design problem for parabolic equations

We assume that $\Omega$ is piecewise $C^1$

**Theorem**

There exists a **unique optimal domain** $\omega^*$

- Quite difficult proof, requiring in particular: Hartung minimax theorem; fine lower estimates of $\phi_j^2$ by J. Apraiz, L. Escauriaza, G. Wang, C. Zhang (JEMS 2014)
- Algorithmic construction of the best observation set $\omega^*$: to be followed (further)
Solving of the optimal design problem for the wave and Schrödinger equations

**Wave and Schrödinger equations**

### Theorem (optimal value)

Under appropriate spectral assumptions:

\[
\sup_{\omega \subset \Omega} \frac{1}{|\omega| = L|\Omega|} \inf_{j \in \mathbb{N}^*} \int_{\omega} \phi_j(x)^2 \, dx = L
\]

- Proof: 1) convexification (relaxation), 2) no-gap (not obvious because not lsc).

- Main spectral assumption:
  - QUE (Quantum Unique Ergodicity): the whole sequence \( \phi_j^2 \, dx \rightharpoonup \frac{dx}{|\Omega|} \) vaguely.

  true in 1D, but in multi-D?
Solving of the optimal design problem for the wave and Schrödinger equations

Wave and Schrödinger equations

Theorem (optimal value)

Under appropriate spectral assumptions:

\[
\sup_{\omega \subset \Omega, |\omega| = L|\Omega|} \inf_{j \in \mathbb{N}^*} \int_{\omega} \phi_j(x)^2 \, dx = L
\]

Relationship to quantum chaos theory:

what are the possible (weak) limits of the probability measures \( \mu_j = \phi_j^2 \, dx \)?

(quantum limits, or semi-classical measures)

- partial answer provided by Shnirelman theorem
- If QUE fails, we may have scars
Solving of the optimal design problem for the wave and Schrödinger equations

**Wave and Schrödinger equations**

**Theorem (optimal value)**

Under appropriate spectral assumptions:

\[
\sup_{\omega \subset \Omega} \left| \omega \right| = L \left| \Omega \right| \inf_{j \in \mathbb{N}^*} \int_{\omega} \phi_j(x)^2 \, dx = L
\]

**Remark:** The above result holds true as well in the disk. Hence the spectral assumptions are not sharp.

(proof: requires the knowledge of all quantum limits in the disk, Privat Hillairet Trélat)

\[\mu_{jk} \rightharpoonup \delta_{r=1}\]

(this is one QL: whispering galleries)
Solving of the optimal design problem for the wave and Schrödinger equations

**Theorem (optimal value)**

Under appropriate spectral assumptions: \[
\sup_{\omega \subset \Omega \atop |\omega| = L|\Omega|} \inf_{j \in \mathbb{N}^*} \int_{\omega} \phi_j(x)^2 \, dx = L
\]

- **Supremum reached?** Open problem in general.
  - in 1D: reached \( \iff L = 1/2 \) (infinite number of optimal sets)
  - in 2D square: reached over Cartesian products \( \iff L \in \{1/4, 1/2, 3/4\} \)
    
    **Conjecture**: Not reached for generic domains \( \Omega \) and generic values of \( L \).

- Construction of a maximizing sequence (by a kind of homogenization)
Theorem

The shape optimization problem above has a unique solution $\omega^*_N$. Moreover, $\omega^N$ is semi-analytic and thus has a finite number of connected components.

$\Gamma$-convergence result: $\lim_{N \to +\infty} \sup_{\chi_\omega \in U_L} J_N(\chi_\omega) = $ optimal value for the relaxed pb.

Spillover phenomenon: the best domain $\omega^N$ for the first $N$ modes is the worst possible for $N+1$ modes.

Wave and Schrödinger equations

The complexity of $\omega^N$ is increasing with $N$.

Illustration of the spillover phenomenon

Parabolic equations

(e.g., heat, Stokes, anomalous diffusions)

Under a slight spectral assumption:

(satisfied, e.g., by $(-\Delta)^{\alpha}$ with $\alpha > 1/2$)

The sequence of optimal sets $\omega^N$ is stationary:

$$\exists N_0 \mid \forall N \geq N_0 \quad \omega^N = \omega^{N_0} = \omega^*$$

with $\omega^*$ the optimal set for all modes.

In particular, $\omega^*$ is semi-analytic and thus has a finite number of connected components.

$\Omega = (0, \pi)^2$

$L = 0.2$

4, 25, 100, 500 eigenmodes
Wave and Schrödinger equations

The complexity of $\omega^N$ is increasing with $N$.

Illustration of the spillover phenomenon

$\Omega =$ unit disk  \hspace{1cm} L = 0.2
1, 25, 100, 400 eigenmodes

Parabolic equations

(e.g., heat, Stokes, anomalous diffusions)

Under a slight spectral assumption:
(satisfied, e.g., by $(-\Delta)^{\alpha}$ with $\alpha > 1/2$)

The sequence of optimal sets $\omega^N$ is stationary:

$$\exists N_0 \mid \forall N \geq N_0 \quad \omega^N = \omega^{N_0} = \omega^*$$

with $\omega^*$ the optimal set for all modes.

In particular, $\omega^*$ is semi-analytic and thus has a finite number of connected components.
Several numerical simulations: $\Omega = \text{unit disk}$

$L = 0.2, \ T = 0.05$, for $1, 4, 9, 16, 25$ and $36$ eigenmodes
Conclusion of this talk

Ongoing works:

- optimal design for boundary observability. (with P. Jounieaux)
  \[ \Omega \text{ being assumed bounded connected and its boundary } C^2, \text{ maximize } \]
  \[ \inf_{j \in \mathbb{N}^*} \frac{1}{\lambda_j(\Omega)} \int_\Sigma \left| \frac{\partial \phi_j}{\partial n} \right|^2 \, dx \]
  over all possible subsets \( \Sigma \subset \partial \Omega \) of given Hausdorff measure.

- new strategies to avoid spillover phenomena when solving optimal design problems (Césaro means).

- Same analysis for the optimal design of the domain of control. (effect of the randomization on the HUM operator?)

- discretization issues. Do the numerical designs converge to the continuous optimal design as the mesh size tends to 0?


Conclusion and perspectives

What can be said for the classical (deterministic) observability constant?

A result for the wave observability constant:

$$\lim_{T \to +\infty} \frac{C_T(\omega)}{T} = \frac{1}{2} \min \left( \inf_{j \in \mathbb{N}^*} \int_{\omega} \phi_j^2 \, dx, \lim_{T \to +\infty} \inf_{\{\gamma \text{ ray}\}} \frac{1}{T} \int_{0}^{T} \chi_\omega(\gamma(t)) \, dt \right)$$

Two quantities:

- spectral
- geometric (rays)

↓

randomized obs. constant

(Humbert Privat Trélat, ongoing)
Thank you for your attention