Inverse Problem for Kinetic Equations

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Mathematics in Optical Imaging
IMA, Minnesota
April 29, 2019

Outline

- Basic Setting;
- Examples: radiative transfer equations (RTE) in 2D
  - Recovery of the absorption coefficient: linear and nonlinear
    - Applications of energy estimates and maximum principle
  - Recovery of the scattering coefficient
    - Application of the velocity averaging lemma
- Future Questions.

Basic Setting

- Consider the stationary RTE

$$v \cdot \nabla_x f = -\sigma_a f + F_f(x)\,.$$

- $(x, v) \in \Omega \times \mathbb{S}^1$.
- $\Omega \subseteq \mathbb{R}^2$ bounded, convex, C^1 boundary.
- $0 \leq \sigma_a = \sigma_a(x) \in L^\infty(\Omega)$.
- F_f depends on the moment(s) of f, where a k^{th} moment of f is given by

$$\int_{\mathbb{V}} v^\alpha f \,\mathrm{d}v\,, \qquad |\alpha| = k\,.$$

- $\langle f \rangle = \int_{\mathbb{S}^1} f \,\mathrm{d}v$: the zeroth moment of f.

Two Examples

- Linear RTE:

$$v \cdot \nabla_x f = -\sigma_a f + \sigma_s \langle f \rangle \,,$$

where $0 \le \sigma_s \le \sigma_a$.

- Nonlinear RTE[1]:

$$v \cdot \nabla_x I = -\sigma_a I + \sigma_a T^4 \,,$$
$$\Delta_x T = \sigma_a T^4 - \sigma_a \langle I \rangle \,.$$

where I is the intensity and T is the temprature.

[1] Chandrasekhar, Radiative Transfer, Oxford University Press, 1950.

Main Framework

- Two components of our method:
  - Isolate the leading order behaviour: singular decomposition and/or iteration .[2]
  - Show that the remainder terms are regular enough to be of higher-order: energy method, velocity averaging lemma etc.

[2] Choulli, Stefanov, Uhlmann, Bal, Monard, Langmore, Ren, $\cdots$

Part I: Recover the Absorption Coefficient

Notation

- $\Gamma_- = \{(x,v)|\, x \in \partial\Omega,\ \ n(x)\cdot v < 0\} \subseteq \partial\Omega \times \mathbb{S}^1$.
- $\Gamma_+ = \{(x,v)|\, x \in \partial\Omega,\ \ n(x)\cdot v > 0\} \subseteq \partial\Omega \times \mathbb{S}^1$.
- $\tau_\pm(x,v)$: forward/backward exit times from point x along the direction v.

Absorption Coefficient: General RTE

- Equation

$$v \cdot \nabla_x f = -\sigma_a f + F_f(x)\,, \tag{0.1}$$

Theorem (Li-S., 2019)

Let Ω be a convex and bounded domain with a C^1 boundary. Suppose $\sigma_a \geq 0$ and $\sigma_a \in C(\bar{\Omega})$. Suppose there exists $p > 1$ such that for any given incoming data ϕ satisfying

$$\phi \in L^p(\Gamma_-)\,, \qquad \phi \geq 0\,,$$

equation (0.7) *has a unique nonnegative solution with the bound*

$$\|F_f\|_{L^p(\Omega\times\mathbb{S}^1)} \leq C_0\,\|\phi\|_{L^p(\Gamma_-)}\,,$$

where C_0 is independent of ϕ and f. Then with proper choices of the incoming data, the absorption coefficient σ_a can be uniquely recovered from the measurement of the outgoing data.

Application: Linear RTE

- Two conditions to verify
  - Forward problem with incoming data is well-posed;
  - The forcing F_f is isotropic and for some $p > 1$,

$$\|F_f\|_{L^p(\Omega\times\mathbb{S}^1)} \leq C_0 \, \|\phi\|_{L^p(\Gamma_-)} \, .$$

- Forcing in the linear RTE

$$F_f((x) = \sigma_s(x)\langle f\rangle(x) \, .$$

- Given $\sigma_s \in L^\infty$, we have

$$\|F_f\|_{L^p(\Omega)} \leq \|\sigma_s\|_{L^\infty} \, \|f\|_{L^p(\mathrm{d}x\,\mathrm{d}v)} \leq \|\sigma_s\|_{L^\infty} \, \|\phi\|_{L^p(\Gamma_-)} \, .$$

- Conclusion: σ_a can be uniquely recovered in the linear RTE.

Application: Nonlinear RTE

- Equation with boundary data

$$v \cdot \nabla_x I = -\sigma_a I + \sigma_a T^4\,, \qquad I\Big|_{\Gamma_-} = \phi(x, v)\,, \tag{0.2}$$

$$\Delta_x T = \sigma_a T^4 - \sigma_a \langle I \rangle, \qquad T\Big|_{\partial\Omega} = 0\,. \tag{0.3}$$

- Forcing: $F_f = \sigma_a T^4$.

Theorem (Well-posedness and bound of forcing)

Suppose $\phi \in L^\infty(\Gamma_-)$ and $\phi \geq 0$ Then

(a) System (0.2)-(0.3) *has a unique solution satisfying that $I \geq 0$, $T \geq 0$,*

$$\|I\|_{L^\infty(\Omega\times\mathbb{S}^1)} \leq \|\phi\|_{L^\infty(\Gamma_-)}\,, \qquad \|T\|_{L^\infty(\Omega)} \leq \|\phi\|^{1/4}_{L^\infty(\Gamma_-)}\,.$$

(b) There exists a constant c_0 independent of ϕ such that

$$\left\|\sigma_a T^4\right\|_{L^2(\Omega)} \leq c_0\, \|\phi\|_{L^2(\Gamma_-)}\,.$$

Nonlinear RTE: Well-posedness

- Main tools
  - Schauder fixed-point
  - Maximum principle for elliptic and RTE
  - Monotonicity argument for elliptic equations.
- Solution set

$$\mathcal{D} = \left\{ T \middle| \ 0 \leq T \leq \|\phi\|_{L^\infty}^{1/4} \right\} \subseteq L^\infty(\Omega) \,.$$

- Mapping: $\mathcal{F} : \mathcal{D} \to \mathcal{D}$
  - Given $H \in \mathcal{D}$, let I_H be the solution such that

$$v \cdot \nabla_x I_H = -\sigma_a I_H + \sigma_a H^4 \,, \qquad I_H\big|_{\Gamma_-} = \phi(x, v) \,.$$

  - Define $\mathcal{F}(H) = T$ where T is the solution to the equation

$$\Delta_x T = \sigma_a T^4 - \sigma_a \langle I_H \rangle, \qquad T\big|_{\partial\Omega} = 0 \,.$$

Nonlinear RTE: Well-posedness

- Maximum principle for I_H:

$$v \cdot \nabla_x I_H = -\sigma_a I_H + \underbrace{\sigma_a H^4}_{\geq 0}, \qquad I_H\big|_{\Gamma_-} = \phi(x,v) \geq 0\,.$$

which gives that $I_H \geq 0$.

- Maximum principle for $I_H - \|\phi\|_{L^\infty}$:

$$v \cdot \nabla_x \left(I_H - \|\phi\|_{L^\infty}\right) = -\sigma_a (I_H - \|\phi\|_{L^\infty}) + \underbrace{\sigma_a \left(H^4 - \|\phi\|_{L^\infty}\right)}_{\leq 0},$$

$$\left(I_H - \|\phi\|_{L^\infty}\right)\big|_{\Gamma_-} = \underbrace{\phi(x,v) - \|\phi\|_{L^\infty}}_{\leq 0}\,.$$

which implies that $I_H \leq \|\phi\|_{L^\infty}$.

- Overall, we have

$$0 \leq I_H \leq \|\phi\|_{L^\infty}\,.$$

Nonlinear RTE: Well-posedness

- Equation for T (given $0 \le I_H \le \|\phi\|_{L^\infty}$)

$$-\Delta_x T = -\sigma_a T^4 + \sigma_a \langle I_H \rangle, \qquad T\Big|_{\partial\Omega} = 0\,. \tag{0.4}$$

- Monotonicity Method
  - Find an increasing sequence T_k

$$0 \le \underline{T} \le T_1 \le \cdots \le T_k \le \cdots \le \overline{T}$$

    where $\underline{T}$ is a subsolution and $\overline{T}$ is a supersolution.
  - Show that $T_k \to T$ and T solves (0.4).

- Maximum principle: at x_0 where T attains its maximum, we have

$$0 \le -\Delta_x T(x_0) = -\sigma_a T^4(x_0) + \sigma_a \langle I_H \rangle(x_0)\,,$$

which implies $T \le \|\phi\|_{L^\infty}^{1/4}$.

Bound of Forcing

- Recall the equation for (I, T)

$$v \cdot \nabla_x I = -\sigma_a I + \sigma_a T^4 , \qquad I\Big|_{\Gamma_-} = \phi(x, v) , \tag{0.5}$$

$$\Delta_x T = \sigma_a T^4 - \sigma_a \langle I \rangle , \qquad T\Big|_{\partial\Omega} = 0 . \tag{0.6}$$

- We use the energy method to show that

$$\left\| \sigma_a T^4 \right\|_{L^2(\Omega)} \leq c_0 \left\| \phi \right\|_{L^2(\Gamma_-)} .$$

- Basic energy estimate[3]

$$\frac{1}{2} \int_{\partial\Omega} \int_{\mathbb{S}^1} (n(x) \cdot v) I^2 + 4 \int_\Omega T^3 \left| \nabla_x T \right|^2 + \int_\Omega \int_{\mathbb{S}^1} \sigma_a \left(\langle I \rangle - T^4 \right)^2 \leq 0 .$$

which implies that

$$\int_\Omega \int_{\mathbb{S}^1} \sigma_a \left(\langle I \rangle - T^4 \right)^2 \mathrm{d}x \, \mathrm{d}v \leq -\frac{1}{2} \int_{\partial\Omega} \int_{\mathbb{S}^1} (n(x) \cdot v) I^2 \, \mathrm{d}v \, \mathrm{d}S_x$$

$$= -\frac{1}{2} \left(\iint_{\Gamma_+} + \iint_{\Gamma_-} \right) \leq \frac{1}{2} \left\| \phi \right\|^2_{\Gamma_-} .$$

[3]Klar-Schmeiser, M3AS, 2001.

Bound of Forcing

- Rewrite the I-equation as

$$v \cdot \nabla_x I = -\sigma_a (I - \langle I \rangle) + \underbrace{\sigma_a (T^4 - \langle I \rangle)}_{forcing} .$$

- Energy Estimate:

$$\|I\|_{L^2(\Omega \times \mathbb{S}^1)} \leq c_0 \left(\left\| \sigma_a (T^4 - \langle I \rangle) \right\|_{L^2(\Omega)} + \|\phi\|_{L^2(\Gamma_-)} \right) \leq c_0 \|\phi\|_{L^2(\Gamma_-)}$$

- Combine the above estimates

$$\begin{aligned} \left\| \sigma_a T^4 \right\|_{L^2(\Omega)} &\leq \left\| \sigma_a (T^4 - \langle I \rangle) \right\|_{L^2(\Omega)} + \|\langle I \rangle\|_{L^2(\Omega)} \\ &\leq \left\| \sigma_a (T^4 - \langle I \rangle) \right\|_{L^2(\Omega)} + \|I\|_{L^2(\Omega \times \mathbb{S}^1)} \\ &\leq c_0 \|\phi\|_{L^2(\Gamma_-)} . \end{aligned}$$

which gives the desired forcing bound.

Absorption Coefficient: General RTE

- Equation

$$v \cdot \nabla_x f = -\sigma_a f + F_f(x)\,, \tag{0.7}$$

Theorem (Li-S., 2019)

Let Ω be a convex and bounded domain with a C^1 boundary. Suppose $\sigma_a \geq 0$ and $\sigma_a \in C(\bar{\Omega})$. Suppose there exists $p > 1$ such that for any given incoming data ϕ satisfying

$$\phi \in L^p(\Gamma_-)\,, \qquad \phi \geq 0\,,$$

equation (0.7) *has a unique nonnegative solution with the bound*

$$\|F_f\|_{L^p(\Omega\times\mathbb{S}^1)} \leq C_0 \, \|\phi\|_{L^p(\Gamma_-)}\ ,$$

where C_0 is independent of ϕ and f. Then with proper choices of the incoming data, the absorption coefficient σ_a can be uniquely recovered from the measurement of the outgoing data.

Idea of Proof

- Incoming data ϕ:

$$\phi(x,v) = \frac{1}{\epsilon\delta}\phi_0\left(\frac{|x-x^{in}|}{\epsilon}\right)\phi_0\left(\frac{|v-v^{in}|}{\delta}\right), \qquad (x,v)\in\Gamma_- .$$

  where
  - $\phi_0 \in C^\infty(\mathbb{R})$ is a compactly supported bump function

$$\phi_0(r) = \begin{cases} 1, & |r| \le 1\,, \\ 0, & |r| \ge 2\,, \\ \text{smooth on } \mathbb{R}\,. \end{cases}$$

  - $\int_{\mathbb{R}} \phi_0(r)\,\mathrm{d}r = 1$.
  - $(x^{in}, v^{in}) \in \Gamma_-$ such that $n(x^{in})\cdot v^{in} < 0$.
- Bound: $\|\phi\|_{L^p(\Gamma_-)} \le c_0\epsilon^{-(1-1/p)}\delta^{-(1-1/p)}$ for any $p \ge 1$.

Measurement

- Total measurement

$$M_\psi = \iint_{\Gamma_+} \psi(x,v) f(x,v)\, \mathrm{d}\Gamma_+ , \qquad \mathrm{d}\Gamma = n(x)\cdot v\, \mathrm{d}S_x\, \mathrm{d}v .$$

where

- the measurement function ψ is

$$\psi(x,v) = \psi_0(x - x^{out})\psi_0\left(\frac{v - v^{out}}{\delta}\right) \qquad (x,v) \in \Gamma_+ .$$

- Choice of (x^{out}, v^{out}):

$$x^{out} = x^{in} + \tau_+(x^{in}, v^{in}) v^{in} , \qquad v^{out} = v^{in} .$$

Decomposition

- Full equation

$$\begin{aligned} v \cdot \nabla_x f &= -\sigma_a f + F_f(x)\,, && x \in \Omega \subseteq \mathbb{R}^2\,, \\ f\big|_{\Gamma_-} &= \phi(x,v)\,, && (x,v) \in \Gamma_-\,. \end{aligned}$$

- Decompose the solution as $f = f_1 + f_2$.
- Leading-order

$$\begin{aligned} v \cdot \nabla_x f_1 &= -\sigma_a f_1\,, && x \in \Omega \subseteq \mathbb{R}^2\,, \\ f_1\big|_{\Gamma_-} &= \phi(x,v)\,, && (x,v) \in \Gamma_-\,. \end{aligned}$$

- Remainder

$$\begin{aligned} v \cdot \nabla_x f_2 &= -\sigma_a f_2 + F_f(x)\,, && x \in \Omega \subseteq \mathbb{R}^2\,, \\ f_2\big|_{\Gamma_-} &= 0\,, && (x,v) \in \Gamma_-\,. \end{aligned}$$

Leading Order

- Formula for f_1:

$$f_1(x,v) = e^{-\int_0^{\tau_-(x,v)} \sigma_a(x-sv)\,\mathrm{d}s}\phi(x-\tau_-(x,v)v, v)\,,$$

for all $(x,v) \in \bar{\Omega} \times \mathbb{S}^1$.

- Contribution of f_1 to the measurement:

$$M_\psi(f_1) = \iint_{\Gamma_+} \psi(x,v) f_1(x,v)\,\mathrm{d}\Gamma_+ = \frac{1}{\epsilon}\int_{\partial\Omega} \psi_0(x - x^{out})\, G_{\epsilon,\delta}(x)\,\mathrm{d}S_x\,.$$

where

$$G_{\epsilon,\delta}(x) = \frac{1}{\delta}\int_{\mathbb{S}^1_{x,+}} e^{-\int_0^{\tau_-(x,v)} \sigma_a(x-sv)\,\mathrm{d}s}\psi_0\left(\frac{|v-v^{in}|}{\delta}\right) \times \phi_0\left(\frac{|x-\tau_-(x,v)v - x^{in}|}{\epsilon}\right)\phi_0\left(\frac{|v-v^{in}|}{\delta}\right) n(x)\cdot v\,\mathrm{d}v\,.$$

- Contribution from f_1:

$$\lim_{\epsilon\to 0}\lim_{\delta\to 0} M_\psi(f_1) = C_{\psi_0,\phi_0}\, e^{-\int_0^{\tau_-(x^{out},v^{out})} \sigma_a(x^{out}-sv^{out})\,\mathrm{d}s}\,.$$

Remainder

- Goal: show that $\lim_{\epsilon\to 0}\lim_{\delta\to 0} M_\psi(f_2) = 0$.
- Formula for f_2

$$f_2(x,v) = \int_0^{\tau_-(x,v)} e^{-\int_0^s \sigma_a(x-\tau v)\,\mathrm{d}\tau} F_f(x-sv)\,\mathrm{d}s\,,$$

  for all $(x,v)\in\bar{\Omega}\times\mathbb{S}^1$.

- Contribution from f_2

$$\begin{aligned} M_\psi(f_2) &= \iint_{\Gamma_+} \psi(x,v) f_2(x,v)\,\mathrm{d}\Gamma_+ \\ &= \iint_{\Gamma_+} \int_0^{\tau_-(x,v)} \psi(x,v) e^{-\int_0^s \sigma_a(x-\tau v)\,\mathrm{d}\tau} F_f(x-sv) n(x)\cdot v\,\mathrm{d}s\,\mathrm{d}S_x\,\mathrm{d}v\,, \end{aligned}$$

- Main ingredients for smallness of $M_\psi(f_2)$
  - Regularity and isotropy of F_f
  - Smallness of the support of integration.

Remainder

- Recall that by assumption,

$$\|F_f\|_{L^p(\Omega\times\mathbb{S}^1)} \le C_0 \|\phi\|_{L^p(\Gamma_-)}\,, \quad \psi(x,v) = \psi_0(x - x^{out})\psi_0\left(\frac{v - v^{out}}{\delta}\right).$$

- Change of variables: $y = x - sv$ such that

$$\iint_{\Gamma_+}\int_0^{\tau_-(x,v)} \cdots n(x)\cdot v\,\mathrm{d}s\,\mathrm{d}S_x\,\mathrm{d}v = \int_{\mathbb{S}^1}\int_\Omega \cdots \mathrm{d}y\,\mathrm{d}v\,.$$

- Estimate

$$\begin{aligned}
&\left|\iint_{\Gamma_+}\psi(x,v)f_2(x,v)\,\mathrm{d}\Gamma\right| \\
&= \iint_{\Gamma_+}\int_0^{\tau_-(x,v)}\psi(x,v)e^{-\int_0^s\sigma_a(x-\tau v)\,\mathrm{d}\tau}F_f(x-sv)n(x)\cdot v\,\mathrm{d}s\,\mathrm{d}S_x\,\mathrm{d}v \\
&\le \int_{\mathbb{S}^1}\int_\Omega \psi_0\left(\frac{v-v^{in}}{\delta}\right)\left|F_f(y)\right|\,\mathrm{d}y\,\mathrm{d}v && \text{(Change of Variable)} \\
&= \left(\int_{\mathbb{S}^1}\psi_0\left(\frac{v-v^{in}}{\delta}\right)\mathrm{d}v\right)\left(\int_\Omega\left|F_f(y)\right|\,\mathrm{d}y\right) && \text{(Isotropy)} \\
&\le c_0\delta\,\|F_f\|_{L^p(\Omega)} \le c_0\delta\,\|\phi\|_{L^p(\Gamma_-)} \le c_0\epsilon^{-(1-1/p)}\delta^{1/p}\,. && \text{(Regularity)}
\end{aligned}$$

- $\lim_{\epsilon\to 0}\lim_{\delta\to 0} M_\psi(f_2) = 0.$

Part II: Recover the Scattering Coefficient in 2D

RTE: Scattering Coefficient

- Equation

$$v \cdot \nabla_x f = -\sigma_a f + \sigma_s \langle f \rangle \,, \tag{0.8}$$

Theorem (Li-S., 2019)

Let $\Omega \subseteq \mathbb{R}^2$ be a convex and bounded domain with a C^1 boundary. Suppose $\sigma_a \in C(\bar{\Omega})$ is given and

$$\sigma_s = \sigma_s(x) \,, \qquad 0 < \sigma_0 \leq \sigma_s \leq \sigma_a \,.$$

Then with proper choices of the incoming data, the scattering coefficient σ_s in (0.8) can be uniquely recovered from the measurement of the outgoing data.

Main Decomposition

- Decompose the solution f into three parts: $f = f_1 + f_2 + f_3$ where

$$
\begin{aligned}
v \cdot \nabla_x f_1 &= -\sigma_a f_1 \,, \qquad && f_1\big|_{\Gamma_-} = \phi(x, v) \,, \\
v \cdot \nabla_x f_2 &= -\sigma_a f_2 + \sigma_s \langle f_1 \rangle \,, \qquad && f_2\big|_{\Gamma_-} = 0 \,, \\
v \cdot \nabla_x f_3 &= -\sigma_a f_3 + \sigma_s \langle f_3 \rangle + \sigma_s \langle f_2 \rangle \,, \qquad && f_3\big|_{\Gamma_-} = 0 \,,
\end{aligned}
$$

- Let M_ψ be the measurement of the outgoing data.
- Main idea: Choose ϕ and ψ in the way such that
  - $M_\psi(f_1) = 0$;
  - $M_\psi(f_2) = \mathcal{O}(1)$;
  - $M_\psi(f_3)$ is of higher order due to the regularity of f_3.
- Use the averaging lemma to show that $\langle f_2 \rangle$ is more regular than f_2.

Averaging Lemma (Simplest Case)

Theorem (Golse-Lions-Perthame-Sentis, JFA, 1988)

Suppose $0 < \sigma_0 \le \sigma_s \le \sigma_a$ *with* $\sigma_a \in C(\bar{\Omega})$. *Suppose* $\phi \in L^p(\Gamma_-)$ *for some* $p > 1$ *and* f *satisfies the equation*

$$v \cdot \nabla_x f = -\sigma_a f + \sigma_s \langle f \rangle + g\,, \qquad f\big|_{\Gamma_-} = \phi(x, v)\,. \tag{0.9}$$

Then for any $\gamma \le \inf\{\frac{1}{p}, 1 - \frac{1}{p}\}$, *the velocity average of* f *satisfies* $\langle f \rangle \in W^{\gamma,p}(\Omega)$ *with the bound*

$$\|\langle f \rangle\|_{W^{\gamma,p}(\Omega)} \le C_0 \left(\|\phi\|_{L^p(\Gamma_-)} + \|g\|_{L^p(\Omega \times \mathbb{S}^1)} \right).$$

In particular, by Sobolev embedding, we have $\langle f \rangle \in L^q(\Omega)$ *where*

$$q = \frac{1}{\frac{1}{p} - \frac{\gamma}{d}} > p\,.$$

Application of Averaging Lemma

- Equations for f_1, f_2, f_3:

$$
\begin{aligned}
v \cdot \nabla_x f_1 &= -\sigma_a f_1 \,, \qquad & f_1\big|_{\Gamma_-} &= \phi(x,v)\,, \\
v \cdot \nabla_x f_2 &= -\sigma_a f_2 + \sigma_s \langle f_1 \rangle \,, \qquad & f_2\big|_{\Gamma_-} &= 0\,, \\
v \cdot \nabla_x f_3 &= -\sigma_a f_3 + \sigma_s \langle f_3 \rangle + \sigma_s \langle f_2 \rangle \,, \qquad & f_3\big|_{\Gamma_-} &= 0\,,
\end{aligned}
$$

- Estimates using the averaging lemma:

$$
\|\langle f_1 \rangle\|_{L^{p_1}(\Omega)} \le c_0 \,\|\langle f_1 \rangle\|_{W^{s_0,r}} \le c_0 \,\|\phi\|_{L^r(\Gamma_-)} \le c_0 \,\epsilon^{-\frac{r-1}{r}} \delta^{-\frac{r-1}{r}} \eta^{\frac{1}{r}} \,,
$$

$$
\|\langle f_2 \rangle\|_{L^{p_2}(\Omega)} \le c_0 \,\|\langle f_2 \rangle\|_{W^{s_1,p_1}(\Omega)} \le c_0 \,\|\langle f_1 \rangle\|_{L^{p_1}(\Omega \times \mathbb{S}^1)} \le c_0 \epsilon^{-\frac{r-1}{r}} \delta^{-\frac{r-1}{r}} \eta^{\frac{1}{r}} \,,
$$

where

$$
s_0 = \inf\left\{\frac{1}{r}, 1 - \frac{1}{r}\right\}, \quad s_1 = \inf\left\{\frac{1}{p_1}, 1 - \frac{1}{p_1}\right\},
$$

$$
p_1 = \frac{1}{\frac{1}{r} - \frac{s_0}{2}} > r \,, \quad p_2 = \frac{1}{\frac{1}{p_1} - \frac{s_1}{2}} > p_1 > r \,.
$$

Data and Measurement

- Let $(x^{in}, v^{in}) \in \Gamma_-$ and $(x^{out}, v^{out}) \in \Gamma_+$ such that $v^{in} \nparallel v^{out}$.
- Let ϕ and ψ be given by [4]

$$\phi(x,v) = \frac{1}{\epsilon\delta}\phi_0\left(\frac{(x - x^{in}) \cdot v_\perp^{in}}{\epsilon\eta}\right)\phi_0\left(\frac{|v - v^{in}|}{\delta}\right), \qquad (x,v) \in \Gamma_-,$$

$$\psi(x,v) = \frac{1}{\theta\beta}\psi_0\left(\frac{|x - x^{out}|}{\theta}\right)\psi_0\left(\frac{|v - v^{out}|}{\beta}\right), \qquad (x,v) \in \Gamma_+.$$

where ϕ_0, ψ_0 are bump functions and

$$v_\perp^{in} \cdot v^{in} = 0 \quad \text{and} \quad \eta = v_\perp^{in} \cdot v^{out} > 0\,.$$

- For $r > 1$ to be chosen later, we have

$$\|\phi\|_{L^r(\Gamma_-)} \le c_0\, \epsilon^{-\frac{r-1}{r}} \delta^{-\frac{r-1}{r}} \eta^{\frac{1}{r}}, \qquad r > 1\,.$$

[4]Choulli-Stefanov, Osaka J. Math., 1998.

Contribution from f_3

- Equation

$$v \cdot \nabla_x f_3 = -\sigma_a f_3 + \underbrace{\sigma_s \langle f_3 \rangle + \sigma_s \langle f_2 \rangle}_{\textit{forcing}}, \qquad f_3\big|_{\Gamma_-} = 0\,,$$

- Contribution to the measurement $M_\psi(f_3)$

$$\left| \iint_{\Gamma_+} \psi(x,v) f_3(x,v)\,\mathrm{d}\Gamma \right| \le c_0 \int_{\mathbb{S}^1} \int_\Omega \psi(y + \tau_+(y,v)v) \left| \langle f_2 \rangle(y) + \langle f_3 \rangle(y) \right| \mathrm{d}y\,\mathrm{d}v$$
$$\le \frac{c_0}{\theta} \left(\int_{\partial\Omega} \psi_0^{p_2'} \left(\frac{|x - x^{out}|}{\theta} \right) \mathrm{d}S_x \right)^{1/p_2'} \|\langle f_2 \rangle\|_{L^{p_2}(\Omega)}$$
$$\le c_0 \theta^{-\frac{1}{p_2}} \epsilon^{-\frac{r-1}{r}} \delta^{-\frac{r-1}{r}} \eta^{\frac{1}{r}}\,.$$

- We will choose proper parameters to make $M_\psi(f_3)$ small.

Contribution from f_1

- Equation

$$v \cdot \nabla_x f_1 = -\sigma_a f_1 \,, \qquad f_1\big|_{\Gamma_-} = \phi(x,v)\,,$$

- Contribution to the measurement $M_\psi(f_1)$

$$\mathcal{M}_1 = \frac{1}{\epsilon\delta}\frac{1}{\theta\beta}\iint_{\Gamma_+} e^{-\int_0^{\tau_-(x,v)}\sigma_a(x-sv)\,\mathrm{d}s}\psi_0\left(\frac{|x-x^{out}|}{\theta}\right)\psi_0\left(\frac{|v-v^{out}|}{\beta}\right)$$
$$\times\,\phi_0\left(\frac{(x-\tau_-(x,v)v-x^{in})\cdot v_\perp^{in}}{\epsilon\eta}\right)\phi_0\left(\frac{|v-v^{in}|}{\delta}\right)\mathrm{d}\Gamma_+\,.$$

- Make $M_\psi(f_1) = 0$ by letting

$$\mathrm{supp}\left(\psi_0\left(\frac{|x-x^{out}|}{\theta}\right)\right)\cap\mathrm{supp}\left(\phi_0\left(\frac{(x-\tau_-(x,v)v-x^{in})\cdot v_\perp^{in}}{\epsilon\eta}\right)\right) = \emptyset\,.$$

- Sufficient condition

$$\left|x^{out} - x_0^{out}\right| \geq c_0\left(\theta+\delta+\epsilon\eta\right)\,.$$

Contribution from f_2

- Equation

$$v \cdot \nabla_x f_2 = -\sigma_a f_2 + \sigma_s \langle f_1 \rangle \,, \qquad f_2\big|_{\Gamma_-} = 0 \,,$$

- Contribution to the measurement $M_\psi(f_2)$

$$M_\psi(f_2) = \iint_{\Gamma_+} \int_0^{\tau_-(x,v)} \psi(x,v) e^{-\int_0^s \sigma_a(x-\tau v)\,\mathrm{d}\tau} \sigma_s(x-sv) \langle f_1 \rangle (x-sv)\,\mathrm{d}s\,\mathrm{d}\Gamma$$
$$\sim \frac{1}{\epsilon\delta} \frac{1}{\theta\beta} \iint_{\Gamma_+} \int_0^{\tau_-(x,v)} \int_{\mathbb{S}^1} H(s, x^{out}, v^{out}, v^{in}) \psi_0\left(\frac{|x - x^{out}|}{\theta}\right) \psi_0\left(\frac{|v - v^{out}|}{\beta}\right)$$
$$\times \phi_0\left(\frac{((x-sv)'_w - x^{in}) \cdot v_\perp^{in}}{\epsilon\eta}\right) \phi_0\left(\frac{|w - v^{in}|}{\delta}\right) \mathrm{d}w\,\mathrm{d}s\,\mathrm{d}\Gamma_+$$

- $H(s,x,v,w) = e^{-\int_0^s \sigma_a(x-\tau v)\,\mathrm{d}\tau} e^{-\int_0^{\tau_-(x-sv,w)} \sigma_a(x-sv-\tau w)\,\mathrm{d}\tau} \sigma_s(x-sv)$.
- $(x-sw)'_v$ is the backward exit point of $x - sw$ along the direction v.

Contribution from f_2

- Focus on the main term

$$\phi_0\left(\frac{((x-sv)'_w - x^{in})\cdot v_\perp^{in}}{\epsilon\eta}\right), \qquad \eta = v_\perp^{in}\cdot v^{out} > 0\,.$$

- Given that $(x, v, w) \sim (x^{out}, v^{out}, v^{in})$, we have

$$(x-sv)'_w \sim (x^{out} - sv^{out}) - \tau_-(x^{out} - sv^{out}, v^{in})v^{in}$$

Together with

$$x^{in} = (x^{out} - s_0 v^{out}) - \tau_-(x^{out} - s_0 v^{out}, v^{in})v^{in}$$

we have

$$\frac{\left|((x-sv)'_w - x^{in})\cdot v_\perp^{in}\right|}{\epsilon\eta} = \frac{|s-s_0|}{\epsilon}\,.$$

- $\lim_{\epsilon,\delta,\theta,\beta\to 0} \mathcal{M}_{2,1} = e^{-\int_0^{\tau_-(x_0,v^{in})}\sigma_a(x_0-\tau x^{in})\,\mathrm{d}\tau}\sigma_s(x_0)$.

Choice of Parameters

- Constraints of parameters

$$\theta^{-\frac{1}{p_2}}\epsilon^{-\frac{r-1}{r}}\delta^{-\frac{r-1}{r}}\eta^{\frac{1}{r}} \ll 1\,, \qquad \theta+\delta+\beta \ll \epsilon\eta\,, \qquad \beta \ll \theta\,. \qquad (0.10)$$

- Borderline case: let

$$\theta = \epsilon\eta\,, \qquad \theta = \delta = \beta\,.$$

In this case we require

$$\frac{1}{p_2} + \frac{2(r-1)}{r} < 1\,.$$

- Example: take $r \in (1, 3/2)$ to obtain that

$$s_0 = 1 - \frac{1}{r}\,, \quad p_1 = \frac{2r}{3-r} < 2\,, \quad s_2 = 1 - \frac{1}{p_1}\,, \quad \frac{1}{p_2} = \frac{3-p_1}{2p_1} = \frac{9}{4r} - \frac{5}{4}\,,$$

which gives

$$\frac{1}{p_2} + \frac{2(r-1)}{r} = 1 + \frac{1}{4}\left(\frac{1}{r} - 1\right) < 1\,.$$

Future Questions

- To name a few...
  - Time-dependent transport equations;
  - Other types of kinetic equations: BGK, Vlasov-type.
  - Multi-scale problems.