Classical knot invariants

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A knot $K$ in $S^3$ is a smooth isotopy class of smooth embeddings of $S^1$ into $S^3$. 

Definition
Figure: The unknot
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Knots

Figure: A knot
Figure: Two knots
Figure: The connected sum of two knots
Crossing number
Definition

The *crossing number* of a knot $K$ is the least number of crossings in a diagram of $K$. 

Example 1: The unknot has crossing number 0.
Definition

The *crossing number* of a knot \( K \) is the least number of crossings in a diagram of \( K \).

Example 1: The unknot has crossing number 0.
Example 2: The trefoil has crossing number 3.
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This can be seen by checking all diagrams with crossing number 0, 1, 2 and noting that they do not give the trefoil. (Except that you need to know that the trefoil is truly knotted.)
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**Theorem**

*A reduced alternating diagram of a knot $K$ realizes the crossing number of $K.*
Fun interlude

Figure: $T(2, 3)$, also known as the trefoil.
Question 1: What is the crossing number of a torus knot, \( T(p, q) \)?
Fun interlude

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Theorem:

$$c(T(p, q)) = \min\{q(p - 1), p(q - 1)\}$$
Fun interlude

Question 2: What is known about the crossing number of a satellite knot?
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Nothing!
Bridge number
Definition

A *height function* on $\mathbb{R}^3$ (or $S^3$) is a smooth function

$$h : \mathbb{R}^3 \to \mathbb{R}$$

(or $h : S^3 \to \mathbb{R}$) without critical points.
A \textit{height function} on $\mathbb{R}^3$ (or $\mathbb{S}^3$) is a smooth function $h : \mathbb{R}^3 \rightarrow \mathbb{R}$ (or $h : \mathbb{S}^3 \rightarrow \mathbb{R}$) without critical points.

Let $K$ be a knot. The \textit{bridge number}, $b(K)$, of $K$, is the least possible number of maxima of $K$ with respect to a height function.
Example 1: The unknot has bridge number 1.
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In fact, the unknot is the only knot with bridge number 1.
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Example 2: The trefoil (a nontrivial knot) has bridge number 2.
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Example 2: The trefoil (a nontrivial knot) has bridge number 2.

Theorem

(Schubert): $b(K_1 \# K_2) = b(K_1) + b(K_2) - 1$
Bridge numbers of knots

Figure: Schematic for bridge number inequality
Fun interlude

**Figure:** $T(2, 3)$, also known as the trefoil.
Question 1: What is the bridge number of a torus knot $T(p, q)$?

\[ \min(p, q). \]

One direction is easy, the other is a theorem of Schubert.

Question 2: What can we say about the bridge number of a satellite knot?
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Answer: $\min(p, q)$. 

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Question 2: What can we say about the bridge number of a satellite knot?

Answer: It's at least the bridge number of the companion knot times the wrapping number of the pattern.
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Tunnel number
Preliminaries for tunnel number

Figure: A handlebody
Preliminaries for tunnel number

Figure: Another handlebody
Tunnel numbers of knots

Definition

A handlebody is a 3-dimensional regular neighborhood of a graph.
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Definition

Let $K$ be a knot. A \textit{tunnel system} for $K$ is a collection of arcs $a_1, \ldots, a_n$ properly embedded in $(S^3, K)$ such that $S^3 - \eta(K \cup a_1 \cup \cdots \cup a_n)$ is a handlebody.
Tunnel numbers of knots

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**Definition**

The *tunnel number* of $K$ is the least number of arcs in a tunnel system for $K$. 
The complement of the unknot is a solid torus, which is a 3-dimensional regular neighborhood of the circle (a graph), hence a handlebody.

Figure: The complement of the unknot
The complement of the unknot is a solid torus, which is a 3-dimensional regular neighborhood of the circle (a graph), hence a handlebody.

Why?
Preliminaries for tunnel number

\[ T^2 = \{(x, y, z, w) \in \mathbb{R}^4 | x^2 + y^2 = \frac{1}{2}, w^2 + z^2 = \frac{1}{2}\} \]
Preliminaries for tunnel number

\[ \mathbb{T}^2 = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 = \frac{1}{2}, w^2 + z^2 = \frac{1}{2}\} \]

The unknot is isotopic to

\[ \{(x, y, \frac{1}{\sqrt{2}}, 0) \in \mathbb{T}^2 \} \in \mathbb{S}^3 \]
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\[ \{(x, y, \frac{1}{\sqrt{2}}, 0) \in \mathbb{T}^2\} \in \mathbb{S}^3 \]

\[ \eta(\text{unknot}) \approx \{(x, y, z, w)| x^2 + y^2 = \frac{1}{2}, z > \frac{1}{2\sqrt{2}}, w < \frac{1}{2\sqrt{2}}\} \in \mathbb{S}^3 \]

\[ C(K) = \mathbb{S}^3 - \eta(\text{unknot}) \]

\[ \approx \{(x, y, z, w)| x^2 + y^2 = \frac{1}{2}, z \leq \frac{1}{2\sqrt{2}}, w \geq \frac{1}{2\sqrt{2}}\} \in \mathbb{S}^3 \]
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Example 2: The trefoil has tunnel number 1.
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Example 2: The trefoil has tunnel number 1.

Example 3: Every 2-bridge knot has a tunnel number 1.
Observation: $t(K) \leq b(K) - 1$
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Tunnel numbers of knots

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Observation: \( t(T(p, q)) = 1 \)

Question: What happens to tunnel number under the operation of connected sum?
Figure: The decomposing annulus
Tunnel numbers of knots

Theorem

\[ t(K \# K') \leq t(K) + t(K') + 1 \]
Theorem

(Morimoto) There is a knot, $K$, such that for any 2-bridge knot $K'$, $t(K \# K') = t(K')$

"2 + 1 = 1"
Tunnel numbers of knots

Theorem

(Morimoto-Sakuma-Yokota) There are knots, $K_1, K_2$, such that

$$t(K \# K') = t(K) + t(K') + 1$$

"$1 + 1 = 3$"
Tunnel numbers of knots

**Theorem**

*(Morimoto-S)* If $K_1$, $K_2$ and $K_2$ are small, then

$$t(K_1 \# K_2) \geq t(K_1) + t(K_2)$$
Theorem

(Scharlemann-S) If $K_1, \ldots, K_n$ are knots and no $K_i$ is a 2-bridge knot, then

$$t(K_1 \# \ldots \# K_n) \geq \frac{2}{5}(t(K_1) + \cdots + t(K_n))$$
Tunnel numbers of knots

**Theorem (Scharlemann-S)** If $K_1, \ldots, K_n$ are knots and no $K_i$ is a 2-bridge knot, then

$$t(K_1 \# \ldots \# K_n) \geq \frac{2}{5}(t(K_1) + \cdots + t(K_n))$$

**Theorem (Scharlemann-S)** If $K_1, \ldots, K_n$ are knots such that $K_1, \ldots, K_p$ are 2-bridge knots and $K_{p+1}, \ldots, K_n$ are not, then

$$t(K_1 \# \ldots \# K_n) \geq \frac{p}{5} + \frac{2}{5}(t(K_{p+1}) + \cdots + t(K_n))$$
Unknotting numbers of knots

Definition

The *unknotting number* of a knot $K$ is the least number of times $K$ needs to pass through itself to become the unknot.
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Example 1: The unknot has unknotting number 0.
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The *unknotting number* of a knot $K$ is the least number of times $K$ needs to pass through itself to become the unknot.

Example 1: The unknot has unknotting number 0.

Example 2: The trefoil has unknotting number 1.
Unknotting numbers of knots

Theorem

\[ u(K) \leq c(K) \]

Theorem

(Scharlemann) Unknotting number 1 knots are prime.
Construction: Given a knot $K$ in $S^3$, we watch $K$ unknot over time.
Unknotting numbers of knots

Construction: Given a knot $K$ in $\mathbb{S}^3$, we watch $K$ unknot over time.

In the product $\mathbb{S}^3 \times [0, 1]$, each $t \in [0, 1]$ corresponds to a particular time, and the knot to the level curve at that time.
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The level curves form a surface in $S^3 \times [0, 1] \subset B^4$. 
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Crossing changes correspond to critical levels in which a closed curve forms a figure 8.
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Above and below the figure 8, $K$ is a single simple closed curve.
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Crossing changes correspond to critical levels in which a closed curve forms a figure 8.

Above and below the figure 8, $K$ is a single simple closed curve.

THUS: A crossing change of $K$ corresponds to attaching a Möbius band to the surface.

The (nonorientable) genus of the surface is exactly the unknotting number of $K$. 
Observation: $u(K) \leq \frac{c(K)}{2}$
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Question: What happens to unknotting number under the operation of connected sum?
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Question: What happens to unknotting number under the operation of connected sum?

Answer: Nobody knows.