Surfaces and more

Jennifer Schultens
Definition

A *closed surface* is a compact space $S$ such that every point in $S$ has a neighborhood homeomorphic to $\mathbb{R}^2$. 
Figure: A polygonal representation of the torus
Theorem

Every closed surface has a polygonal representation.
Figure: A polygonal representation of the 2-sphere
Definition

Let $S$ be an orientable closed surface. The *Euler characteristic* of $S$ is the following number:

$$\#\text{vertices} - \#\text{edges} + \#\text{faces}$$

(This number is computed in the surface - the bottom picture - invariant under different choices of polygonal representations.)
Theorem

(Classification of surfaces) Every orientable closed surface is a connected sum of tori.
Theorem

(Classification of surfaces) Every orientable closed surface is a connected sum of tori.

Definition

Let $S$ be an orientable closed surface. The number of tori in a connected sum representing $S$ is called the genus of $S$. (It is uniquely determined.)
Theorem

(Classification of surfaces) Every orientable closed surface is a connected sum of tori.

Definition

Let $S$ be an orientable closed surface. The number of tori in a connected sum representing $S$ is called the \textit{genus} of $S$. (It is uniquely determined.)

Computation: If $S$ has genus $g$, then it is the quotient of a $4n$-gon.

$$\text{genus}(S) = g \implies \chi(S) = 2 - 2g$$
I spy an orientable closed surface with Euler characteristic -2.
I spy an orientable closed surface with Euler characteristic -2.

It’s a genus 2 surface!
I spy an orientable closed surface with Euler characteristic -2.

It’s a genus 2 surface!

I spy an orientable closed surface with Euler characteristic -12.
I spy an orientable closed surface with Euler characteristic -2.

It’s a genus 2 surface!

I spy an orientable closed surface with Euler characteristic -12.

It’s a genus 7 surface!
I spy an orientable closed surface with Euler characteristic -2.

It’s a genus 2 surface!

I spy an orientable closed surface with Euler characteristic -12.

It’s a genus 7 surface!

I spy an orientable closed surface with Euler characteristic -11.
I spy an orientable closed surface with Euler characteristic -2.

It’s a genus 2 surface!

I spy an orientable closed surface with Euler characteristic -12.

It’s a genus 7 surface!

I spy an orientable closed surface with Euler characteristic -11.

No you don’t!
Geometrization for surfaces
Definition

We say that a manifold $M$ exhibits a certain geometry if it is the quotient of the geometry by a “nice” group (i.e., a discrete group of isometries).
**Figure:** The torus is a quotient of the Cartesian plane by a translation group
Theorem

(Geometrization for surfaces) Any orientable closed surface is either spherical, Euclidean, or hyperbolic.
Circles inscribed in triangles in the hyperbolic plane are *not* arbitrarily large!
The curve complex
The curve complex, defined by Harvey in the 1980s, established itself as a tool for studying 3-manifolds through the work of Howie Masur, Yair Minsky, who showed that it is “quasi-hyperbolic” and of John Hempel who used it to analyze 3-manifolds.
The curve complex

Definition

A *simplicial complex* $\mathcal{K}$ is a set of simplices that satisfies the following:

- Every face of a simplex of $\mathcal{K}$ is also in $\mathcal{K}$;
- The intersection of any two simplices $\sigma_1, \sigma_2 \in \mathcal{K}$ is either empty or a face of both $\sigma_1$ and $\sigma_2$. 

Jennifer Schultens  
Surfaces and more
The curve complex

Definition

The *dimension* of a simplicial complex is the highest dimension occurring among its simplices.
The curve complex

**Definition**

The *dimension* of a simplicial complex is the highest dimension occurring among its simplices.

**Definition**

The *k*-skeleton of a simplicial complex is the subcomplex consisting of vertices, 1-simplices, . . . , *k*-simplices.
The curve complex

**Definition**

The *dimension* of a simplicial complex is the highest dimension occurring among its simplices.

**Definition**

The $k$-skeleton of a simplicial complex is the subcomplex consisting of vertices, 1-simplices, $\ldots$, $k$-simplices.

**Definition**

A simplicial complex $\mathcal{K}$ is *flag* if any simplex whose 1-skeleton lies in $\mathcal{K}$ is also in $\mathcal{K}$. 
The curve complex

**Definition**

Let $S$ be an orientable closed surface. The *curve complex* of $S$, denoted $\mathcal{K}(S)$, is the flag complex whose vertices are isotopy classes of essential simple closed curves in $S$ and whose edges are pairs of distinct vertices that admit disjoint representatives.
The curve complex

Definition
Let $S$ be an orientable closed surface. The curve complex of $S$, denoted $\mathcal{K}(S)$, is the flag complex whose vertices are isotopy classes of essential simple closed curves in $S$ and whose edges are pairs of distinct vertices that admit disjoint representatives.

Definition
The distance between two vertices $v, w$ in the curve complex is the number of edges in an edge-path between $v$ and $w$. 
Figure: A simplex in the curve complex of the genus 2 surface
Figure: A maximal simplex in the curve complex of the genus g surface
The curve complex

Theorem

\[ \dim(\mathcal{K}(S)) = 3\text{genus}(S) - 3 \]
Case study: The curve complex of the torus

Figure: The torus knot $T(2, 3)$
Observation 1: Torus knots correspond to rational numbers.
Observation 1: Torus knots correspond to rational numbers.

Observation 2: Non isotopic torus knots are not disjoint.
Observation 1: Torus knots correspond to rational numbers.

Observation 2: Non isotopic torus knots are not disjoint.

Observation 3: $T(p, q)$ and $T(r, s)$ meet in a single point $\iff ps - qr = \pm 1$
Case study: The curve complex of the torus

Definition

The (modified) curve complex of the torus, denoted $\mathcal{K}(T^2)$, is the flag complex whose vertices are isotopy classes of essential simple closed curves in the torus and whose edges are pairs of distinct vertices that admit representatives that meet exactly once non tangentially.
Case study: The curve complex of the torus

Figure: The Farey graph is the curve complex of the torus
Coarse geometry concerns properties of metric spaces from a ‘large scale’ point of view.
Definition

Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces. A quasi-isometry between \((X, d_X)\) and \((Y, d_Y)\) is a (not necessarily continuous) function

\[ f : X \rightarrow Y \]

such that there are constants \(A, C\) with

\[
\frac{1}{C}d(x, y) - A \leq d(f(x), f(y)) \leq Cd(x, y) + A
\]

If there is a quasi-isometry between \((X, d_X)\) and \((Y, d_Y)\), then we say that \((X, d_X)\) and \((Y, d_Y)\) are quasi-isometric.
Example 1: Every compact space is quasi-isometric to a point.
Example 1: Every compact space is quasi-isometric to a point.

Example 2: The ‘integer lattice’ is quasi-isometric to the Cartesian plane.
Example 1: Every compact space is quasi-isometric to a point.

Example 2: The ‘integer lattice’ is quasi-isometric to the Cartesian plane.

**Definition**

A space is *quasi-Euclidean* if it is quasi-isometric to the Cartesian plane.
Figure: The Cartesian plane with the integer lattice
Coarse geometry

Definition
A triangle in a metric space is $\delta$-thin if each edge lies in a $\delta$-neighborhood of the other two.
A $(X, d_X)$ Gromov hyperbolic if there exists a $\delta$ such that all triangles are $\delta$-thin.
Figure: A $\delta$-thin triangle
Theorem

(Masur-Minsky) The curve complex of an orientable closed surface of genus at least two is Gromov hyperbolic.
Definition

Let $M$ be a 3-manifold. If $M = V \cup W$, where $V, W$ are handlebodies and $\partial V = \partial W$ is an orientable closed surface $S$, then we call $M = V \cup_{S} W$ a Heegaard splitting of $M$ with splitting surface $S$. 

The genus of a Heegaard splitting is the genus of its splitting surface.
Heegaard splittings

Definition
Let $M$ be a 3-manifold. If $M = V \cup W$, where $V, W$ are handlebodies and $\partial V = \partial W$ is an orientable closed surface $S$, then we call $M = V \cup_S W$ a Heegaard splitting of $M$ with splitting surface $S$.

Definition
The genus of a Heegaard splitting is the genus of its splitting surface.
Heegaard splittings

Theorem

(Moise, Bing) Every 3-manifold admits a Heegaard splitting.
Example 1: $S^3$ has a genus 0 Heegaard splitting.
Heegaard splittings

Example 1: $\mathbb{S}^3$ has a genus 0 Heegaard splitting.

Example 2: $\mathbb{S}^3$ has a genus 1 Heegaard splitting.
Heegaard splittings

Example 1: $\mathbb{S}^3$ has a genus 0 Heegaard splitting.

Example 2: $\mathbb{S}^3$ has a genus 1 Heegaard splitting.

Example 3: $\mathbb{T}^3$ has a genus 3 Heegaard splitting.
Heegaard splittings

Definition

Let $M$ be a 3-manifold. If $M = V \cup W$, where $V, W$ are handlebodies and $\partial V = \partial W$ is an orientable closed surface $S$, then we call $M = V \cup_S W$ a Heegaard splitting of $M$ with splitting surface $S$. 
Heegaard splittings

Definition
Let $M$ be a 3-manifold. If $M = V \cup W$, where $V, W$ are handlebodies and $\partial V = \partial W$ is an orientable closed surface $S$, then we call $M = V \cup_S W$ a *Heegaard splitting* of $M$ with *splitting surface* $S$.

Definition
The *genus* of a Heegaard splitting is the genus of its splitting surface.
Definition

The *distance* of a Heegaard splitting $M = V \cup_S W$ is the minimal distance, in the curve complex of $S$, between a simple closed curve in $S$ that bounds a disk in $V$ and a simple closed curve in $S$ that bounds a disk in $W$. 