Effective constructions in algebraic topology and topological data analysis

Celebrating Gunnar’s birthday in Minnesota

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August 2, 2022
A goal of algebraic topology
To construct invariants of spaces up to some notion of equivalence.

Today
CW complexes and homotopy equivalence.

A basic tension
Computability vs strength of invariants.

Example
Cohomology vs homotopy.

A more subtle one
Effectiveness vs functoriality of their constructions.

Example
Cohomology via chain complex vs maps to Eilenberg-Maclane spaces.
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Effectively defined cohomology

Poincaré's idea
Break spaces into contractible combinatorial pieces:
Simplices, cubes, ...

Kan–Quillen's idea
Replace spaces by functors with a geometric realization:
Simplicial sets, cubical sets, ...

Compute cohomology
Using a chain complex assembled from the standard chain complexes:
$C(\Delta^n)$, $C(I^n)$, ...

Our goals (loosely stated)
Understand the diagonal map of these standard complexes better to:
1) Present effective/local computations of finer invariants in cohomology.
2) Describe explicit algebraic models of the homotopy type of spaces.
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As graded vector spaces
\[ H^\bullet \left( \mathbb{R}P^2; \mathbb{F}_2 \right) \cong H^\bullet \left( S^1 \vee S^2; \mathbb{F}_2 \right) \]
Similarly, as graded abelian groups
\[ H^\bullet \left( C\mathbb{P}^2; \mathbb{Z} \right) \cong H^\bullet \left( S^2 \vee S^4; \mathbb{Z} \right) \]
These can be distinguished by the product structure in
\[ H^\bullet \]
Defined by dualizing an explicit chain approximation to the diagonal
\[ C(\Delta^n) \to C(\Delta^n) \otimes C(\Delta^n) \]
due to Alexander and Whitney.
Similarly, Cartan and Serre constructed
\[ C(I^n) \to C(I^n) \otimes C(I^n) \]
Shortcomings of cohomology I

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As graded vector spaces

$$H^\bullet(\mathbb{RP}^2; \mathbb{F}_2) \cong H^\bullet(S^1 \vee S^2; \mathbb{F}_2).$$

Similarly, as graded abelian groups

$$H^\bullet(\mathbb{CP}^2; \mathbb{Z}) \cong H^\bullet(S^2 \vee S^4; \mathbb{Z}).$$

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\[ C(\mathbb{I}^n) \to C(\mathbb{I}^n) \otimes C(\mathbb{I}^n). \]
Let \( \Sigma \) denote suspension, for example \( \Sigma(S^1) \) is a suspension. As graded rings, 
\[
H^\bullet(\Sigma(CP^2)) \cong H^\bullet(\Sigma(S^2 \vee S^4))
\]
These can be distinguished by the action of the Steenrod algebra on \( H^\bullet \). From the spectral viewpoint this structure is present by definition. Question: Can it be described explicitly at the chain level?
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From the spectral viewpoint this structure is present by definition.

**Question:** Can it be described explicitly at the chain level?
Steenrod construction

Unlike the diagonal of spaces, chain approximations to it are not invariant under $x \otimes y \mapsto y \otimes x$. For example in $C(\Delta^n) \to C(\Delta^n) \otimes C(\Delta^n)$ we have

To correct homotopically the breaking of this symmetry, Steenrod introduced explicit maps $\Delta_i: C(\Delta^n) \to C(\Delta^n) \otimes 2$ satisfying

The cup-i coproducts. These define the Steenrod squares as $Sq^k: H^\bullet(X; F_2) \to H^\bullet(X; F_2)$ $[\alpha] \mapsto [\alpha \otimes \alpha] \Delta_i(-)$
Unlike the diagonal of spaces, chain approxs to it are not invariant under

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For example in \( C(\mathbb{1}) \to C(\mathbb{1}) \otimes C(\mathbb{1}) \) we have

\[ \begin{array}{c}
\text{\hspace{1cm}}
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\[ \text{Sq}^k : H^\bullet(X; \mathbb{F}_2) \to H^\bullet(X; \mathbb{F}_2) \]

\[ [\alpha] \mapsto [(\alpha \otimes \alpha) \Delta_i(-)] \]
A new description of Steenrod’s construction

Notation:
\[
\begin{align*}
\{v_0, \ldots, v_m\} &= \{\hat{v}_0 u, \ldots, v_m\} \\
\forall U = \{u_1 < \cdots < u_q\} \in P_{nq} &= \{U \subseteq \{0, \ldots, n\} : |U| = q\} \\
d_U &= d_{u_1} \cdots d_{u_q} \\
\epsilon &= \{u_i \in U | u_i + i \equiv \epsilon \mod 2\}
\end{align*}
\]

Definition (Med.):
For a basis element \(x \in C_m(\Delta_n, F_2)\)

\[
\Delta_i(x) = \sum_{U \in P_{n-m-i}} d_U 0(x) \otimes d_U 1(x)
\]

Example:
\[
\Delta_0[0, 1, 2] = \begin{pmatrix} d_{12} \otimes \text{id} + d_{2} \otimes d_{0} + \text{id} \otimes d_{01} \end{pmatrix} \begin{pmatrix} [0, 1, 2] \otimes 2 \end{pmatrix} = [0] \otimes [0, 1, 2] + [0] \otimes [1, 2] + [0] \otimes [2].
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Fast computation of Steenrod squares
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Comparing with SAGE: (algorithm based on EZ-AW contraction)
Fast computation of Steenrod squares

Comparing with SAGE: (algorithm based on EZ-AW contraction)

\[ \text{Sq}^1 \text{ on } \Sigma^i \mathbb{R}P^2 \]  
\( (i^{th} \text{ suspension of the real projective plane}) \)

Number of simplices in the \( i \)-th suspension of \( \mathbb{R}P^2 \) for \( i = 0, 1, \ldots, 10 \)

Execution time in milliseconds
Steenrod barcodes

Given a filtered simplicial complex $X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_n$. Cohomology induces a persistent module, its barcode is a summary of how Betti numbers are consecutively shared.

$$H^\bullet(X_n; \mathbb{F}_2) \cdots H^\bullet(X_{n-1}; \mathbb{F}_2) H^\bullet(X_0; \mathbb{F}_2)$$
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A cohomology operation induces an endomorphism

$$
\begin{align*}
H^\bullet(X_n; \mathbb{F}_2) &\to \cdots \to H^\bullet(X_{n-1}; \mathbb{F}_2) \to H^\bullet(X_0; \mathbb{F}_2) \\
\xrightarrow{Sq^k} &\xrightarrow{Sq^k} \xrightarrow{Sq^k}
\end{align*}
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With the support of K. Hess-Bellwald, jointly with U. Lupo and G. Tauzin from giotto-tda's team we developed steenroder.
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The $Sq^k$-barcode of $X$ is the barcode of $\text{img } Sq^k$. 

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\[ \xymatrix{ H^\bullet(X_n; \mathbb{F}_2) \ar[r] & \cdots \ar[r] & H^\bullet(X_{n-1}; \mathbb{F}_2) \ar[r] & H^\bullet(X_0; \mathbb{F}_2) \ar[u]^{Sq^k} } \]

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Comparing persistent $Sq^2$-modules
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Filtrations of the cone on the suspension of $S^2 \vee S^4$ and $\mathbb{C}P^2$. 

img(Sq^2) ∩ H_4 

H_4 (C \Sigma C_\mathbb{P}^2)
Comparing persistent $\text{Sq}^2$-modules

Filtrations of the cone on the suspension of $S^2 \vee S^4$ and $\mathbb{C}P^2$.

(a) $C \Sigma(S^2 \vee S^4)$

(b) $C \Sigma \mathbb{C}P^2$
Space of conformations of $C_8H_{16}$
Space of conformations of $C_{8}H_{16}$

Points in $\mathbb{R}^{24}$ (positions of 8 carbons in $\mathbb{R}^{3}$)
Space of conformations of $C_{8}H_{16}$

Points in $\mathbb{R}^{24}$ (positions of 8 carbons in $\mathbb{R}^{3}$)

Computing $Sq^{1}$ barcode of a “smooth component” of this point cloud

Consistent with a Klein bottle component.
More on cup-$i$ constructions

\begin{itemize}
\item Theorem (Med.)
\end{itemize}

All cup-$i$ constructions in the literature are equal up isomorphism:

\[ \Delta \sim \Delta' \iff \forall i \in \mathbb{N}, \Delta_i = \Delta'_i \lor \Delta_i = T \Delta'_i. \] (Proven via an axiomatic characterization.)

\begin{itemize}
\item Theorem (Cantero-Moran)
\end{itemize}

Adaptation of the new cup-$i$ formulas to define operations in Khovanov homology of knots and links.

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Steenrod's cup-$i$ construction defines the nerve of higher categories.

\begin{itemize}
\item Theorem (Laplante-Anfossi–Med.–Vallette)
\end{itemize}

Let $P \subset \mathbb{R}^n$ be an $n$-dim convex polytope. A generic orthogonal ordered basis of $\mathbb{R}^n$ defines a cellular cup-$i$ construction $S_\infty \times P \to P \times P$.

\begin{itemize}
\item Theorem (Med.)
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Sheaves on a simplicial complex $X$ fully faithfully modeled by comodules over Steenrod cup-$i$ coalgebra $\mathbb{C}(X)$ using the Ranicki–Weiss assembly.
Theorem (Med.)
All cup-$i$ constructions in the literature are equal up to isomorphism:

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Sheaves on a simplicial complex \(X\) fully faithfully modeled by comodules over Steenrod cup-\(i\) coalgebra \(C(X)\) using the Ranicki–Weiss assembly.
Relations

There are two main Steenrod square relations:

- **Cartan**
  \[ Sq^k(\alpha \beta) = \sum_{i+j=k} Sq^i(\alpha) Sq^j(\beta) \]

- **Adem**
  \[ Sq^i Sq^j = \left\lfloor \frac{i}{2} \right\rfloor \sum_{k=0} (j-k-1)(i-2k) Sq^{i+j-k} \]

**Construction (Brumfiel–Med.–Morgan)**
Explicit cochains witnessing these relations at the cochain level.

**Application (Gaiotto, Kapustin, Thorngren and others)**
Classification of (low dim. symm. protected fermionic) top. phases.

**Vague idea**
Cochains as fields on triangulated spacetime with actions defined using cup-i products and these secondary structures.
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**Adem**

\[ \text{Sq}^i \text{Sq}^j = \sum_{k=0}^{[i/2]} \binom{j - k - 1}{i - 2k} \text{Sq}^{i+j-k} \text{Sq}^k. \]
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\[ \text{Sq}^i \text{Sq}^j = \left\lfloor \frac{i}{2} \right\rfloor \sum_{k=0}^{[i/2]} \binom{j-k-1}{i-2k} \text{Sq}^{i+j-k} \text{Sq}^k. \]

**Construction (Brumfiel–Med.–Morgan)**

Explicit cochains witnessing these relations at the cochain level.
Relations

There are two main Steenrod square relations:

**Cartan**

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Classification of (low dim. symm. protected fermionic) top. phases.
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**Vague idea**
Cochains as fields on triangulated spacetime with actions defined using cup-$i$ products and these secondary structures.
Operations at odd primes

Steenrod squares come from the symmetry of the binary diagonal. Steenrod, and more generally May, also defined operations $P^k: H^\ast(X; \mathbb{F}_p) \to H^\ast(X; \mathbb{F}_p)$ from the symmetry of the diagonal $X \to X \times \cdots \times X$.

Note: indirect group homology definition. No generalizations of cup-i.

Construction (Kaufmann-Med.)

Explicit cup-$(p,i)$ coproducts defining these operations.

Example

Using the computer algebra system ComCH we have

\[
\Delta^3,2[0,1,2] = \Delta^1[0,1][0,1,2][0,1] + \Delta^2[0,2][0,1] + \Delta^1[0,2][0,2][0,1,2] - \Delta^2[0,1,2][0,1,2][1] - \Delta^1[0,2][0,1,2][1,2] + \Delta^2[0,1,2][1,2][1,2] - \Delta^1[0,1][1,2][0,1,2] - \Delta^3[2][0,1,2] - \Delta^2[0][0,1,2][0,1,2]
\]
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\[ \Delta^3,2[0,1,2] = \Delta \Delta^2[0,1] + \Delta^2[0,1,2] + \Delta^2[0,2] - \Delta^2[0,1,2][1] - \Delta^2[0,2][1,2] + \Delta^2[1,2][1,2] - \Delta[0,1][1,2][0,1,2] - \Delta^2[2][0,1,2] - \Delta[0][0,1,2][0,1,2] \]
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**Example**

Using the computer algebra system **ComCH** we have \( \Delta_{3,2}[0,1,2] = \)

\[- [0,1] [0,1,2] [0,1] + [0,1,2] [0,2] [0,1] + [0,2] [0,2] [0,1,2] \\
- [0,1,2] [0,1,2] [1] - [0,2] [0,1,2] [1,2] + [0,1,2] [1,2] [1,2] \\
- [0,1] [1,2] [0,1,2] - [0,1,2] [2] [0,1,2] - [0] [0,1,2] [0,1,2] \]
A more abstract viewpoint

Operads control algebraic structures. The operad $C$ controls cocommutative and coassociative coalgebras. An $E_\infty$-operad is an $S$-cofibrant resolution of $C$. Controls coalgebras cocommutative and coassociative up to coherent homotopies. Fact: Fully deriving its diagonal map, the chains of a space form an $E_\infty$-coalgebra. Principle (Quillen, Sullivan, Mandell, ...): All homotopy information of spaces is in this algebraic model. Question: How explicit can this $E_\infty$-structure be made?
A more abstract viewpoint

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All homotopy information of spaces is in this algebraic model.

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Explicit $E_{\infty}$-structure on (co)chains
Explicit $E_\infty$-structure on (co)chains

Theorem (Med.)
The collection of maps $C(\Delta^n) \rightarrow C(\Delta^n)^\otimes r$ obtained from compositions of

$$\Delta: C(\Delta^n) \rightarrow C(\Delta^n)^\otimes 2$$ (AW diagonal)

$$*: C(\Delta^n)^\otimes 2 \rightarrow C(\Delta^n)$$ (Join map)

defines an $E_\infty$-coalgebra on simplicial chains.
Explicit $E_\infty$-structure on (co)chains

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The collection of maps $C(\Delta^n) \to C(\Delta^n)^{\otimes r}$ obtained from compositions of

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defines an $E_\infty$-coalgebra on simplicial chains.

Join map

\[
\begin{array}{c}
\begin{array}{c}
\bullet 0 \\
\bullet 1 \\
\bullet 2
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\bullet 0 \\
\bullet 1 \\
\bullet 2
\end{array}
\end{array}
\ast
\begin{array}{c}
\begin{array}{c}
\bullet 0 \\
\bullet 1 \\
\bullet 2
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\bullet 0 \\
\bullet 1 \\
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\end{array}
\end{array}
\]

Other versions
1) Cubical (Kaufmann–Med.)
2) Multisimplicial (Med.–Pizzi–Salvatore).
Explicit $E_\infty$-structure on (co)chains

Theorem (Med.)
The collection of maps $C(\Delta^n) \to C(\Delta^n)^{\otimes r}$ obtained from compositions of

$\Delta : C(\Delta^n) \to C(\Delta^n)^{\otimes 2}$ \hspace{1cm} (AW diagonal)

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Join map

Other versions
1) Cubical (Kaufmann–Med.)
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A finitely presented $\mathcal{E}_\infty$-prop
A finitely presented $E_\infty$-prop

Consider the prop $\mathcal{M}$ generated by

\[
\begin{array}{c}
1 \\
\bullet \quad 1 \quad 2 \quad \begin{array}{c}
1 \\
\downarrow \quad \downarrow
\end{array}
\end{array}
\]

in degrees 0, 0 & 1 with non-zero boundary

\[\partial \Upsilon = \downarrow \downarrow - \downarrow \downarrow\]

and relators

\[
\begin{array}{c}
\quad = \quad = \\
\bullet \quad \bullet \quad \bullet
\end{array}
\]

and

\[\Upsilon = 0.\]
A finitely presented $E_\infty$-prop

Consider the prop $\mathcal{M}$ generated by

\begin{align*}
\begin{array}{ccc}
 & 1 \\
\downarrow & & \downarrow \\
1 & 2 & 1^2
\end{array}
\end{align*}

in degrees 0, 0 & 1 with non-zero boundary

$$\partial \Upsilon = \begin{array}{c} \begin{array}{c} \downarrow \end{array} \\
\begin{array}{c} \downarrow \\
\end{array} \end{array}$$

and relators

$$\begin{array}{ccc}
\begin{array}{c} \downarrow \end{array} &=& \begin{array}{c} \downarrow \end{array} \\
\begin{array}{c} \downarrow \end{array} &=& \begin{array}{c} \downarrow \end{array} \\
\Upsilon &=& 0.
\end{array}$$

**Theorem (Med.)**
The operad associated to $\mathcal{M}$, defined by

$$U(\mathcal{M}) = \{ \mathcal{M}(1, r) \}_{r \geq 1},$$

is a (cofibrant and Hopf) $E_\infty$-operad.
Proof (time permitting)

Basic case: $M(s,0) \sim = \mathbb{Z}\{1 \ldots s\}$

Contraction: $\partial \circ h = \text{id} - p \circ i + h \circ \partial M(s,r - 1)$

Then $(\partial \circ h)_{s \rightarrow r} = \partial_{s \rightarrow r} = s_r - 1 + (h \circ \partial)_{s \rightarrow r}$
Proof (time permitting)

Basic case: $\mathcal{M}(s, 0) \cong \mathbb{Z}\{\frac{1}{\cdot}^{s}\}$
Proof (time permitting)

Basic case: $\mathcal{M}(s, 0) \cong \mathbb{Z}\{ \cdot^s \cdot \}$

Contraction: $\partial \circ h = \text{id} - p \circ i + h \circ \partial$

\[\begin{align*}
\mathcal{M}(s, r - 1) &\xleftarrow{i} \mathcal{M}(s, r) \xrightarrow{p} \mathcal{M}(s, r) &\circlearrowleft h
\end{align*}\]
Proof (time permitting)

Basic case: $\mathcal{M}(s, 0) \cong \mathbb{Z}\{\frac{1}{\cdots}\}$

Contraction: $\partial \circ h = \text{id} - p \circ i + h \circ \partial$

$$\mathcal{M}(s, r - 1) \xleftarrow{i} \xrightarrow{p} \mathcal{M}(s, r) \xrightarrow{h}$$
Proof (time permitting)

Basic case: \( \mathcal{M}(s, 0) \cong \mathbb{Z}\{\overline{1^s}\} \)

Contraction: \( \partial \circ h = \text{id} - p \circ i + h \circ \partial \)

\[ \begin{array}{cccc}
\mathcal{M}(s, r - 1) & \xrightarrow{i} & \mathcal{M}(s, r) & \xrightarrow{h} \\
\begin{array}{c}
1 \\
\vdots \\
1 \quad r - 1
\end{array} & \quad \xrightarrow{p} & \begin{array}{c}
1 \\
\vdots \\
1 \quad 2 \quad r
\end{array}& \\
\end{array} \]

Then

\[ (\partial \circ h) \]

\[ \begin{array}{c}
1 \\
\vdots \\
1 \quad r
\end{array} \]
Proof (time permitting)

Basic case: $\mathcal{M}(s, 0) \cong \mathbb{Z}\{\ldots 1 \ldots s\}$

Contraction: $\partial \circ h = \text{id} - p \circ i + h \circ \partial$

Then

$$(\partial \circ h) \begin{array}{c} 1 \\ s \\ \vdots \\ r \end{array} = \partial \begin{array}{c} 1 \\ s \\ \vdots \\ r \end{array}$$
Proof (time permitting)

Basic case: \( \mathcal{M}(s, 0) \cong \mathbb{Z} \{ \cdot \ldots \cdot \} \)

Contraction: \( \partial \circ h = \text{id} - p \circ i + h \circ \partial \)

\[
\mathcal{M}(s, r - 1) \xrightarrow{i} \mathcal{M}(s, r) \xrightarrow{\partial} h
\]

Then

\[
(\partial \circ h) = \partial \left( \begin{array}{c}
\begin{array}{c}
1 \\
1 \\
r - 1
\end{array}
\begin{array}{c}
1 \\
1 \\
r
\end{array}
\begin{array}{c}
s \\
s \\
1
\end{array}
\end{array} \right) = \begin{array}{c}
\begin{array}{c}
1 \\
1 \\
r - 1
\end{array}
\begin{array}{c}
1 \\
1 \\
r
\end{array}
\begin{array}{c}
s \\
s \\
1
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
1 \\
1 \\
r
\end{array}
\begin{array}{c}
1 \\
1 \\
r
\end{array}
\begin{array}{c}
s \\
s \\
1
\end{array}
\end{array} + (h \circ \partial) \begin{array}{c}
\begin{array}{c}
1 \\
1 \\
r
\end{array}
\begin{array}{c}
1 \\
1 \\
r
\end{array}
\begin{array}{c}
s \\
s \\
1
\end{array}
\end{array}
\]
Proof (time permitting)

Basic case: \( \mathcal{M}(s, 0) \cong \mathbb{Z}\{ \begin{array}{c} 1 \\ \vdots \\ s \end{array} \} \)

Contraction: \( \partial \circ h = \text{id} - p \circ i + h \circ \partial \)

\[
\mathcal{M}(s, r - 1) \xleftarrow{i} \xrightarrow{p} \mathcal{M}(s, r)
\]

Then

\[
(\partial \circ h) = \partial \left( \begin{array}{c} 1 \\ 1 \\ s \\ 1 \\ r \end{array} \right) - \begin{array}{c} 1 \\ 1 \\ s \\ 1 \\ r \end{array} + (h \circ \partial) = (\text{id} - i \circ p + h \circ \partial)
\]
Two $E_\infty$-bialgebra models for loop spaces
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Construction (Adams)

Based space $(X, x)$
Two $E_\infty$-bialgebra models for loop spaces

Construction (Adams)

Based space $(X, x)$

\[ \downarrow \]

Two algebras and a q-iso of algebras:

\[ \Omega S^\triangle (X, x) \xrightarrow{\theta X} S^\square (\Omega_x X) \]

Cobar const. on based simplicial sing. chains
Cubical sing. chains on based loop space
Two $E_\infty$-bialgebra models for loop spaces

Construction (Adams)

Based space $(X, x)$

\[ \Downarrow \]

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Cubical sing. chains on based loop space

Theorem (Baues)

$\theta_X$ is a map of bialgebras.
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$\downarrow$

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Theorem (Med.–Rivera)

$\theta_X$ is a map of $E_\infty$-bialgebras.
Conclusions and future directions

Slogan
Homotopy theory requires additional structure to be used "concretely."

Today's focus
The algebro-homotopical diagonal of spaces.

I. Steenrod cup-(p,i)coproducts
Applications
Steenrod barcodes, knot invariants, lattice TFTs, K- and L-theory, connections to: higher categories, convex geometry.

Future
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II. Finitely presented prop M modeling E∞ Applications
Adams cobar as an E∞-bialgebra model of based loop spaces.

Future
Double and free loop spaces.
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Happy birthday Gunnar!