Speculations

Institute for Mathematics and its Applications

Gunnar Carlsson

Stanford University

August 5, 2022
Agenda

1. Local to Global Methods
2. Parametrized Topology
3. Motion Planning and Evasion
4. Geometry of Feature Space
Local to Global Methods

- Mayer-Vietoris exact sequences and spectral sequences
- Should be used as a tool for parallelization
- How to extend to persistent homology?
Local to Global Methods

- Bounded $K$-theory is a functor from locally compact metric spaces and proper maps to spectra
Local to Global Methods

- Bounded $K$ -theory is a functor from locally compact metric spaces and proper maps to spectra
- It is a coarse functor, depending only on large scale distances

For contractible spaces, it should be equivalent to Borel-Moore homology.

How to compute with local to global methods?
Local to Global Methods

- Bounded $K$-theory is a functor from locally compact metric spaces and proper maps to spectra
- It is a coarse functor, depending only on large scale distances
- $\mathbb{Z} \hookrightarrow \mathbb{R}$ induces an equivalence
Local to Global Methods

- Bounded $K$-theory is a functor from locally compact metric spaces and proper maps to spectra
- It is a coarse functor, depending only on large scale distances
- $\mathbb{Z} \to \mathbb{R}$ induces an equivalence
- Captures behavior at “infinity”
Local to Global Methods

- Bounded $K$-theory is a functor from locally compact metric spaces and proper maps to spectra
- It is a coarse functor, depending only on large scale distances
- $\mathbb{Z} \rightarrow \mathbb{R}$ induces an equivalence
- Captures behavior at “infinity”
- For contractible spaces, it should be equivalent to Borel-Moore homology.
Local to Global Methods

- Bounded $K$-theory is a functor from locally compact metric spaces and proper maps to spectra.
- It is a coarse functor, depending only on large scale distances.
- $\mathbb{Z} \rightarrow \mathbb{R}$ induces an equivalence.
- Captures behavior at “infinity.”
- For contractible spaces, it should be equivalent to Borel-Moore homology.
- How to compute with local to global methods?
Local to Global Methods

There is a natural map from the pushout of the diagram

\[ \ast \simeq K([0, +\infty)) \leftarrow K(\{0\}) \rightarrow K((-, 0]) \simeq \ast \]

to \( K(\mathbb{R}) \), but can’t prove it to be an equivalence.
Local to Global Methods

▶ There is a natural map from the pushout of the diagram

\[ * \simeq K([0, +\infty)) \leftarrow K(\{0\}) \rightarrow K((-\infty, 0]) \simeq * \]

to \( K(\mathbb{R}) \), but can’t prove it to be an equivalence.

▶ Colimit as \( R \to \infty \) of pushouts works

\[
\begin{array}{ccc}
K([-R, +\infty)) & \leftarrow & K([-R, R]) \rightarrow K((-\infty, R]) \\
& \downarrow & \\
K([-R', +\infty)) & \leftarrow & K([-R', +R']) \rightarrow K((-\infty, +R])
\end{array}
\]
Local to Global Methods

Failure of Mayer-Vietoris for Persistent Homology
Local to Global Methods

Patch: Persistent Coverings
Local to Global Methods

Typical Case for Data in the Plane
Local to Global Methods

\[ \mathcal{U} = \bigsqcup_{\alpha \in A} \mathcal{U}_\alpha \]

Consider maps of coverings

\[ \mathcal{U} \rightarrow \mathcal{U} \times_{\times} \mathcal{U} \xrightarrow{d_0, d_1, d_2} \mathcal{U} \times_{\times} \mathcal{U} \xrightarrow{d_0, d_1} \mathcal{U} \]
Consider maps of coverings $\coprod_{\alpha \in A} U_\alpha \to \coprod_{\beta \in B} V_\beta$
Consider maps of coverings \( \coprod_{\alpha \in A} U_\alpha \to \coprod_{\beta \in B} V_\beta \).

Under suitable conditions, the map from the colimit to \( X \) can be proved to give an equivalence.
Local to Global Methods

▶ All levels of spectral sequence can be computed in parallel
Local to Global Methods

- All levels of spectral sequence can be computed in parallel
- Output is more informative than just PH

Persistent coverings
Local to Global Methods

- All levels of spectral sequence can be computed in parallel
- Output is more informative than just PH
- Can find "scale of origin" of a class
Local to Global Methods

- All levels of spectral sequence can be computed in parallel
- Output is more informative than just PH
- Can find “scale of origin” of a class
- Need to find right definition of coverings
Local to Global Methods

- All levels of spectral sequence can be computed in parallel
- Output is more informative than just PH
- Can find “scale of origin” of a class
- Need to find right definition of coverings
- Persistent coverings
Etale homotopy theory assigns a space $X^{et}$ to an algebraic variety over a field $K$, say $\mathbb{Q}$.
Etale homotopy theory assigns a space $X^{et}$ to an algebraic variety over a field $K$, say $\mathbb{Q}$.

If $K = \mathbb{C}$ is algebraically closed, then $X^{et}$ roughly captures the topology of the set of complex points of $X$. 
Etale homotopy theory assigns a space $X^{et}$ to an algebraic variety over a field $K$, say $\mathbb{Q}$.

If $K = \mathbb{C}$ is algebraically closed, then $X^{et}$ roughly captures the topology of the set of complex points of $X$.

For general $K$, $X^{et}$ is equipped with a reference map $\pi : X^{et} \rightarrow BG_K$, where $BG_K$ denotes the “classifying space” of the absolute Galois group $G_K$ of $K$. 
Etale homotopy theory assigns a space $X^{et}$ to an algebraic variety over a field $K$, say $\mathbb{Q}$.

If $K = \mathbb{C}$ is algebraically closed, then $X^{et}$ roughly captures the topology of the set of complex points of $X$.

For general $K$, $X^{et}$ is equipped with a reference map $\pi : X^{et} \rightarrow BG_K$, where $BG_K$ denotes the “classifying space” of the absolute Galois group $G_K$ of $K$.

A $K$-rational point $x$ on $X$ produces a map $s_x : BG_K \rightarrow X^{et}$ so that $\pi \circ s_x = id_{BG_K}$, i.e. a section of $\pi$. 
Topology Over a Base
Analogous situation in time dependent motion planning
Analogous situation in time dependent motion planning

One considers a situation where there is a moving set of obstacles within a region $R$, and asks whether there is a path that avoids obstacles.
Analogous situation in time dependent motion planning
One considers a situation where there is a moving set of obstacles within a region $R$, and asks whether there is a path that avoids obstacles
Comes up in the study of moving sensor nets
Topography Over a Base

- Analogous situation in time dependent motion planning
- One considers a situation where there is a moving set of obstacles within a region $R$, and asks whether there is a path that avoids obstacles
- Comes up in the study of moving sensor nets
- The set of points away from the obstacles at each time $t$ gives a space with reference map to $\mathbb{R}$, given by the time coordinate. The possible paths avoiding obstacles are exactly sections of the reference map
Topology Over a Base

(a)

(b)
All of homotopy theory and topology can be done in this context, with the objects being spaces $X$ with a reference map to a fixed base space $B$. Sections are analogous to base points in the absolute setting. $H^\ast(X)$ becomes a module over $H^\ast(B)$, so have a much richer set of invariants. Are they useful?
Topology Over a Base

- All of homotopy theory and topology can be done in this context, with the objects being spaces $X$ with a reference map to a fixed base space $B$
- Sections are analogous to base points in the absolute setting
Topology Over a Base

- All of homotopy theory and topology can be done in this context, with the objects being spaces $X$ with a reference map to a fixed base space $B$
- Sections are analogous to base points in the absolute setting
- $H^*(X)$ becomes a module over $H^*(B)$, so have a much richer set of invariants. Are they useful?
Traditionally topologists have restricted themselves to $\pi : X \to B$ where the map $\pi$ is a fibration.
Traditionally topologists have restricted themselves to \(\pi : X \to B\) where the map \(\pi\) is a fibration.

Works well in the etale situation, and is more convenient computationally.

Handling them requires substantial technical effort, which Wyatt Mackey has carried out.
Traditionally topologists have restricted themselves to $\pi : X \rightarrow B$ where the map $\pi$ is a fibration.

Works well in the etale situation, and is more convenient computationally.

Is not useful for the motion planning setting. The obstacles and their complements should not be expected to produce fibrations.
Traditionally topologists have restricted themselves to \( \pi : X \to B \) where the map \( \pi \) is a fibration. Works well in the etale situation, and is more convenient computationally. Is not useful for the motion planning setting. The obstacles and their complements should not be expected to produce fibrations. Handling them requires substantial technical effort, which Wyatt Mackey has carried out.
What are the problems one might attempt to answer concerning a parametrized space $\pi : X \to B$?
What are the problems one might attempt to answer concerning a parametrized space $\pi : X \to B$?

Is there a section?
What are the problems one might attempt to answer concerning a parametrized space \( \pi : X \to B \)?

- Is there a section?
- If there is one, what is the homotopy classification of sections?
What are the problems one might attempt to answer concerning a parametrized space $\pi : X \rightarrow B$?

- Is there a section?
- If there is one, what is the homotopy classification of sections?
- What is the homotopy type of the space of sections?
Motion Planning and Evasion

- First question is clearly important. If there is no section, then one is happy if one is trying to catch evaders and unhappy if one is an evader.
Motion Planning and Evasion

- First question is clearly important. If there is no section, then one is happy if one is trying to catch evaders and unhappy if one is an evader.
- The second question is important if one is trying to perform optimization on the set of sections, to obtain shortest or least energy paths.
Motion Planning and Evasion

- First question is clearly important. If there is no section, then one is happy if one is trying to catch evaders and unhappy if one is an evader.
- The second question is important if one is trying to perform optimization on the set of sections, to obtain shortest or least energy paths.
- A key problem in optimization is the existence of multiple basins of attraction, and the enumeration of a reasonable class of paths is a good starting point.
Motion Planning and Evasion

- First question is clearly important. If there is no section, then one is happy if one is trying to catch evaders and unhappy if one is an evader.
- The second question is important if one is trying to perform optimization on the set of sections, to obtain shortest or least energy paths.
- A key problem in optimization is the existence of multiple basins of attraction, and the enumeration of a reasonable class of paths is a good starting point.
- The third problem can also help with optimization.
Motion Planning and Evasion

- Sara Kalisnik et al have developed a version of Alexander duality in this parametrized setting. This permits the identification of the cohomology of the uncovered region from the homology of the covered region.

- It is critical that the cohomology of the uncovered region be obtained with cup products.

- This can be done computationally with reasonable assumptions about how the obstacle set is presented (Brad Nelson).

- There is now a reasonable approach to the problems above.

- Why are the cup products important?
Sara Kalisnik et al have developed a version of Alexander duality in this parametrized setting. This permits the identification of the cohomology of the uncovered region from the homology of the covered region.

It is critical that the cohomology of the uncovered region be obtained \textit{with cup products}.
Motion Planning and Evasion

- Sara Kalisnik et al have developed a version of Alexander duality in this parametrized setting. This permits the identification of the cohomology of the uncovered region from the homology of the covered region.
- It is critical that the cohomology of the uncovered region be obtained \textit{with cup products}
- This can be done computationally with reasonable assumptions about how the obstacle set is presented (Brad Nelson).
Sara Kalisnik et al have developed a version of Alexander duality in this parametrized setting. This permits the identification of the cohomology of the uncovered region from the homology of the covered region.

It is critical that the cohomology of the uncovered region be obtained with cup products.

This can be done computationally with reasonable assumptions about how the obstacle set is presented (Brad Nelson).

There is now a reasonable approach to the problems above.
Sara Kalisnik et al have developed a version of Alexander duality in this parametrized setting. This permits the identification of the cohomology of the uncovered region from the homology of the covered region.

It is critical that the cohomology of the uncovered region be obtained with cup products.

This can be done computationally with reasonable assumptions about how the obstacle set is presented (Brad Nelson).

There is now a reasonable approach to the problems above.

Why are the cup products important?
Motion Planning and Evasion

- $H_0$ or $H^0$ (as vector spaces) determine the cardinality of $\pi_0$, but not as a set valued functor.
Motion Planning and Evasion

- $H_0$ or $H^0$ (as vector spaces) determine the cardinality of $\pi_0$, but not as a set valued functor
- $H^0$, as a ring under cup products does.
Motion Planning and Evasion

- $H_0$ or $H^0$ (as vector spaces) determine the cardinality of $\pi_0$, but not as a set valued functor.
- $H^0$, as a ring under cup products does.
- $\pi_0(X) \cong Hom_{Rings}(H^0(X), k)$, where $k$ is the coefficient ring.
Motion Planning and Evasion

- $H_0$ or $H^0$ (as vector spaces) determine the cardinality of $\pi_0$, but not as a set valued functor.
- $H^0$, as a ring under cup products does.
- $\pi_0(X) \cong Hom_{\text{Rings}}(H^0(X), k)$, where $k$ is the coefficient ring.
- Functorial identification of $\pi_0$ is critical in the description of the homotopy classes of sections.
Given $\pi : X \to B$, one can consider the space $\Pi_0(X)$ over $B$ obtained by replacing each fiber $F_x$ with $\pi_0(F_x)$. Problem 1 above can now be answered, since there is a section for $X$ if and only if there is a section of $\Pi_0(X)$. Problem two above is answered in terms of a short exact section of sets $\ast \to \lim_1(\pi_1(F_x)) \to \pi_0(\Gamma(X)) \to \lim_0(\pi_0(F_x)) \to \ast$. 
Motion Planning and Evasion

- Given $\pi : X \to B$, one can consider the space $\Pi_0(X)$ over $B$ obtained by replacing each fiber $F_x$ with $\pi_0(F_x)$.
- Problem 1 above can now be answered, since there is a section for $X$ if and only if there is a section of $\Pi_0(X)$. 

Problem two above is answered in terms of a short exact section of sets $\ast \to \lim_1(\pi_1(F_x)) \to \pi_0(\Gamma(X)) \to \lim_0(\pi_0(F_x)) \to \ast$. 
Motion Planning and Evasion

Given $\pi : X \rightarrow B$, one can consider the space $\Pi_0(X)$ over $B$ obtained by replacing each fiber $F_x$ with $\pi_0(F_x)$

Problem 1 above can now be answered, since there is a section for $X$ if and only if there is a section of $\Pi_0(X)$

Problem two above is answered in terms of a short exact section of sets

$$
* \rightarrow \lim^1(\pi_1(F_x)) \rightarrow \pi_0(\Gamma(X)) \rightarrow \lim^0(\pi_0(F_x)) \rightarrow *
$$
Motion Planning and Evasion
How to get at $\pi_1(F_x)$?

There is an approach to computing homotopy from homology, called the Adams spectral sequence. Originally developed for the stable situation, there is a more involved version which computes unstable homotopy. Often intractable in high dimensions, but manageable in low dimensions. There is a version of this spectral sequence that works for spaces over $\mathbb{R}$ or $[0, 1]$. May not be relevant for optimization problems, but offers hope for Problem 3.
How to get at $\pi_1(F_x)$?

There is an approach to computing homotopy from homology, called the *Adams spectral sequence*. Originally developed for the stable situation, there is a more involved version which computes unstable homotopy. Often intractable in high dimensions, but manageable in low dimensions. There is a version of this spectral sequence that works for spaces over $\mathbb{R}$ or $[0,1]$. May not be relevant for optimization problems, but offers hope for Problem 3.
Motion Planning and Evasion

- How to get at $\pi_1(F_x)$?
- There is an approach to computing homotopy from homology, called the *Adams spectral sequence*
- Originally developed for the stable situation, there is a more involved version which computes unstable homotopy

May not be relevant for optimization problems, but offers hope for Problem 3.
Motion Planning and Evasion

- How to get at $\pi_1(F_x)$?
- There is an approach to computing homotopy from homology, called the Adams spectral sequence
- Originally developed for the stable situation, there is a more involved version which computes unstable homotopy
- Often intractable in high dimensions, but manageable in low dimensions.
How to get at $\pi_1(F_x)$?

There is an approach to computing homotopy from homology, called the *Adams spectral sequence*

Originally developed for the stable situation, there is a more involved version which computes unstable homotopy

Often intractable in high dimensions, but manageable in low dimensions.

There is a version of this spectral sequence that works for spaces over $\mathbb{R}$ or $[0, 1]$. 
Motion Planning and Evasion

- How to get at $\pi_1(F_x)$?
- There is an approach to computing homotopy from homology, called the *Adams spectral sequence*
- Originally developed for the stable situation, there is a more involved version which computes unstable homotopy
- Often intractable in high dimensions, but manageable in low dimensions.
- There is a version of this spectral sequence that works for spaces over $\mathbb{R}$ or $[0, 1]$.
- May not be relevant for optimization problems, but offers hope for Problem 3.
Topological Structures on Feature Space

- **Mapper** is a method for producing graph structures on finite metric spaces. It can enable the study of the taxonomy for a data set and view data from various points of view and levels of resolution.

  It is the nerve construction applied to a certain cover of the data set, so nodes correspond to collections of data points.
Mapper is a method for producing graph structures finite metric spaces.
As such, it can enable the study of the taxonomy for a data set.
Topological Structures on Feature Space

- *Mapper* is a method for producing graph structures finite metric spaces
- As such, it can enable the study of the taxonomy for a data set
- It can view data from various points of view and levels of resolution.
Mapper is a method for producing graph structures finite metric spaces.

As such, it can enable the study of the taxonomy for a data set.

It can view data from various points of view and levels of resolution.

It is the nerve construction applied to a certain cover of the data set, so nodes correspond to collections of data points.
Topological Structures on Feature Space
Topological Structures on Feature Space

Diagram of gene expression profiles for breast cancer
M. Nicolau, A. Levine, and G. Carlsson, PNAS 2011

NKI Breast Cancer Data set
We have seen topological models of the set of rows of data sets.
Topological Structures on Feature Space

- We have seen topological models of the set of rows of data sets.
- It is also useful to build topological models of the set of columns, or features.
Topological Structures on Feature Space

▶ We have seen topological models of the set of rows of data sets.
▶ It is also useful to build topological models of the set of columns, or features.
▶ Data points or sets of data points can be regarded as functions on the vertices of this model.
Topological Structures on Feature Space

▶ We have seen topological models of the set of rows of data sets
▶ It is also useful to build topological models of the set of columns, or features
▶ Data points or sets of data points can be regarded as functions on the vertices of this model
▶ They can then be displayed as colorings of the topological model of the set of features
Topological Structures on Feature Space

- We have seen topological models of the set of rows of data sets.
- It is also useful to build topological models of the set of columns, or features.
- Data points or sets of data points can be regarded as functions on the vertices of this model.
- They can then be displayed as colorings of the topological model of the set of features.
- They often have good continuity properties.
Topological Models of Feature Space

Cohort 1  Cohort 2  Cohort 3

Colorings of cohorts in NKI data set
Topological Models of Feature Space

Gut biome data from L. Smarr (UCSD)
Thank You Facundo, Kirsten, and Matt!